

Research Article

A Sixth-order Hybrid Block Method for the Direct Solution of 3rd-order Initial Value Problems of Ordinary Differential Equations

Duromola Monday Kolawole^{1,*} , Akinmoladun Olusegun Mayowa² ,
Fafesobi Gbenga Olatunde¹ 

¹Department of Mathematical Sciences, Federal University of Technology, Akure, Nigeria

²Department of Mathematical Sciences, Lagos State University of Science and Technology, Ikorodu, Nigeria

Abstract

This work focuses on the derivation, analysis and numerical application of a hybrid linear multistep formula for directly solving third-order initial value problems (IVPs) ordinary differential equations (ODEs). The method is derived by adopting the collocation and interpolation procedure with a power series polynomial of degree eight as the basis function. The procedure resulted in the construction of a continuous hybrid linear multistep method by collocating the third derivative at nodal and five off-nodal points to generate a system of linear equations for the determination of the unknown parameters, from which the formulas that constitute the proposed hybrid block method are obtained. The suggested method is fully hybrid, which is an important property that contributed to the good accuracy and minimum errors associated with the results of the method. The analysis of the basic properties of the suggested method reveals that the method is of theoretical order six, stable, consistent and convergent. Four numerical examples are considered for the numerical experiment to confirm the accuracy of the proposed method. The experiment shows that the present method is very efficient for the numerical approximation of third-order initial value problems for ordinary differential equations. The method also performs favorably well when compared with the results of some cited methods in the literature.

Keywords

Power Series Polynomials, Stable, Consistent, Accuracy, Interpolation and Collocation

1. Introduction

The modelling of real-life situation often gives rise to third-order IVPs of ODEs which are generally written as:

$$y''' = f(x, y, y', y''), y(x_0) = \alpha_0, y'(x_0) = \alpha_1, y''(x_0) = \alpha_2. \text{ where } \alpha_0, \alpha_1, \alpha_2 \in \mathbb{R} \quad (1)$$

*Correspondence: Duromola Monday Kolawole (mkduromola@futa.edu.ng)

Received: 7 June 2026; Accepted: 22 June 2026; Published: 17 July 2026



Copyright: © The Author(s), 2026. Published by Science Publishing Group. This is an **Open Access** article, distributed under the terms of the Creative Commons Attribution 4.0 License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution and reproduction in any medium, provided the original work is properly cited.

This work proposed method which is easily implemented and can also provide high order of accuracy. Higher-order ODEs are frequently written as equivalent systems of first-order ODEs to get numerical solutions, which may subsequently be solved using proven numerical techniques, and this has assisted in obtaining solutions to higher-order ODEs discussed by [1-4]. However, because of the longer runtime and additional computing effort, this transformation may increase computational cost in some cases, particularly for large systems or stiff problems, as [5] pointed out. Notably, significant attention has been given to the development of methods that directly solve IVPs of higher-order ODEs without reducing them to systems of first-order. This approach has been discussed by researchers such as [6-11]. Recently, [12] examined the numerical solution of third-order IVPs of ODEs using a set of linear multistep method with block extension. [13] discussed the P-stable single-step block method of solving third-order IVPs: nine orders of accuracy, and additionally, the modified fourth derivative block approach and its direct applicability to third-order initial value problems were covered in detail by [14]. In this research, third-order ordinary differential equations (IVPs) were numerically integrated using a hybrid linear multistep approach with five off-steps. By improving accuracy, stability, and computational time, this technique seeks to provide a more potent substitute for reduction-based methods.

2. Derivation of the Method

This section presents a detailed derivation of the proposed scheme by applying interpolation and collocation procedures to a power series polynomial, which serves as the trial function. The power series function is given as:

$$y(x) = \sum_{j=0}^{(r+s)-1} a_j x^j \quad (2)$$

where r and s are respectively number of interpolation and collocation points, a_j 's are parameters to be obtained

uniquely and x is differentiable over the $[a, b]$. The third derivative of (2) gives

$$y'''(x) = \sum_{j=0}^{(r+s)-1} j(j-1)(j-2)a_j x^{j-3} \quad (3)$$

Equating (1) and (3) yields the differential system

$$\sum_{j=0}^{(r+s)-1} j(j-1)(j-2)a_j x^{j-3} = f(x, y(x), y'(x), y''(x)) \quad (4)$$

By collocating (4) at grid points $x = x_{n+\frac{j}{180}}$, $j = 0(1)5$ and interpolating Eq. (2) at $x = x_{n+\frac{j}{180}}$, $j = 2(1)4$ yield a system of equations for interpolation equation,

$$\sum_{j=0}^{(r+s)-1} a_j x^j = y_{n+\frac{j}{180}}, \quad j = 2(1)4 \quad (5)$$

and for collocation,

$$\sum_{j=2}^{(r+s)-1} j(j-1)(j-2)a_j x^{j-3} = f_{n+\frac{j}{180}}, \quad j = 0(1)5. \quad (6)$$

By putting equations (5) and (6) in matrix form and solving them to determine the parameter values a_j 's, $j = 0, 1, \dots, 8$, which when substituted in (3) yields, after a some simplification, thus provides a continuous hybrid linear multistep formula of the form:

$$y(t) = \sum_{j=2}^4 \alpha_{\frac{j}{180}} y_{n+\frac{j}{180}}(t) + h^3 \sum_{j=0}^5 \beta_{\frac{j}{180}} f_{n+\frac{j}{180}}(t) \quad (7)$$

where $\alpha_j(x)$ and $\beta_j(x)$ are the coefficients that defined the method and are obtained as:

$$\alpha_{\frac{1}{45}}(t) = 16200t^2 - 450t + 3, \quad \alpha_{\frac{1}{60}}(t) = -32400t^2 + 1080t - 8,$$

$$\alpha_{\frac{1}{90}}(t) = 16200t^2 - 630t + 6, \quad \beta_0(t) = -\frac{32805000}{7}t^8 + \frac{437400}{7}t^7 - 34425t^6 + \frac{2025}{2}t^5 - \frac{137}{8}t^4 + \frac{1}{6}t^3 - \frac{6383}{7257600}t^2 + \frac{479}{21772800}t - \frac{1}{466560000},$$

$$\beta_{\frac{1}{180}}(t) = \frac{164025000}{7}t^8 - 291600t^7 + 143775t^6 - 3465t^5 + \frac{75}{2}t^4 - \frac{9307}{2419200}t^2 + \frac{593}{18662400}t - \frac{109}{1399680000},$$

$$\beta_{\frac{1}{90}}(t) = -\frac{32805000}{7}t^8 + \frac{37908000}{7}t^7 - 238950t^6 + 4815t^5 - \frac{75}{2}t^4 + \frac{143}{80640}t^2 - \frac{19847}{32659200}t - \frac{247}{699940000},$$

$$\beta_{\frac{1}{60}}(t) = \frac{328050000}{7}t^8 - \frac{34992000}{7}t^7 + 198450t^6 - 3510t^5 + 25t^4 - \frac{7397}{3628800}t^2 + \frac{53}{134400}t - \frac{59}{233280000},$$

$$\beta_{\frac{1}{45}}(t) = -\frac{164025000}{7}t^8 + \frac{16038000}{7}t^7 - 83025t^6 + \frac{2745}{2}t^5 - \frac{75}{8}t^4 + \frac{599}{2419200}t^2 - \frac{83}{130636800}t + \frac{1}{1399680000},$$

$$\beta_{\frac{1}{36}}(t) = \frac{3280500}{7}t^8 - \frac{2916000}{7}t^7 + 14175t^6 - 225t^5 + \frac{3}{2}t^4 - \frac{103}{2419200}t^2 + \frac{127}{653184000}t - \frac{1}{1399680000},$$

where $t = \frac{x - x_n}{h}$. Evaluate Eq. (8) at $t = 0, \frac{1}{180}, \frac{1}{36}$ to obtain the discrete one thirty-sixth step formulas

$$y_n - 6y_{n+\frac{1}{90}} + 8y_{n+\frac{1}{60}} - 3y_{n+\frac{1}{45}} = \frac{-h^3}{1399680000} \left(\begin{array}{l} 3f_n + 109f_{n+\frac{1}{180}} + 494f_{n+\frac{1}{90}} \\ + 354f_{n+\frac{1}{60}} - f_{n+\frac{1}{45}} + f_{n+\frac{1}{36}} \end{array} \right) \quad (8)$$

$$y_{n+\frac{1}{180}} - 3y_{n+\frac{1}{90}} + 3y_{n+\frac{1}{60}} - y_{n+\frac{1}{45}} = \frac{-h^3}{2799360000} \left(\begin{array}{l} f_n - 3f_{n+\frac{1}{180}} + 242f_{n+\frac{1}{90}} \\ + 242f_{n+\frac{1}{60}} - 3f_{n+\frac{1}{45}} + f_{n+\frac{1}{36}} \end{array} \right) \quad (9)$$

$$y_{\frac{1}{36}} - 3y_{n+\frac{1}{90}} + 3y_{n+\frac{1}{60}} - y_{n+\frac{1}{45}} = \frac{-h^3}{2799360000} \left(\begin{array}{l} f_n - 7f_{n+\frac{1}{180}} + 18f_{n+\frac{1}{90}} \\ - 262f_{n+\frac{1}{60}} - 227f_{n+\frac{1}{45}} - 3f_{n+\frac{1}{36}} \end{array} \right) \quad (10)$$

In order to implement the derived method in block mode, we adopted the block formula proposed in Awoyemi *et al.* [15]. The formula in its normalized form is given as

$$A^{(0)}Y_m = ey_n + h^{\mu-\lambda}df(y_m) + h^{\mu-\lambda}bF(y_m) \quad (11)$$

By evaluating Eq. (7) at $t = \frac{1}{36}$; the first and second derivatives at $x = x_{n+i}$, $i = 0 \left(\frac{1}{180} \right) \frac{1}{36}$ and substituting into Eq. (11) gives the coefficients matrices as:

$$y_{n+\frac{1}{180}} = y_n + \frac{1}{180}hy'_n + \frac{1}{64800}h^2y''_n + \frac{h^3}{235146240000} \left[3929f_n + 475f_{n+\frac{1}{180}} - 3862f_{n+\frac{1}{90}} + 2422f_{n+\frac{1}{60}} - 883f_{n+\frac{1}{45}} + 139f_{n+\frac{1}{36}} \right], \quad (12)$$

$$y_{n+\frac{1}{90}} = y_n + \frac{1}{180}hy'_n + \frac{1}{16200}h^2y''_n + \frac{h^3}{36741600000} \left[317f_n + 734f_{n+\frac{1}{180}} - 380f_{n+\frac{1}{90}} + 244f_{n+\frac{1}{60}} - 89f_{n+\frac{1}{45}} + 14f_{n+\frac{1}{36}} \right] \quad (13)$$

$$y_{n+\frac{1}{60}} = y_n + \frac{1}{60}hy'_n + \frac{1}{7200}h^2y''_n + \frac{h^3}{2903040000} \left[609f_n + 1710f_{n+\frac{1}{180}} - 486f_{n+\frac{1}{90}} + 470f_{n+\frac{1}{60}} - 171f_{n+\frac{1}{45}} + 27f_{n+\frac{1}{36}} \right] \quad (14)$$

$$y_{n+\frac{1}{45}} = y_n + \frac{1}{45}hy'_n + \frac{1}{4050}h^2y''_n + \frac{h^3}{229635000} \left[89f_n + 292f_{n+\frac{1}{180}} - 28f_{n+\frac{1}{90}} + 88f_{n+\frac{1}{60}} - 25f_{n+\frac{1}{45}} + 4f_{n+\frac{1}{36}} \right] \quad (15)$$

$$y_{n+\frac{1}{36}} = y_n + \frac{1}{36}hy'_n + \frac{1}{2592}h^2y''_n + \frac{h^3}{9405849600} \left[\begin{array}{l} 5825f_n + 20375f_{n+\frac{1}{180}} + 250f_{n+\frac{1}{90}} \\ + 7750f_{n+\frac{1}{60}} - 707f_{n+\frac{1}{45}} + 275f_{n+\frac{1}{36}} \end{array} \right] \quad (16)$$

$$y'_{n+\frac{1}{180}} = hy'_n + \frac{1}{180}h^2y''_n + \frac{h^2}{1777\ 640256000} \left[134011584f_n + 23486545f_{n+\frac{1}{180}} - 16568492f_{n+\frac{1}{90}} + 10243726f_{n+\frac{1}{60}} - 3712126f_{n+\frac{1}{45}} + 582401f_{n+\frac{1}{36}} \right] \quad (17)$$

$$y'_{n+\frac{1}{90}} = hy'_n + \frac{1}{90}h^2y''_n + \frac{h^2}{20412000} \left[355f_n + 1088f_{n+\frac{1}{180}} - 370f_{n+\frac{1}{90}} + 272f_{n+\frac{1}{60}} - 101f_{n+\frac{1}{45}} + 16f_{n+\frac{1}{36}} \right] \quad (18)$$

$$y'_{n+\frac{1}{60}} = hy'_n + \frac{1}{60}h^2y''_n + \frac{h^2}{12096000} \left[328f_n + 1167f_{n+\frac{1}{180}} - 24f_{n+\frac{1}{90}} + 290f_{n+\frac{1}{60}} - 96f_{n+\frac{1}{45}} + 15f_{n+\frac{1}{36}} \right] \quad (19)$$

$$y'_{n+\frac{1}{45}} = hy'_n + \frac{1}{45}h^2y''_n + \frac{h^2}{2020759\ 178256000} \left[74\ 446938176f_n + 281\ 947978624f_{n+\frac{1}{180}} + 204\ 419581821f_{n+\frac{1}{90}} + 120\ 382283008f_{n+\frac{1}{60}} - 1583971200f_{n+\frac{1}{45}} + 3167954816f_{n+\frac{1}{36}} \right] \quad (20)$$

$$y'_{n+\frac{1}{36}} = hy'_n + \frac{1}{36}h^2y''_n + \frac{h^2}{2612736} \left[122f_n + 475f_{n+\frac{1}{180}} + 100f_{n+\frac{1}{90}} + 250f_{n+\frac{1}{60}} + 250f_{n+\frac{1}{45}} + 11f_{n+\frac{1}{36}} \right] \quad (21)$$

$$y''_{n+\frac{1}{180}} = h^2y''_n + \frac{h}{259200} \left[475f_n + 1427f_{n+\frac{1}{180}} - 798f_{n+\frac{1}{90}} + 482f_{n+\frac{1}{60}} - 173f_{n+\frac{1}{45}} + 27f_{n+\frac{1}{36}} \right] \quad (22)$$

$$y''_{n+\frac{1}{90}} = h^2y''_n + \frac{h}{16200} \left[28f_n + 129f_{n+\frac{1}{180}} + 14f_{n+\frac{1}{90}} + 14f_{n+\frac{1}{60}} - 6f_{n+\frac{1}{45}} + f_{n+\frac{1}{36}} \right] \quad (23)$$

$$y''_{n+\frac{1}{60}} = h^2y''_n + \frac{h}{9600} \left[17f_n + 73f_{n+\frac{1}{180}} + 38f_{n+\frac{1}{90}} + 38f_{n+\frac{1}{60}} - 7f_{n+\frac{1}{45}} + f_{n+\frac{1}{36}} \right] \quad (24)$$

$$y''_{n+\frac{1}{45}} = h^2y''_n + \frac{h}{4050} \left[7f_n + 32f_{n+\frac{1}{180}} + 12f_{n+\frac{1}{90}} + 32f_{n+\frac{1}{60}} + 7f_{n+\frac{1}{45}} \right] \quad (25)$$

$$y''_{n+\frac{1}{36}} = h^2y''_n + \frac{h}{10368} \left[19f_n + 75f_{n+\frac{1}{180}} + 50f_{n+\frac{1}{90}} + 50f_{n+\frac{1}{60}} + 75f_{n+\frac{1}{45}} + 19f_{n+\frac{1}{36}} \right] \quad (26)$$

Equations (12)-(26) are the discrete formulas of one-thirty-six off-steps hybrid block method suggested in the work for the direct solution of third-order IVPs of ordinary differential equations which require no predictor to start the corrector.

the propose method was carried out as follows.

3.1. Order and Error Constant of the Method

The linear multistep formula in equation (10) in its conventional form can be express as

$$\sum_{j=2}^4 \alpha_j y_{n+\frac{j}{180}} = h^3 \sum_{j=0}^5 \beta_j y'''_{n+\frac{j}{180}} \quad (27)$$

3. Analysis of the Basic Properties of the Proposed Method

This section presents the analysis of the basic properties of

The local truncation error associated with equation (27), going the work of Lambert [2], is defined by the difference operator.

$$L_{\frac{j}{180}} \left\{ y(x) : h \right\} = \sum_{j=2}^4 \left\{ \alpha_{\frac{j}{180}} y \left(x_n + \frac{j}{180} h \right) \right\} - h^3 \sum_{j=0}^5 \left\{ \beta_{\frac{j}{180}} y''' \left(x_n + \frac{j}{180} h \right) \right\} \quad (28)$$

where $y(x)$ is continuously differentiable. Expanding (10) in Taylor series about the point x yields the expression of the local truncation error

$$L_{\frac{j}{180}} \left\{ y(x) : h \right\} = c_0 y(x) + c_1 h y'(x) + c_2 h^2 y''(x) + \dots + c_{p+2} h^{p+2} y^{(p+2)}(x) + c_{p+3} h^{p+3} y^{(p+3)}(x) \quad (29)$$

The term c_{p+3} is the error constant while the local truncation error is given by:

$$t_{n+k} = c_{p+3} h^{(p+3)} y^{(p+3)}(x_n) + O(h^{(p+4)}) \quad (30)$$

since $c_0 = c_1 = \dots = c_{p+2} = 0, c_{p+3} \neq 0$. The procedure reveals that the suggested method has order $p = 6$ with error constant $c_{p+3} = \frac{1}{187449529397760000000000}$, see [16].

3.2. Zero Stability of the Proposed Method

Definition 3.1 (Fatula [17])

The block method presented in (12) - (26) is zero Stable provided the roots $z_j, j = 1(1)k$ of the first characteristic polynomial $\rho(r)$ specified as

$$\rho(z) = \det \left[\sum_{j=0}^k A^{(j)} Z^{k-j} \right] = 0, A^{(0)} = -1 \quad (31)$$

Satisfies $|z_j| \leq 1$, and for those roots with $|z_j| = 1$, the

$$\sigma(r) = \frac{1}{2799360000} - \frac{7}{2799360000} r^{\frac{1}{180}} + \frac{18}{2799360000} r^{\frac{1}{90}} - \frac{222}{2799360000} r^{\frac{1}{60}} - \frac{227}{2799360000} r^{\frac{1}{45}} - \frac{3}{2799360000} r^{\frac{1}{36}}$$

This implies that the method presented in this report is consistent since it satisfies the following conditions:

- The order of the method is $p = 6 > 1$ which is obvious.
- For the method, $\alpha_1 = 1, \alpha_2 = -3, \alpha_3 = 3, \text{ and } \alpha_4 = -1$,

thus

$$\sum_{j=1}^4 \alpha_j = 1 - 3 + 3 - 1 = 0, \text{ show the condition (ii) is satisfied.}$$

- If

multiplicity must not exceed 2.

By applying definition (3.1), the block is found to be zero stable since the roots of the characteristic polynomial of the proposed block method satisfy $|z| \leq 1$ and the root $|z| = 1$ has multiplicity not exceeding the order of the ODE. This is established with the following calculation; as $h'' \rightarrow 0, \rho(z) = z^{r-\mu} (\lambda - 1)^\mu$, where μ is the order of the differential equation, for the block method, $r = 15$, and $\mu = 3$

$$\rho(z) = \lambda^{12} (\lambda - 1)^3 = 0 \Rightarrow \lambda = 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1$$

Hence, since all roots satisfy the roots condition, the method is Zero Stable.

3.3. Consistency of the Method

From Eq. (10), the first and second characteristics polynomials of the method are given by

$$\rho(r) = r^{\frac{1}{36}} - 3r^{\frac{1}{45}} + 3r^{\frac{1}{60}} - r^{\frac{1}{90}}$$

$$\rho(r) = r^{\frac{1}{36}} - 3r^{\frac{1}{45}} + 3r^{\frac{1}{60}} - r^{\frac{1}{90}}$$

and

$$\rho'(r) = \frac{1}{36} r^{-\frac{35}{36}} - \frac{1}{15} r^{-\frac{44}{45}} + \frac{1}{20} r^{-\frac{59}{60}} - \frac{1}{90} r^{-\frac{89}{90}}$$

It follows from here that $\rho(1) = 0 = \rho'(1)$. Show that the condition (iii) is satisfied as well

- Note that

$$\rho'''(r) = \frac{2485}{46656} r^{-\frac{107}{36}} - \frac{3916}{30375} r^{-\frac{134}{45}} + \frac{7021}{72000} r^{-\frac{179}{60}} - \frac{15931}{729000} r^{-\frac{269}{90}} \Rightarrow \rho'''(1) = \frac{1}{972000} = 3! \sigma(1)$$

Thus, the condition (iv) is satisfied. Hence the method is consistent.

$$\pi(r, \bar{h}) = \rho(r) - \bar{h} \sigma(r) = 0 \quad (32)$$

3.4. Convergence of the Method

As reported in [18], the method (10) requires both consistency and zero stability to achieve convergence; Thus, since it has been successful shown in (3.2) and (3.3) above respectively. Therefore, (10) converges.

Where $\bar{h} = h^2 \lambda$ and $\lambda = \frac{\partial f}{\partial y}$ is assumed constant. The first and second characteristics polynomials of Eq. (10) are given by

$$\rho(r) = r^{\frac{1}{36}} - 3r^{\frac{1}{45}} + 3r^{\frac{1}{60}} - r^{\frac{1}{90}}$$

3.5. Region of Absolute Stability of the Method

Worth considering the stability polynomial in the general

$$\sigma(r) = \frac{1}{2799360000} - \frac{7}{2799360000} r^{\frac{1}{180}} + \frac{18}{2799360000} r^{\frac{1}{90}} - \frac{222}{2799360000} r^{\frac{1}{60}} - \frac{227}{2799360000} r^{\frac{1}{45}} - \frac{3}{2799360000} r^{\frac{1}{36}}$$

The boundary of the region of the absolute stability is

$$\bar{h} = \frac{\rho(r)}{\sigma(r)} = \frac{-2799360000 \left(r^{\frac{1}{36}} - 3r^{\frac{1}{45}} + 3r^{\frac{1}{60}} - r^{\frac{1}{90}} \right)}{1 - 7r^{\frac{1}{180}} + 18r^{\frac{1}{90}} - 222r^{\frac{1}{60}} - 227r^{\frac{1}{45}} - 3r^{\frac{1}{36}}} \quad (33)$$

By setting $r = e^{i\theta}$, then Eq. (33) becomes

$$\bar{h}(\theta) = \frac{-2799360000 \left(e^{\frac{1}{36}i\theta} + 10e^{\frac{3}{4}i\theta} - 10e^{\frac{5}{8}i\theta} + 5e^{\frac{1}{2}i\theta} - e^{\frac{3}{8}i\theta} \right)}{19 - 212e^{\frac{1}{8}i\theta} + 1072e^{\frac{1}{4}i\theta} - 3044e^{\frac{3}{8}i\theta} - 25010e^{\frac{1}{2}i\theta} - 338204e^{\frac{5}{8}i\theta} - 329048e^{\frac{3}{4}i\theta} - 31532e^{\frac{7}{8}i\theta} + 199e^{i\theta}}$$

$$\frac{-2799360000 \left(r^{\frac{1}{36}} - 3r^{\frac{1}{45}} + 3r^{\frac{1}{60}} - r^{\frac{1}{90}} \right)}{1 - 7r^{\frac{1}{180}} + 18r^{\frac{1}{90}} - 222r^{\frac{1}{60}} - 227r^{\frac{1}{45}} - 3r^{\frac{1}{36}}} \quad (34)$$

Evaluate Eq. (34), and equate the imaginary part to zero gives

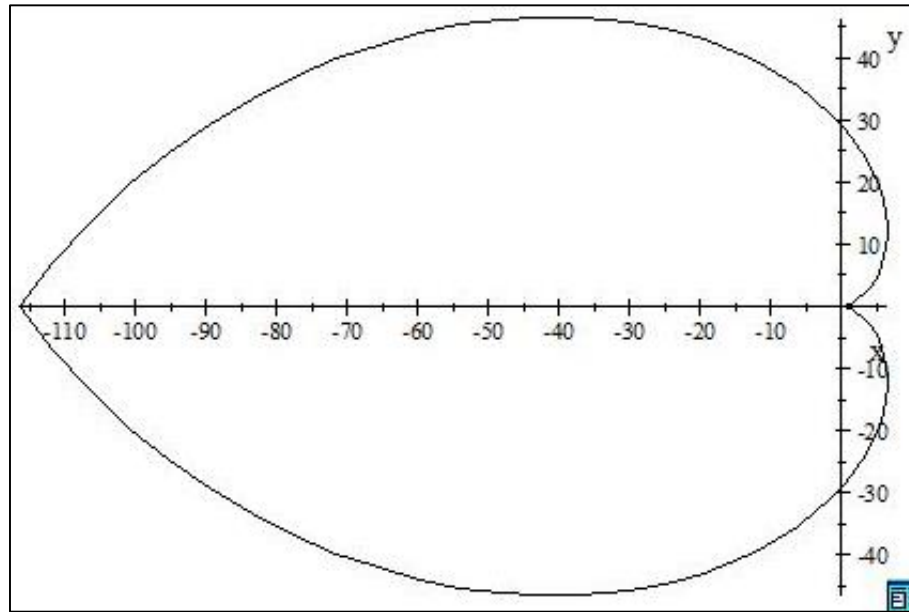
$$\bar{h}(\theta) = \frac{-23781703680 \left(\begin{aligned} &19 \cos \theta - 226369 \cos \frac{1}{8} \theta + 485216 \cos \frac{1}{4} \theta - 85995 \cos \frac{3}{8} \theta \\ &- 291386 \cos \frac{1}{2} \theta - 10913 \cos \frac{5}{8} \theta + 2322 \cos \frac{3}{4} \theta - 307 \cos \frac{6}{7} \theta + 127413 \end{aligned} \right)}{\begin{aligned} &7562 \cos \theta + 260219207780 \cos \frac{1}{8} \theta + 39711472112 \cos \frac{1}{4} \theta + 2731248864 \cos \frac{3}{8} \theta - 381017960 \cos \frac{1}{2} \theta \\ &+ 57848480 \cos \frac{5}{8} \theta + 1292400 \cos \frac{3}{4} \theta - 1282592 \cos \frac{7}{8} \theta + 116120485352 \end{aligned}} \quad (35)$$

Evaluating Eq. (35) at the interval of 30° gives the following results of the boundaries for the region of absolute stability of the method as tabulated below;

Table 1. Boundaries for region of absolute stability.

θ	0°	30°	60°	90°	120°	150°	180°
$\bar{h}(\theta)$	0	3.2384	12.953	29.088	51.656	80.640	116.060

From Table 1 above, it could be deduced that the region of absolute stability of the method is given by $x(\theta) = (0, 116.060)$ which suggests potential P-stability according to [19], subject to further spectral verification.

**Figure 1.** Domain of stability (i.e. absolute) of the proposed method.

4. Numerical Results

The section presents numerical experiment that involves numerical solutions of third-order initial value of ordinary differential equations to confirm the accuracy and viability of the method.

The Table show that $E_{rc} = |y_{ex} - y_c|$ where y_{ex} is the exact solution, y_c is the computed result and absolute error defined as E_{rc} .

4.1. Problem 1: We Consider the Initial Value Problem (IVP) of the Form

$$y''' = e^x, y(0) = 3, y'(0) = 1, y''(0) = 5, h = 0.1$$

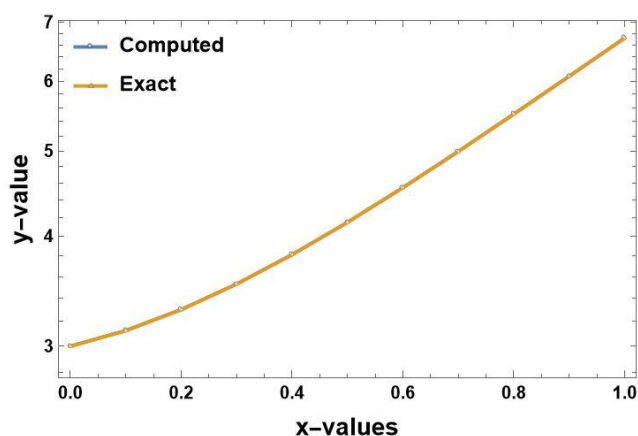
Exact solution: $y(x) = 2 + 2x^2 - e^x$

Source: Kuboye et al., (2020)

The solutions were obtained within 10 iterations following the procedure described in Ref. [20]. Table 2 compares the exact results (ERC) with the numerical results (NRC) generated by the newly developed method for solving ordinary differential equation initial value problems (IVPs). For all values of the independent variable (XVC), the numerical results show excellent agreement with the exact solutions, achieving accuracy close to machine precision. The computed errors (ERR) of the proposed method remain extremely small, consistently lying between (10^{-13}) and (10^{-16}) , which reflects the high precision and numerical stability of the method. In comparison, the errors obtained using the method in Ref. [20] are noticeably larger, increasing gradually from (10^{-12}) at $(XVC = 0.1)$ to (10^{-10}) at $(XVC = 1.0)$. As illustrated in Figure 2, the proposed method provides several orders of magnitude improvement in accuracy over the method reported in Ref. [20]. These results therefore establish the proposed method as an efficient, reliable, and highly accurate technique for solving higher-order initial value problems in ordinary differential equations.

Table 2. Numerical result for problem 1.

XVC	ERC	NRC	ERR	ERR in [20]
0.10	3.12517	3.12517	0	1.5227811E-12
0.20	3.30140	3.30140	2.22044604925031E-16	9.6922224E-12
0.3	3.52986	3.52986	2.22044604925031E-15	2.4267699E-11
0.4	3.81182	3.81182	7.993605777301127-15	4.5451198E-11
0.5	4.14872	4.14872	1.68753899743023E-14	7.8387963E-11
0.6	4.54212	4.54212	3.08642000845793E-14	1.3159340E-10
0.7	4.99375	4.99375	5.329070518200751E-14	2.0471091E-10
0.8	5.50554	5.50554	8.26005930321116E-14	2.9804159E-10
0.9	6.07960	6.07960	1.225686219186172E-13	4.1925841E-10
1.0	6.71828	6.71828	1.771915947301749E-13	5.8107297E-10

**Figure 2.** Numerical and exact solution of problem 1.

4.2. Problem 2: Consider the Third Order Linear Problem

$$y''' + e^x = 0, y(0) = 1, y'(0) = -1, y''(0) = 3, h = 0.1$$

Exact solution: $y(x) = 2(1 + x^2) - e^x$

Source: Ogunware et al., (2020)

Table 3; presents a detailed comparison between the ERC and NRC obtained by adopting the developed block formula for solving ordinary differential equation initial value problems. The numerical results agree very closely with the exact solutions for all values of the independent variable XVC, with only negligible differences observed. The ERR produced by the suggested approach are very small, between (10^{-13}) to (10^{-16}) , demonstrating the method's high numerical precision and stability. As illustrated in **Figure 3**, the proposed method consistently outperforms the method of Ogunware et al. [21], particularly as the values of increase. For example, at $(XVC = 1.0)$, the proposed method produces an error of only (1.79×10^{-13}) , whereas the method in Ref. [21] gives an error of (4.87×10^{-13}) . This substantial reduction in error highlights the improved accuracy of the newly developed method. In all, the results confirm that the suggested formula is consistent, stable, and highly accurate option to existing numerical techniques for third-order ODEs.

Table 3. Numerical result for problem 2.

XVC	ERC	NRC	ERR	ERR in [21]
0.1	0.914829	0.914829	0.000000E+000	3.473600E-14
0.2	0.858597	0.858597	6.66133814775093910E-16	3.326900E-13
0.3	0.830141	0.830141	2.886579864025407E-15	3.709100E-14
0.4	0.828175	0.828175	7.549516567451064E-15	5.791840E-13

XVC	ERC	NRC	ERR	ERR in [21]
0.5	0.851279	0.851279	1.7319479184152442E-14	3.581010E-13
0.6	0.897881	0.897881	3.241851231905457E-14	1.209298E-12
0.7	0.966247	0.966247	5.3734794391857577E-14	1.179995E-12
0.8	1.054460	1.054460	8.393286066166183E-14	2.514500E-12
0.9	1.160400	1.160400	1.2523315717771766E-13	2.409110E-12
1.0	1.281720	1.281720	1.794120407794253E-13	4.870670E-12

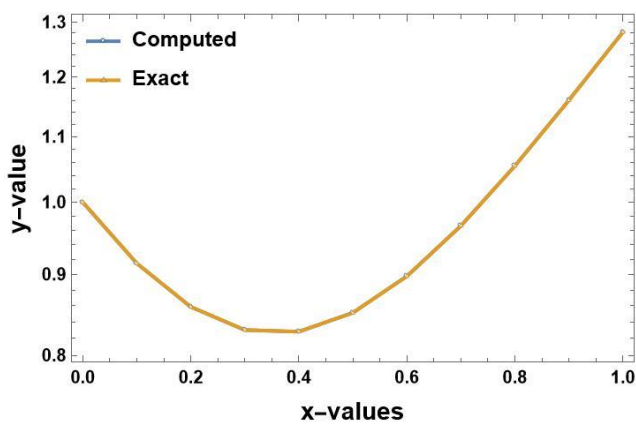


Figure 3. Numerical and exact solution of problem 2.

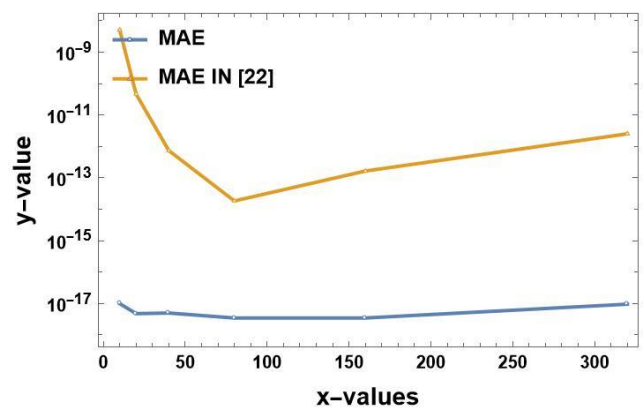


Figure 4. Numerical and exact solution of problem 3.

4.3. Problem 3: We Consider the Non-linear Third Order IVP

$$y'''(x) + 2e^{-3y(x)} = 4(1+x)^{-3} \quad y(0) = 0, y'(0) = 1, y''(0) = -1$$

Exact solution: $y(x) = \ln(1+x)$

Source: Orakwelu et al. (2023)

Table 4. Numerical result for problem 3.

N	MAE	MAE in [22]
10	9.97466E-18	2.91E-09
20	4.77049E-18	4.65E-11
40	4.98733E-18	7.39E-13
80	3.46945E-18	1.84E-14
160	3.46945E-18	1.65E-13
320	9.54945E-18	2.52E-12

Table 4 and Figure 4 present a comparison of the Maximum Absolute Errors (MAE) obtained using the newly developed method for solving ordinary differential equation initial value problems. The numerical results show excellent agreement with the exact solutions, with the MAE values remaining extremely small throughout the computation, specifically within the range of (10^{-18}) . This reflects the high level of accuracy and precision achieved by the proposed method. In comparison with the results reported by Orakwelu et al. [22], the proposed method consistently produces smaller MAE values, indicating a clear improvement in performance. As observed from Table 4 and Figure 4, the analysis demonstrates that the new method provides superior numerical accuracy and offers a more efficient and reliable approach for solving higher-order differential equation initial value problems.

4.4. Problem 4: Consider the Third-order IVP

$$y''' = 3 \sin x. \quad y(0) = 1, y'(0) = 0, y''(0) = -2, h = 0.1$$

$$\text{Exact solution: } y(x) = 3 \cos x + \frac{x^2}{2} - 2 = 2$$

Source: Kuboye et al., (2020)

Figure 5 compares the exact results (ERC) with the numerical results (NRC) obtained from our newly developed method

for solving an initial value problem in ordinary differential equations. The numerical values match the exact ones very closely across all tested points, with errors (ERR) falling between 10^{-16} and 10^{-14} which is impressively tiny and speaks to the remarkable accuracy of the proposed method. That said, the error does tend to increase slightly as XVC gets

larger, but the growth is so slow and well-behaved that it hardly raises any concern, confirming that the method stays efficient even over longer intervals. All in all, the analysis shows that our new numerical approach is not only highly accurate but also robust and reliable for solving initial value problems.

Table 5. Numerical result for problem 4.

XVC	ERC	NRC	ERR	ERR in [20]
0.10	0.990012	0.990012	0	6.8911543E-13
0.20	0.960200	0.960200	0	4.4015902E-12
0.30	0.911009	0.911009	0	1.0999868E-11
0.40	0.843183	0.843183	8.88178E-16	2.0601632E-11
0.50	0.757748	0.757748	2.22045E-15	3.6853520E-11
0.60	0.656007	0.656007	4.88498E-15	6.726841E-11
0.70	0.539527	0.539527	8.88178E-15	1.1150603E-10
0.80	0.410120	0.410120	1.50990E-14	1.6985002E-10
0.90	0.269830	0.269830	2.37588E-14	2.4948449E-10
1.00	0.120907	0.120907	3.59712E-14	3.6226498E-10

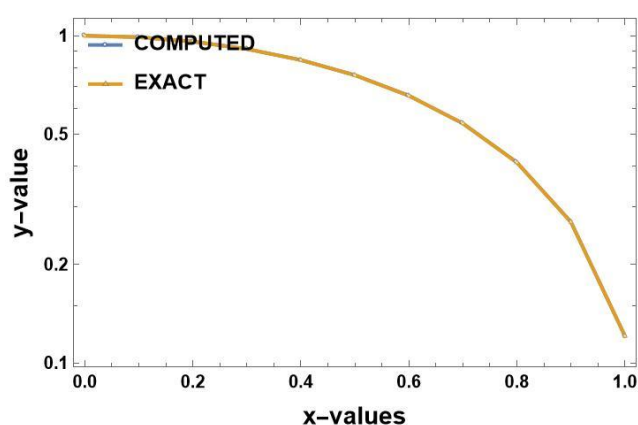


Figure 5. Numerical and exact solution of problem 4.

While the absolute error appears to be increasing slightly due to the increasing size of the XVC values, the errors are still significantly low and are within accepted computational limits, demonstrating that the method is table over the entire interval $([0, 1])$. Therefore, we can conclude that our newly developed method is highly accurate, reliable, and numerically robust and is therefore a viable option for use in scientific computing for handling direct solution of third-order ODEs.

5. Discussion of Results

The current paper discusses an approach based on collocation and interpolation procedure which resulted in a hybrid multistep technique for solving IVPs arising from third-order ODEs. The new method appears to have both high accuracy and computational efficiency, so it could easily handle large and complex initial value problems. Results presented in [Tables 2, 3, 4 and 5](#) indicate that the new technique outperformed previous techniques as proposed by Kuboye *et al.* [20], Ogunware *et al.* [21], and Orakwelu *et al.* [22]. The relationship between numerical solutions derived through use of the proposed technique and the corresponding exact solutions is shown in [Figures 2-5](#), indicating close correlation between these two forms of solution. This new sixth order method has increased accuracy and stability than previously proposed methods in terms of their ability to minimize errors and/or converge.

6. Conclusion

As depicted in our finding, a sixth-order hybrid block approach for the direct computation of Third-Order IVPs of ODEs. The formulas of the suggested method are gotten from a continuous approach that utilizes concepts of interpolation

and collocation to derive the main and additional formulae for the proposed block method. It was established that the hybrid block method is of order six, exhibits zero stability, consistency, convergence and has a significantly large region where numerical results will exhibit absolute stability. The overall approach is presented as a single block (combined) method used for numerical resolution of Third-Order ODE's. The approximate solutions as obtained using the developed scheme, reflect the relevance and satisfaction in solving general types of Third-Order ODEs. The difference between the proposed method and published articles indicates that the proposed method produces results that are comparable to and will perform similarly to, those results from published references both in terms of accuracy and effectiveness.

Abbreviations

XVC	Value of the Independent Variable Where a Numerical Value Is Taken
ERC	Exact Result at XVC
NRC	Numerical Result of XVC
ERR	Error in Proposed Method at XVC
N	Steps
MAE	Maximum Absolute Errors

Author Contributions

Duromola Monday Kolawole: Conceptualization, Methodology, Resources, Supervision, Writing – original draft

Akinmoladun Olusegun Mayowa: Validation, Writing – review & editing

Fafesobi Gbenga Olatunde: Formal Analysis, Investigation, Software

Conflicts of Interest

The authors declare no conflicts of interest.

References

- [1] Butcher, J. C. (1965): A modified multistep method for the numerical integration of ordinary differential equations. *J Assoc. Comput. Math.* 12; 124-135.
- [2] Lambert, J. D. (1973). *Computational Methods in Ordinary Differential Equations*. John Wiley, New York.
- [3] Chu, M. T. and Hamiton, H. (1987): Parallel Solution of Ordinary Differential Equations by Multiblock Methods. *SIAM Journal of Scientific and Statistical Computation*. 8; 342 – 553.
- [4] Fatunla, S. O. (1988). *Numerical Methods for Initial value Problems in Ordinary Differential Equations*. Academic Press Inc., New York.
- [5] Awoyemi, D. O (1999): A class of continuous Methods for general second order initial value problems in ordinary differential equations. *International journal of computer mathematics* 72; 29-39. <https://doi.org/10.9734/arjom/2022/v18i130354>
- [6] Olabode B. T. (2009): An accurate scheme by block method for the third order ordinary differential equations. *Pacific Journal of Science and Technology*. 2009; 10(1).
- [7] Bolarinwa Bolaji, Ademiluyi, R. A., Olaseni Tunde and Duromola M. K (2012): A New Implicit Hybrid Block Method for the Direct Solution of initial value problems of general third order ordinary differential equations. *Canadian Journal of Science and Engineering*, 3; 86 - 97.
- [8] Ademiluyi, R. A., Duromola, M. K. & Bolarinwa Bolaji. (2014). Modified block method for the direct solution of initial value problems of fourth order Ordinary differential equations. *Australian Journal of Basic and Applied Sciences*, 8(10) July 2014; 389-394.
- [9] Adesanya, A. O., Abdulqadri, B. and Ibrahim, Y. S. (2014) Hybrid One Step Block. Method for the Solution of Third Order Initial Value Problems of Ordinary Differential Equations. *International Journal of Pure and Applied Mathematics*, 97, 1-11. <https://doi.org/10.12732/ijpam.v97i1.1>
- [10] Duromola, M. K., Lawal, R. S. and Akinmoladun, O. M. (2024): Numerical integration of linear hybrid multistep block method for third-order ordinary differential equations (IVPs). *Scientific African (Elsevier) ScienceDirect* <https://doi.org/10.1016/j.sciaf.2024.e02129>
- [11] Duromola, M. K. (2025) An efficient A-stable linear multistep hybrid block method for solving fifth-order initial value problems in ordinary differential equations. *American Journal of Applied Mathematics*. 13(3): 174-193 <https://doi.org/10.11648/j.ajam.20251303.11>
- [12] Duromola, M. K and Momoh, A. L. Hybrid numerical method with block extension for direct solution of third order ordinary differential equations. *American Journal of Computational Mathematics*. 2019; 9: 68-80.
- [13] Duromola M. K. (2022) Single-Step Block Method of P-stable for Solving Third Order Differential Equations (IVPs): Ninth Order of Accuracy. *American Journal of Applied Mathematics and Statistics*, 2022, Vol. 10, No. 1. Pp 4-1.3 <http://pubs.sciepub.com/ajams/10/1/2>
- [14] Duromola, M. K., Momoh, A. L. and Kusoro O. O. (2024). A modified fourth derivative block method and its direct applications to third-order initial value problems. *Journal of Mathematical Analysis and Modeling*, 4(3): 60-75. <https://doi.org/10.48185/jmam.v4i3.897>
- [15] D. O. Awoyemi, E. A. Adebile, A. O. Adesanya and T. A. Anake (2011). Modified block method for the direct solution of second order ordinary differential equations. *International Journal of Applied Mathematics and Computation*. 3(3), pp 181-188.
- [16] Y. A. Yahaya and A. M. Badmus (2009). *A class of collocation for General Second Order Ordinary Differential Equations*. *African Journal of Mathematics and Computer Science Research*, 2(4): 069-072.

- [17] Fatunla, S. O. (1991). Block method for Second Order IVPs. *International Journal of Computer Mathematics*, 41(9); 55-63.
- [18] Henrici P. (1962). *Discrete Variable Method in Ordinary Differential Equations*. John Wiley and Sons, New York.
- [19] Shampine, L. F. and Watts, H. A. (1969): Block Implicit one step methods. *Mathematics of Computation*, 23 (108); 731-740.
- [20] J. O. Kuboye, O. F. Quadri, O. R. Elusakin (2020): Solving Third Order Ordinary Differential Equations Directly Using Hybrid Numerical Models. *Journal of the Nigerian Society of Physical Sciences*, 2 (2020) 69-76.
- [21] Abolarin, O. O., Kuboye, J. O., Adeyefa, E. O., and Ogunware, G. (2020): New Efficient Numerical Model for Solving Second, third and Fourth Order Ordinary Differential Equations Directly. *Journal of Science*. 33(4): 821-833 (2020). <https://doi.org/10.35378/gujs.627677>
- [22] Orakwelu, M. G., Otegbeye, O., and Mambili-Mamboundou, H. (2023). A class of single- step hybrid block methods with equally spaced points for general third-order ordinary differential equations. *Journal of the Nigerian Society of Physical Sciences*, 5(2023) 1484. <https://doi.org/10.46481/jnsps.2023.1484>