

Review Article

# Application of Curvilinear Analytical Surfaces in Forms of Architectural Objects and Machine Building Products

**Bock Hyeng Christian Alain<sup>1,\*</sup>** , **Krivoshapko Sergey Nikolaiv<sup>2</sup>** ,  
**Kouamou Nguessi Arnaud<sup>3</sup>** , **Yamb Bell Emmanuel<sup>4</sup>**, **Bahel Benjamin<sup>5</sup>**

<sup>1</sup>National Advanced School of Engineering, University of Yaounde 1, Yaoundé, Cameroon

<sup>2</sup>Russian People Friendship, University of Moscow, Moscow, Russia

<sup>3</sup>High Technical Teacher Training College, University of Bamenda, Bamenda, Cameroon

<sup>4</sup>High Technical Teacher Training College, University of Douala, Douala, Cameroon

<sup>5</sup>Higher Technical Teacher Training College, University of Buea, Buea, Cameroon

## Abstract

The principal achievements of science and engineering in the sphere of static and dynamic analysis of thin-walled objects, structures, and shells in the shape of analytic surfaces are used for practical needs of people. Classes of surfaces that found the application in forms of architectural erections and machine building products are considered. This is confirmed by presented illustrations of real products and erections. Classes of surfaces which did not attract the attention of architects and designers working with curvilinear forms are pointed out too. The presented materials confirm conclusions of most scientists, structural engineers, and architects on increasing interest to design and building of objects of curvilinear forms. The analysis noted the end of the recession of interest in thin shell and parish structures in the 21st century. These shell structures are produced due to the presence of new structural materials and the expansion of the list of analytical, point, spline, and frame surfaces that can be used as middle surfaces of shells. These surfaces are used for the study of certain physical processes, for solving particular mathematical problems and for determining surfaces isometric to surfaces of revolution, thus allowing the creation of more precise methods of calculating the strength of shells. This review paper contains 47 references, and these are practically all original sources dealing with applications, classifications, definition of analytical surfaces, and analysis of shells with middle surfaces in the form of analytical surfaces on strength, stability, and dynamic. It gives to use the opportunity of parametrical architecture.

## Keywords

Analytical Surface, Shell, Parametrical Architecture, Architectural Erections, Machine, Strength Analysis

## 1. Introduction

The latest publications mention repeatedly that only 40 surfaces of 600 known analytical surfaces have found appli-

cation [1, 2]. The history of civil engineering has repeatedly shown that new types of structures have been built before their

\*Corresponding author: [hyengbock@hotmail.com](mailto:hyengbock@hotmail.com) (Bock Hyeng Christian Alain)

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behavior was fully understood [3]. The encyclopedia [4] contains 37 classes of analytical surfaces. These classes subdivide into subclasses and groups and several classes, subclasses, and groups of the surfaces presented by geometers did not find practical application neither in architecture nor in machine building, but they are used in subsequent geometrical research [5], or for the description of some natural phenomenon [6], physical processes [7], and so on. The structural analysis of shells has had a long and difficult history. Shells were developed and reached their popularity before the ready availability of computers and the FE method. Until the 21<sup>st</sup> century shells have lost their popularity compared to their heyday in 1950s and 1960s, when architects adopted them as a new mean for artistic expression. But at present, shells are attracting interest among the new generation of architects and engineers [3]. At last ten years, several published papers appeared, where positive influence of curvilinearity of surrounding buildings on general feeling of man and his behavior, is investigated [8].

### 1.1. Aim of Research

Using the classification of analytical surfaces offered in the encyclopedia [4], the author wants to reveal all surfaces, found application in practice, and thus to attract attention of architects, designers, and machine building engineers to using, in case of need, new or already applied earlier surfaces for formation of architectural objects and machine building products. Every surface offered by geometers possesses some advantages as applied to practical problems.

### 1.2. Explanations to The Contents of a Paper

The author does not raise a task of detailed description of analytical surfaces or treatment of the scale of application of one or another surface on practice. Application of every surface pointed out in a work [4] will be illustrated by only one figure with indication of original sources where application of considered surfaces are illustrated by large number of real erections or products. The principal works on analysis of shells of considered forms on strength, stability, and dynamic with great number of references are pointed out.

For every class of surfaces, it is pointed out in round parentheses quantity of analytical surfaces, found confirmed application in different areas of man activities. The second cipher gives quantity of analytical surfaces of the same class, presented and described in scientific literature. For some classes of surfaces, one or two ciphers are absent. It means that these positions are impossible to count.

## 2. Analytical Surfaces That Have Found Application in Forms of Architectural Objects and Machine Building Products

### 2.1. Ruled Surfaces

Ruled surfaces can be only by surfaces of zero or negative Gaussian curvature.

### 2.2. Ruled Surfaces of Zero Gaussian Curvature ( $K = 0$ )

*Torse surfaces (torses) (3 of 29)*



a) evolvent helicoid (Archimedes Screw)



b) a torse composition obtained by parabolic bending of the plane metal list, Ireland



c) a torse surface with an cuspidal edge on a circular cone (spiral-shaped roof)



b) parabolic cylindrical surface (pavilion «Mechanization», Moscow, 1939)

**Figure 1.** Torse products.

Non-degenerated torse surfaces i. e. torses with cuspidal edge find wide application in ship building, in aircraft construction, in agricultural machine building (Figure 1a), in cartography, and in erection of sculpture forms (Figure 1b). The application of thin-walled torse shells in building can give positive effect in some cases (Figure 1c). The most researchers support this opinion, but at present time, only sketches of torse coverings and recommendations on application of torses in building are offered. The most complete information on torse surfaces and shells is given in a monograph [9]. Georg Glaeser [10] notes that in practice, developable surfaces are of value in avantgarde architecture, where smooth surfaces are requested, and construction prices should not exceed certain limits. Several surfaces put together create new structures that are completely developable.

#### *Cylindrical surfaces (8 of 21)*

Cylindrical surfaces were well studied and found application already many hundreds of years ago. They have great popularity and now (Figure 2). Great number of monographs and scientific papers are devoted to strength analysis, stability, vibration [11], to architecture and application [12] of these surfaces and shells.



c) sine cylindrical surface (the car station, Sochi, Russia)



d) elliptical cylindrical surface (metro station «Babushkinskaya», Moscow, 1978)



a) round cylindrical surface (Modern car museum, Munich, Germany)



e) cylindrical-and-conical strip (the Great Mosque, Samarra, Iraq, 848)





f) a cylinder with directrix curve of evolvent type (the Azerbaijan National Museum of carpet, 1967)



a) circular conical surface (The Lesotho National Museum in Maseru, Lesotho)



g) cylindrical surface with a director curve of variable curvature (Capital Green-house, the botanical gardens, Moscow)

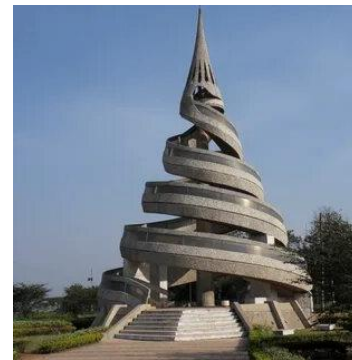


b) circular inclined conical surface (Museum of Glass, downtown Tacoma, Washington, USA)  
[[http://senapa.livejournal.com/tag/\\*tacoma](http://senapa.livejournal.com/tag/*tacoma)]



h) cylindrical helical strip (a car ramp, Italy)

**Figure 2.** Cylindrical buildings and erections.



c) Reunification Monument, Yaoundé, Cameroon (a spiral strip on the cone of revolution)

**Figure 3.** Conical erections.

#### Conical surfaces (4 of 11)

Conical surfaces as well as cylindrical surfaces find wide application at our time (Figure 3). Subject of application of circular conical surfaces in different branches of machine building and civil and industrial building was exposed in detail in review paper [13]. Conical surfaces of the high orders did not find deserving application yet. Only elliptical cone has found application in machine building [14].

### 2.3. Ruled Surfaces of Negative Gaussian Curvature ( $K < 0$ )

One has 16 surfaces belonging to other classes besides ruled surfaces of the negative Gaussian curvature considered below. For example, right helicoid will be taken into account in the class “Minimal Surfaces”, oblique helicoid will be in the class “Helical Surfaces”.

#### Conoids (4 of 12)

Shells in the form of conoids and cylindroids are applied less than conical, spherical shells, than parabolic shells of revolution or shells in the form one sheet hyperboloid of revolution and some others. Firstly, conoidal shells appeared in France, after that in Czechoslovakia, Italy, and especially in Poland due to V. Zalewski who studied them. A. Gaudi was first who put into practice surface of conoid (Figure 4a).



a) sinusoidal conoid (the temporary children's schoolhouse, Barcelona, Spain, A. Gaudi, 1909–1936)



b) similar-to-sine right conoid (the Bodegas Ysios winery building, Spain, S. Calatrava)



c) inclined parabolic conoid (Dakshin Delhi Kalibari Temple, South Delhi, India)



d) parabolic conoid (Church at Atlántida, Uruguay)

**Figure 4.** Conoids.

The latest information on application of conoids in architecture and building is given in a paper [15]. In Figure 4, the forms of conoidal shells assimilated by architects are given. Scientific works devoted to analysis of conoidal shells on strength, stability, and dynamics are presented also in a review [15] with 35 references.

#### *Cylindroids (4 of 6)*

The form of cylindroid was used less than form of conoid. Three types of cylindroids found application in architecture are known (Figure 5). The well-known architect F. Candela liked to work with them. Cylindroids are used also in machine building [16]. For example, Ball's cylindroid has important part in screw theory. In 1875, R. S. Ball has built a model of this cylindroid from steel wires.



a) the experimental cylindroid (Mexico City, 1950, F. Candela)



b) intersecting cylindroids (the State Mosque of Negeri Sembilan, Малайзия, 1967, Chen Voon Fee, Lim Chong Keat, and William Lim Siew Wai)





c) roofs from corrugated iron in the form of cylindroids (Ishibashi, Tokugawa & Associates, Токио, Japan)

**Figure 5.** Cylindroids.

#### Catalan surfaces (1 of 13)

This subclass includes Whitney umbrella, pseudo-developable helicoid, ruled rotor cylindroid, and 10 ruled surfaces with  $K < 0$ , belonging to other classes. Only hyperbolic paraboloid from 13 Catalan surfaces has found application.

#### Twice oblique cylindroid (1 of 1)

Only oblique trochoid cylindroid belongs to this subclass. This surface is formed in the process of gear cutting with arch gearing.

#### Surfaces of revolution (11 of 59)

Surfaces of revolution are well known to architects and machine building engineers and there is no point in adducing illustrations demonstrating the using of them in practice. Let us list several examples of their application presenting only one example for every subclass of surfaces of revolution: 1) sphere (a trade-and-entertaining center «PITERLAND», SPb, 2012), 2) ellipsoid of revolution (tea-house, Gissar, Tajikistan), 3) paraboloid of revolution [17], 4) one sheet hyperboloid of revolution (cool towers), 5) a surface of revolution «Egg», 6) circular torus (project «Ark», bio-climatic building with autonomous life-support system, 2010), 7) surface of revolution of cycloidal curve (a fairing of cycloidal type) [18], 8) Bullet Nose (upper part of “30 St Mary Axe” after the 17<sup>th</sup> floor, Foster and Partners), 9) pseudo-sphere is presented in a class “Surfaces of constant Gaussian curvature”, 10) catenoid is presented in a class “Minimal surfaces”, 11) drop-shaped surface (the Folk Theater in the form of a drop, China).

59 analytical surfaces of revolution, described in scientific literature, are pointed out in a paper [19] but not all of them found the application in practice. But in spite of large choice of revolution surfaces, there is need in design of revolution shells with new middle surfaces [20]. The selection of revolution shells with using of 23 criteria of optimality is offered in a paper [21].

## 2.4. Translation Surfaces

Surfaces of translation are available in seven classes of 37. One can consider that translation surfaces belong to kinematical surfaces.

#### Surfaces of right translation (Figure 6)

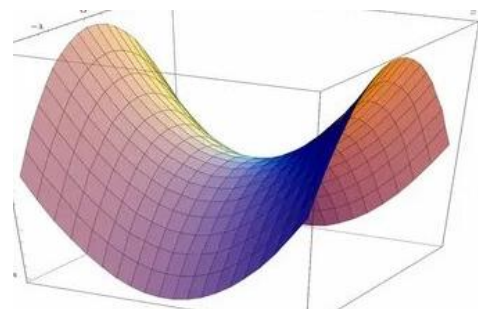
These are very functional surfaces and therefore architects and civil engineers like them [22]. Strength analysis of shells of right translation does not contain any difficulties. It can be carried out by both analytical methods expanding unknown displacements of middle surface of the shell into a series and by numerical methods using typical computer complexes [23].



a) The Cheryomushkinsky market (circular translation surface), Moscow



b) surface of translation of a parabola over another parabola (the parabolic sun concentrator)



c) surface of translation of a concave parabola along the convex parabola (hyperbolic paraboloid)

**Figure 6.** Surfaces of right translation.

#### Surfaces of oblique translation (1 of 3)

This surface is formed by parallel translation of a plane curve and two of its symmetrical points touch the plane contour continuously. The surfaces created on a contour with symmetry, for example, circumferences, ellipses, squares, ovals, rectangles, are of the greatest interest. For example, in Hungary, a sports hall was built on an elliptical plan with  $93 \times$

61 m dimensions in axes and a height of 10.5 m. The system of parabolic steel tubular arches located along the elliptic paraboloid translation networks with a step of 6 m in both directions was chosen as the supporting structure of the coating.

*Funicular (velaroidal) surfaces (3 of 6)*

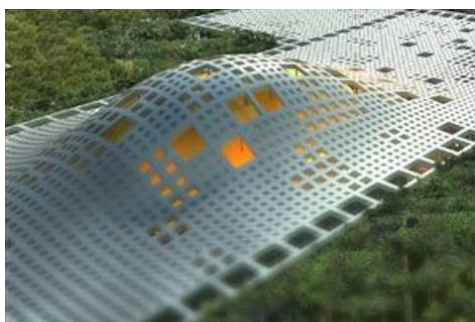
Velaroidal surface is called also funicular surface. Now, three types of velaroidal surfaces are known that can be used for coverings of building erections (Figure 7). In the old time in Georgia, builders have built stone velaroidal domes [24].



a) the funicular shell formed by the chain lines (Heinz Isler)



b) the shallow reinforced concrete covering of the Nekrasov market, SPb, Russia, 1960)



c) the covering of culture center in Muscat, Oman, Architecture-Studio Architects, Paris

**Figure 7.** Funicular surfaces.

Now, researchers study the experience of application of velaroidal shells [25].

## 2.5. Surfaces of Velaroidal Type

About ten of scientific papers are devoted to this subclass of surfaces. One marks that these surfaces found the application in forms of hulls of river and sea ships [26]. Examples of their application in building are absent.

*Minimal surfaces (3 of 15)*

Right helicoid is only ruled minimal surface. It is used often as a model of helical staircases (Figure 8a) and car ramps. Catenoid is only minimal surface of revolution which did not use widely in building and machine building yet (Figure 8b, c). B. Janett presented innovative project of three-floor pavilion with hollow pillars in the form of catenoid. This shape was chosen for the guaranteeing of smooth transition from the walls to the ceiling. Catenoids of National Theatre of Taiwan are the first examples of real application of catenoids in architecture (Figure 8b). The Olympic Stadium in Munich was erected in 1972 with application of minimal surfaces similar to Schwarz surface.

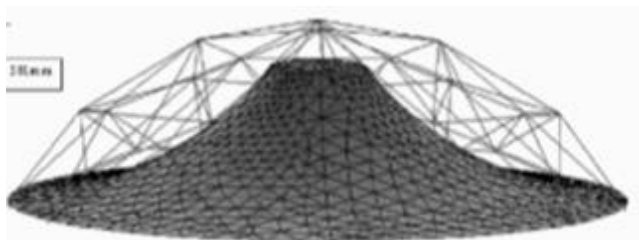
Architects maintain the idea of using of minimal surfaces in architecture. It has appeared architectural direction “Minimal Surface Architecture” [27]. But majority of ideas were realized only in projects. Fifteen minimal surfaces are known. They have parametrical form of definition but only three minimal surfaces namely right helicoid, catenoid, and Schwarz surface have found the application.



a) right helicoid (the hotel Seven Visions the Dvin, Yerevan, Armenia, 2022)



b) catenoids (Taichung Metropolitan Opera House, T. Ito, Da-ju Architects)



c) catenoid (the model of the pavilion coverage in the park of the Queretaro City, Mexico)

**Figure 8.** Minimal surfaces.

### Helical surfaces

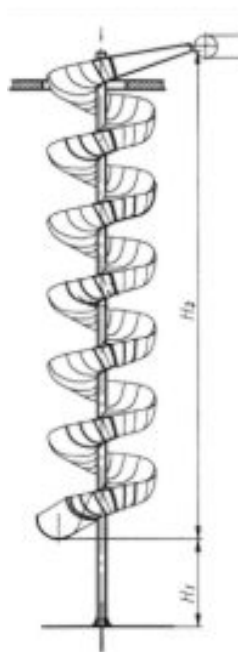
Usually, helical surfaces are divided into two subclasses, that are ordinary helical surfaces and helical surfaces of variable lead.

## 2.6. Ordinary Helical Surfaces

This group of helical surfaces contains *ruled helical surfaces* which включают в себя 8 types of surfaces and *helical surfaces with arbitrary plane generatrix curves* which насчитывают 8 surfaces at present time. Evolvent helicoids (Figure 1a), right helicoids (Figure 8a), and cylindrical helical strip (Figure 2h) from a group “Ruled helical surfaces” are widely used. The rest five surfaces did not find deserving application on practice [28].

A group “Circular helical surfaces” can be found also in a class “Cyclical surfaces” where they will be under consideration in detail.

*Helical surfaces with arbitrary plane generatrix curves* (Figure 9)



a) A helix chute of ООО «PentaDesign»



b) NEXUS Multiple in-feed Gravity Roller Spiral Conveyor



c) a helical surface with a congruent generatrix curve



d) Spiral chute Baltmechkom [baltmechcom.ru]

**Figure 9.** Helical surfaces with arbitrary plane generatrix curves.

Dini helicoid belonging to this group of helical surfaces has the most popularity. It possesses a number of remarkable properties with geometrical point of view [4].

*Helical surfaces of variable lead (pitch)*

Machine building industry has examples of application of helical surfaces of variable pitch [28].



In this work, it is shown that machine building is the main sphere of their application.

#### *Helix-Shaped surfaces (4 of 19)*

Surfaces, formed by generatrix curves executing besides ordinary helical motion relatively the helical axis any additional motion or deforming at certain law, are attributed to helical-shaped surfaces. Trajectories of points of a generatrix curve when helical-shaped movement will not be cylindrical helical curves (Figure 10). The most complete information on helical and helical-shaped erections and products one can know from a work [29]. Great number of helical-shaped surfaces one can see in the forms of water zoom and attractions (Figure 10). Helical-shaped surfaces one can find in structures of building machines and mechanisms, screw hoists, in the forms of teeth and threads of skew gearings, in heat exchanger [30]. Helix-shaped surfaces can be seen in the forms of blades in ship-, aircraft- and other branches of machine building, but majority of forms of these details were obtained by experimental methods.



a) complex compound structure at children's playground in Canberra, Australia  
[<https://rope-park.com/19-samyih-interesnyih-detskih-ploshhdok-v-mire>]



b) slide wheel, China



c) space twister: water attraction, Austria



d) helical-shaped surface (spiral camera of hydraulic turbine of water-power station is a settling name but with geometrical point of view this is not correctly)

*Figure 10. Helical-shaped surface.*

## 2.7. Spiral Surfaces

Producers called their products, presented in Figure 9b, d, spirals but they must be rightly called helix chutes.

The trajectories of points of the generatrix curve under spiral motion are disposed on the circular cones (Figure 11). The projections of the points of the trajectories of the points of the generatrix curve on a plane perpendicular to the axis of a cone will be logarithmic spirals.



a) spiral bent (attraction in the park "Sokolniki", Moscow)



b) the thatch spiral roof of dwelling-house, Danish ecological settlement Fri & Fro



c) ruled spiral surface (water tower "Kizuminami", Kizu, Japan, 1999 [bugaga.ru/interesting])

*Figure 11. Spiral surfaces.*

Ruled spiral surfaces were presented also on [Figure 2e](#) and on [Figure 3c](#).

## 2.8. Spiral-Shaped Surfaces

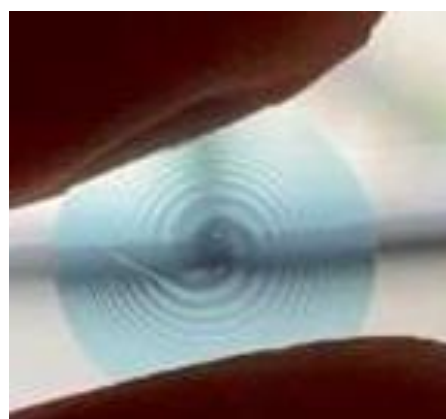
*Spiral-shaped surfaces* bear a resemblance to spiral surfaces, but these surfaces cannot be related to the same class because the spiral surface has the directrix curve only in the form of a spiral on a right circular cone. For a directrix curve of any spiral-shaped surface, one may take arbitrary spiral curve laying on any surface ([Figure 12](#)). The descriptions of some spiral-shaped skyscrapers, the forms of which after some assumptions one can add on to spiral-shaped surfaces, are given in a work [31]. Researchers have developed a new type of lens that uses a spiral-shaped surface to maintain a clear focus at different distances in varying light conditions ([Figure 12c](#)).



a) spiral-shaped surface («White Cyclone»: wooden roller coaster. Nagashima Spa Land, Japan, 1995)



b) a project of footpath of the studio Glissfeldt Kloster, Denmark, with a director spirale on the one sheet hyperboloid of revolution



c) spiral-shaped surface (a spiral-shaped lens, by Optica, 02.26.2024, [scitechdaily.com])

*Figure 12. Spiral-shaped surfaces.*

### *One-sided surfaces (2 of 6)*

In the encyclopedia [4], six one-sided surfaces are shown and only models of a Mobius strip have found the application in architecture of small forms (рис. 13a) and Klein surface is used for studies ([Figure 13b](#)). Immersed Klein bottles are presented in the Science Museum in London.



a) one-sided surface ("Mobius strip", Ekaterinburg, RF.)





b) Klein surface (Klein bottle)

**Figure 13.** One-sided surfaces.*Surfaces of the constant gaussian curvature (2 of 6)*

Three surfaces of constant positive Gaussian curvature are known, and sphere is one of them which is considered in a class "Surfaces of revolution".



**Figure 14.** Laminated wood sculpture entitled "Pseudosphere" by Ruth Vollmer and metal and wire sculpture "Funicular Polygon of Revolution - Pseudosphere" by Robert Le Ricolais [32]

Pseudosphere, Dini helicoid, and Kuen surface form a subclass "Surfaces of constant negative Gaussian curvature". Pseudosphere is a surface of revolution which is находится в центре внимания of geometers after the 19<sup>th</sup> century, but it is practically unknown for architects and designers. There are no examples of its application in building industry and in architecture. One can see it only in objects of small or garden architecture or as educational supply in universities (Figure 14).

*Surfaces Of The Constant Mean Curvature (3 Из 5)*

Sphere ( $H = R$ ), cylindrical surface of revolution ( $H = R/2$ ), and minimal surfaces ( $H = 0$ ) are the most known surfaces of the constant mean curvature. These surfaces found application both in architecture and machine building. *Nodoid* and *unduloid surfaces of revolution* that attract fixed attention of mathematicians worked out the problems of conjugacy of two revolution surfaces are known less [33].

## 2.9. Umbrella Surfaces and Cyclical Surfaces

*a) Umbrella Surfaces*

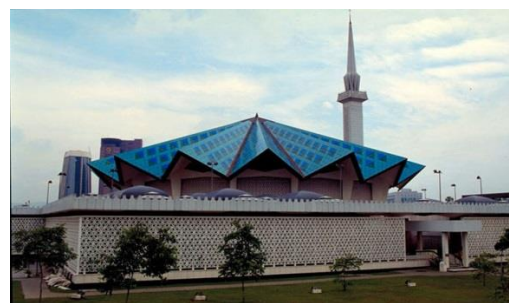
Umbrella shells are used widely in town-planning practice. A cyclic symmetrical spatial structure formed from several identical elements is called an umbrella dome. Curves obtained as a result of the intersection of their middle surface are

the generatrix curves of any dome-shaped surface of revolution.

*Surfaces of umbrella type* are cyclically symmetrical surfaces consisting of several identical elements. But the whole surface of umbrella type and all surfaces constituting its identical elements is defined with the help of the same explicit, implicit, or parametrical equations. In literature, surfaces of umbrella type are called often also *waving or wave-shaped domes*.

Terms «multiple, pumpkin, melon, scalloped, or parachute domes» are used for the designation of umbrella domes and domes of umbrella type too.

It is practically impossible to discern umbrella shells and shells of umbrella type separate. This can be made only by designers of these shells. Umbrella shells are divided into umbrella shells made of elements of negative Gaussian curvature and umbrella shells made of elements of positive Gaussian curvature (Figure 15).

a) cafe «Pearl», Baku, [<http://photomir.ucoz.ru/photo/4-6-0-0-2>]

b) Hasan-Uddin Khan. Kuala Lumpur State Mosque, Kuala Lumpur, Malaysia, 1994 [archnet.org]



c) the complex of erections «Oceanographic», 1994–2002, Valencia, Spain





d) the umbrella roof for reservoir, FRP Dome, American Structures, Inc

**Figure 15.** Umbrella shells.

Additional information on umbrella shells and shells of umbrella type can be taken in papers [34, 3]. But now shells of umbrella type were not discovered in nature, though a class of surfaces of umbrella type includes in itself 18 denominations. All of them consist only in projects.

#### b) Cyclical Surfaces

*Cyclic surfaces* are formed by motion of a circle of variable or constant radius on arbitrary law in space [35]. Four subclasses form a class “Cyclical Surfaces”. The latest information on classification of cyclic surfaces, on description of all groups of cyclic surfaces, on bibliography devoted to geometry of cyclic surfaces and analysis of cyclic shells on strength, stability, and dynamic are presented in works [4, 35, 36].

Translation surfaces, circular helical surfaces, and pipe surfaces with arbitrary line of centers are the most used cyclical surfaces (Figure 16). The works [36-38] report in detail about application of cyclical surfaces in architecture and machine building.

One may add the shells given on Figure 3a, b; Figure 6a, Figure 8c, Figure 10a, c, d, and on Figure 14 to cyclic shells.



Conical spring of pressure



Helical cylindrical spring of pressure



Water attraction



Normal cyclic surface with plane line of the centers and with a generatrix circle of variable radius (Dmitrov town, Moscow region)

**Figure 16.** Shell structures in the form of cyclic surfaces.

## 2.10. Wave-Shaped, Waving, and Corrugated Surfaces

In an encyclopedia [4], 37 surfaces were included in this class. Assume the definitions of these three groups of surfaces from this encyclopedia.

*Wave-shaped surfaces* are formed by translation-and-oscillatory motion of a rigid generatrix curve vibrating about a basic surface, a plane or a line taken in advance. Hence the generatrix curves of the wave-shaped surfaces are congruent to each other. Consequently, these surfaces may be included into a class of *surfaces of the congruent cross sec-*

tions as well. At literature, there are received other names of the waved-shaped surfaces, for example, *wave surfaces*.

*Waving surfaces* are formed by translational-and-oscillatory motion of the generatrix curves which do not only vibrate about basic surfaces, planes or lines chosen in advance, but they deform themselves remaining at one and the same class of curves [39].

*Riffled surfaces (rifle-shaped surfaces)* have taken this name due to the English word “*riffle*”. So, the riffled surfaces are surfaces with hollows or bulges disposed on them in order. The riffled surfaces are widely used in machine-building.



The waving surface: Sage Gateshead, UK, 2004 (N. Foster and Partners)



The waving surface: the waving roof of “Fold House”, Ontario, Canada, 2020



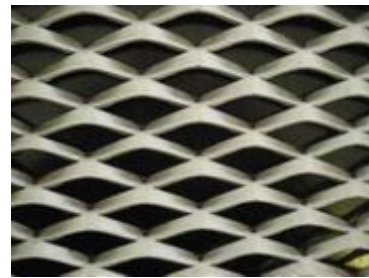
The wave-shaped surface: dwelling complex “The Wave”, Vejle, Denmark, 2006-2015.



The wave-shaped surface: the glass-reinforced plastic umbrella shell GRP (OAO «Avangard»)



The wave-shaped surface: Sport Palace of I. Vinner-USmanova in Luzhniki, Moscow

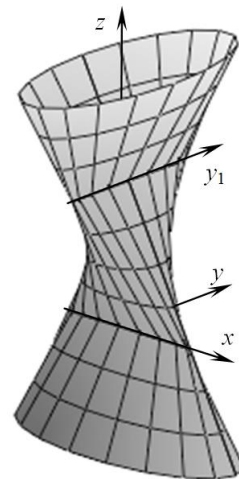


The riffled surface

**Figure 17.** Wave-shaped, waving, and corrugated surfaces.

Some shapes of wave-shaped and waving erections are shown in Figure 17.

#### *Kinematical Surfaces of General Type*



**Figure 18.** The ruled rotative surface of Lusta.

Kinematical surfaces of general type subdivide into 1) translation surfaces, 2) rotative surfaces [40], and 3) spiroidal surfaces [41]. Translation surfaces were considered earlier. Rotative and spiroidal surfaces are used in theory of motion of one body on another. Only one example of application of a rotative surface in civil engineering branch is described in a work [42]. Here, a rotative surface was used in the process of design of coverings of water-sport complex in Rostov-na-Donu and in the trade center in Kamensk-town. Lusta [4] recommended to apply the ruled rotative surface shown in Figure 18 for transmission of load from one beam on another beam lying crosswise. A fragment of the surface between straight lines is called “Die Milchtüte”.

It should be noted that architects must pay attention to rotative surfaces. Interesting umbrella and waving conical structures can be created with the help of these surfaces and with the help of epicycloid and hypocycloid cylinders.

#### *The Second Order Surfaces (7 Of 9)*

Four cylindrical surfaces of the second order belong also to a subclass “Cylindrical surfaces” and two surfaces were illustrated in Figure 2d (elliptical cylinder) and in Figure 2b (parabolical cylinder).

Let us illustrate an application of the rest surfaces of the second order (quadric) on concrete examples: ellipsoid (the observation deck of “Scenic Bridge” in the form of three-axial ellipsoid, Moscow, arch. N. Shumakov, 2007), one sheet hyperboloid, elliptical paraboloid (Utkal University, Bhubaneswar, Orissa, India), hyperbolic paraboloid (Courtesy of Princeton University Tedesco Archive, Denver, Colorado, 1959), and elliptical conical surface ([14, 43]).

Two sheet hyperboloid and hyperbolic cylinder did not find application in building and machine building.

Six surfaces of revolution are the particular case of six surfaces of the second order of general type and all of them are presented in the section «Surfaces of Revolution».

Having taken into consideration wide dissemination of shells outlined on the 2<sup>nd</sup> order surfaces one has given large attention to their analysis on strength, stability, and dynamic both in the first middle of the 20<sup>th</sup> century [44] and in the 21<sup>st</sup> century [45].

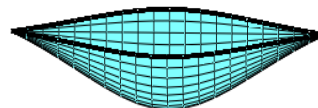
#### *Algebraic Surfaces of the High Orders With $n > 2$ (4 Of 104)*

A surface defined by an algebraic equation with the  $n$  degree in some system of rectangular coordinates is called an *algebraic surface of the  $n$  order*. Algebraic surfaces may be related into one or at once into several other classes of surfaces. For example, an algebraic surface of the fourth order that is a circular torus may be related also to surfaces of revolution or to cyclic surfaces. In present section, algebraic surfaces of the high orders, not included into other parts of the paper, are described.

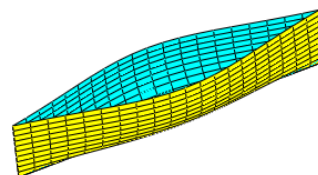
A brief additional information on algebraic surfaces such as «Miter Surface», «Burkhardt Quartic», «Clebsch Diagonal Cubic», «Cushion», «Desmic Surface», «Endraß Octic», «Heptic Surface», «Labs Septic», «Nonic Surface», «Octic

Surface», «Paraboloid Geodesic», «Septic Surface», «Symmetroid», «Tetrahedroid» was presented by Eric W. Weisstein [46]. But these surfaces did not find any application in architecture, building, and machine building.

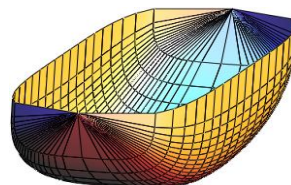
Algebraic surfaces of the high orders with  $n > 2$  were presented in tens of scientific papers. Authors of these papers recommended them for ship hulls on early stage of construction (Figure 19). However, concrete examples of application in building were not found, except surfaces pointed out in other classes.



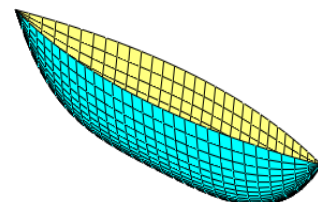
The 6<sup>th</sup> order surface with Agnesi curl, ellipse, Agnesi curl formed by the family of ellipses lying in the planes  $x = \text{const}$



The 7<sup>th</sup> order surface with Agnesi curl, Lamé's curve of the third order, straight line lying in three principal coordinate sections formed by a family of the Agnesi curls



The algebraical surface of the 8<sup>th</sup> order formed by Lamé's curves of the 8<sup>th</sup> order



Гидродинамическая поверхность с каркасом из параболы, параболы 4-го порядка и параболы 4-го порядка

**Figure 19.** Algebraic surfaces of the high orders with  $n > 2$ .

#### *Interesting Erections Realized In Nature*

Having studied the opinions of well-known architects and designers, it is possible to make a list of the most interesting curvilinear erections realized in nature after 2000<sup>th</sup>. The most interesting projects were presented and realized by Podgor-



nov E. V. (trade center «PITERLAND», RF, 2012, sphere), Finnish firm «Fecsim» with chief architect of the project L. A. Il'chik (Capital Green-house SBG RAS, Moscow, 2016, cylindrical surface, Figure 2g), Helmut Jahn (center SONY, FRG, 2000), S. Calatrava (Palau de les Arts Reina Sofia and planetarium of the Center "The City of Arts and Sciences", Valencian, Spain, 2005, sphere), Paul Andreu (National Grand Theater of China, 2007; Osaka Maritime Museum, Japan, 2000), Zaha Hadid (G. Aliev' Cultural Center in Baku, Azerbaijan, 2012), F. O. Gehry (Marques De Riscal Hotel, Elciego, 2006 г.), Pei I. M. (German Historical Museum, Berlin, 2001 г.), Hiroshi Nakamura (futuristic wedding chapel in the form of two spiral staircases with the steel frame wood-sheathed, Onomichi, Japan, 2013), Oscar Niemeyer (Museu Nacional Honestino Guimarães By Oscar Niemeyer, 2006, sphere), Sou Fujimoto (House of Hungarian Music in Budapest with the openings in covering for trees, model ecology on standards of BREEAM, 2021), Halili Nader («Eco dome» made of clay and thatch, ecological clean dwelling for Africa), V. Kramarenko (the glass dome of the hall "Victory" of the Museum of History of the 2<sup>nd</sup> World War, Minsk, 2014, sphere), Sh. Videchnik («Borisov-Arena», 70 km from Minsk, Belorussian, 2014), Simon Vélez (bamboo architecture of shell structures, Colombia), Sh. Killa (Killa Design, steel structure of the Museum of Future in Dubai, 2022, a shape in the form of surface of plane-and-parallel translation of a changing generatrix ellipse along the vertical director ellipse).

Everyone will find in this list a favorite direction in architecture of curvilinear structure. But well-known scientist I. A. Bondarenko warns that it is not necessary to roll down to populism, but it is necessary to keep sensing of proportion [47].

### 3. Conclusion

Every large-span structure or shell of unusual form is innovative and unique erection. Many countries are proud of their iconic large-span structures and shells. The analysis conducted by various researchers concluded that at the beginning of the 21<sup>st</sup> century, a recession in interest in shell structures and thin-walled shells came to an end. This happened due to the presence of new structural materials, and expansion of the list of analytical, point, spline, and frame surfaces suitable for use as middle surfaces of shells, due to the creation of more accurate methods of calculating shells for strength and on their basis of standard computer computational complexes, and most importantly, there was an increased demand for the creation of the structures under consideration. The study of scientific and reference literature, materials from internet showed that some classes did not find application in building and machine building branches yet. These are *pseudo-minimal surfaces*, majority of surfaces from classes «Affine minimal surfaces», «Peterson surfaces», except surfaces of translation of the second order curves; «Bonnet surfaces», «Blutel surfaces», except the second order

surfaces; «Hoshimoto surfaces», «Weingarten surfaces», except revolution surfaces. These surfaces are used for study of some physical processes, for solving of particular mathematical problems, and for determination of surfaces isometrical to surfaces of revolution.

Architects and building engineers have great potential in the choice of shape, materials, methods of analysis, constructive solutions, styles, and examples of application of large-span thin-walled shell structures. More than one-hundred-year experience of erection of these objects shows undamped interest and need of man in them.

### Abbreviations

GRP	Glass Reinforced Plastic
FRP	Fibre Reinforced Plastics
SPb	St. Petersburg
hypar	Hyperbolic Paraboloids

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### Authors Contributions

**Bock Hyeng Christian Alain:** Project administration, Writing – original draft

**Krivoshapko Sergey Nikolaiv:** Validation, Writing – review & editing

**Kouamou Nguessi Arnaud:** Data curation, Methodology, Visualization, Writing – review & editing

**Yamb Bell Emmanuel:** Data curation, Methodology, Visualization, Writing – review & editing

**Bahel Benjamin:** Validation

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