

Research Article

Injective Envelopes of Real C^* - and AW^* -Algebras

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Abstract

Injective (complex and real) W^* - and C^* - algebras, in particular, factors have been studied quite well. On the other hand, in an arbitrary case, i.e., in the non-injective case, it is quite difficult to study (up to isomorphism) the W^* -algebras, in particular, it is known that there is a continuum of pairwise non-isomorphic non-injective factors of type II_1 . Therefore, it seems interesting to study the so called maximal injective W^* and C^* -subalgebras or what is equivalent, the smallest injective C^* -algebra containing a given algebra, which is called an injective envelope of C^* - algebra. It is shown that every outer $*$ -automorphism of a real C^* -algebra can be uniquely extended to an injective envelope of real C^* -algebra. It is proven that if a real C^* -algebra is a simple, then its injective envelope is also simple, and it is a real AW^* -factor. The example of a real C^* -algebra that is not real AW^* -algebra and the injective envelope is a real AW^* -factor of type III, which is not a real W^* -algebra is constructed. This leads to the interesting result that up to isomorphism, the class of injective real (resp. complex) AW^* -factors of type III is at least one larger than the class injective real (resp. complex) W^* -factors of type III.

Keywords

C^* - Algebras, AW^* -algebras, Injective Envelopes of Real C^* -Algebras

1. Introduction

As is known, injective (complex and real) W^* - and C^* -algebras, in particular, factors have been studied quite well. On the other hand, in an arbitrary case, i.e., in the non-injective case, it is quite difficult to study (up to isomorphism) the W^* -algebras, in particular, it is known that there is a continuum of pairwise non-isomorphic non-injective factors of type II_1 . Therefore, it seems interesting to study the so called maximal injective W^* and C^* -subalgebras or what is equivalent, the smallest injective C^* -algebra containing a given algebra, which is called an injective envelope of C^* -algebra. In the complex case, such algebras were considered

in the works M.Hamana, K.Saito, M.Wright, R.Kadison, J.Fang. In this paper, we consider the existence and uniqueness of injective envelope real C^* -algebras for given real C^* -algebras. By analogy with Saito and Joita we will show that every outer $*$ -automorphism of a real C^* -algebra can be uniquely extended to an injective envelope of real C^* -algebra [1, 2]. Similar to Hamana, we will prove that if a real C^* -algebra is a simple, then its injective envelope is also simple and it is a real AW^* -factor [3]. We will build an example of a real C^* -algebra that is not real AW^* - algebra and the injective envelope is a real AW^* -factor of type III, which is not a real

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W^* -algebra. This leads to the interesting result that up to isomorphism, the class of injective real (resp. complex) AW^* -factors of type III is at least one larger than the class injective real (resp. complex) W^* -factors of type III.

2. Injective Envelopes C^* -Algebras

2.1. Preliminaries

Let $B(H)$ be the algebra of all bounded linear operators acting on a complex Hilbert space H . Recall that a real $*$ -subalgebra $R \subset B(H)$ with the identity $\mathbb{1}$ is called a *real W^* -algebra*, if it is weakly closed and $R \cap iR = \{0\}$.

Let A be a Banach $*$ -algebra over the field \mathbb{C} . The algebra A is called a *C^* -algebra*, if $\|aa^*\| = \|a\|^2$, for any $a \in A$. A real Banach $*$ -algebra R is called a *real C^* -algebra*, if $\|aa^*\| = \|a\|^2$ and an element $\mathbb{1} + aa^*$ is invertible for any $a \in R$. It is easy to see that R is a real C^* -algebra if and only if a norm on R can be extended onto the complexification $A = R + iR$ of the algebra R so that algebra A is a C^* -algebra (see [4, 5]).

Denote by $M_n(A)$ algebra of all $n \times n$ matrices over A which is also a C^* -algebra, relative to ordinary matrix operations. An element $a \in A$ is called *positive* and denote by $a \geq 0$, if there exists a self-adjoint element $b \in A$, such that $a = b^2$. The set of all positive elements of A denoted by A^+ .

A continuous linear map φ between two C^* -algebras A and B is called *completely positive*, if for any $n \geq 1$, the natural map φ_n from the C^* -algebra $A \otimes M_n$ to the C^* -algebra $B \otimes M_n$, defining by

$$\varphi_n \left((a_{ij})_{i,j=1}^n \right) = \left(\varphi(a_{ij}) \right)_{i,j=1}^n$$

is positive, where M_n is the C^* -algebra of $n \times n$ matrices over \mathbb{C} .

A C^* -algebra A with the unit 1 is said to be *injective* if whenever B is a unital C^* -algebra and S is a self-adjoint subspace of B containing the unit, then each completely positive map $\varphi: S \rightarrow A$ can be extended to a completely positive map $\bar{\varphi}: B \rightarrow A$. It was proved that for each Hilbert space H , the algebra of bounded operators on H is injective, i.e., $B(H)$ is injective.

It is known that any C^* -algebra can be isomorphically embedded into some algebra $B(H)$ in which it is uniformly closed. In 1967 in the work of Hakeda-Tomiyama, C^* -algebras with property E are considered: A C^* -algebra A has the *property E* (an extensional property) if there is a projection $P: B(H) \rightarrow A$ such that $\|P\| = 1$ and $P(1) = 1$. In this case the map P is completely positive. It is known that these two definitions are equivalent to each other.

Thus, every C^* -algebra lies in the injective C^* -algebra $B(H)$ and the smallest injective C^* -algebra containing this algebra is called an *injective enveloping C^* -algebra*. In [3],

the author proved that every C^* -algebra A with identity has a unique injective enveloping C^* -algebra, which is denoted as $I(A)$, i.e., the smallest injective C^* -algebra containing A as a C^* -subalgebra [6-9].

2.2. Outer $*$ -Automorphisms of Real C^* -Algebras

Let A be a real or complex algebra. A subspace I of an algebra A is called an *left ideal* (resp. *right ideal*) if $xy \in I$ (resp. $yx \in I$), for all $x \in A$ and $y \in I$. A left and right ideal is called a *two-sided ideal* or *ideal*. An algebra A is said to be *simple* if it contains no non-trivial two-sided ideals and the multiplication operation is not zero (that is, there are a and b with $ab \neq 0$). If α is an automorphism of A , then A said to be α -simple if the only an α -invariant, closed, proper, two-sided ideal of A is 0 .

Let A be (complex) an $*$ -algebra. A real subspace I of A is called a *real ideal* of A if $I \cdot A, A \cdot I \subset I_c$, where $I_c = I + iI$. Since each complex subspace of A is a real subspace, any complex ideal is automatically a real ideal of A . Let I be a real ideal of A . If there exists a real $*$ -subalgebra R of A with $R + iR = A$, such that $I \subset R$, then I is called a *pure real ideal* of A . In this case, it is obvious that we have $I \cdot R \subset I$. Note that, the reverse is not true, i.e., from $I \cdot R \subset I$ it does not follow $I \subset R$. But a complex subspace $J = I + iI$ always is a complex ideal of A . On the other hand if $I \subset R$ is a real subspace of A and $I + iI$ is a complex ideal, then I is a pure real ideal, i.e., we obtain $I \cdot R \subset I$.

Let, now I and Q be pure real ideals of A . In general, the set $I + iQ$ is not a (complex) subspace. More precisely the set $I + iQ$ is a complex subspace if and only if $I = Q$. Therefore we consider the smallest complex subspace J of A , containing I and Q . Obviously J is equal to $(I + Q) + i(I + Q)$. Thus, if I and Q are real ideals, then $J = (I + Q) + i(I + Q)$ is a complex ideal.

Theorem 3.1. *Let R be a real algebra. Then R is simple if, and only if $R + iR$ is simple.*

Proof. Assume that real subalgebra I of a algebra R be a non-trivial two-sided ideal of a algebra R . Then, obviously that complex subspace $I + iI$ is be a non-trivial two-sided ideal of a algebra $R + iR$. Inversely, let J is be non-trivial two-sided ideal in $R + iR$ and let $a + ib \in J$, where $a, b \in R$. Since J is a two-sided ideal, then for $a - ib \in R + iR$ we have

$$x = (a + ib)(a - ib) = a^2 + b^2 + iba - iab \in J$$

$$y = (a - ib)(a + ib) = a^2 + b^2 - iba + iab \in J$$

Hence, $x + y = 2a^2 + 2b^2 \in J$, therefore $J \cap R \neq \emptyset$. Assume that $I := J \cap R$. It is easy to see, that real subspace I be a non-trivial two-sided ideal of algebra R . The proof is completed.

Let α be a $*$ -automorphism of real $*$ -algebra R . By $\tilde{\alpha}$ we denote the linear extension of α to $= R + iR$, which is defined as $\tilde{\alpha}(x + iy) = \alpha(x) + i\alpha(y)$, where $x, y \in R$.

Proposition 3.1. *R is an α -simple if and only if A is an $\tilde{\alpha}$ -simple.*

Proof. Let I be a subspace of R . Then $I_c = I + iI$ is (complex) subspace of A . It's obvious that I is ideal of R if and only if I_c is ideal of A . Moreover, I is closed if and only if I_c is closed. It's easy to see that if $\alpha(I) \subset I$, then $\tilde{\alpha}(I_c) = \alpha(I) + i\alpha(I) \subset I_c$, and obversely, $\tilde{\alpha}(I_c) \subset I_c$ implies $\alpha(I) \subset I$. Finally, it is obvious that $I \neq R$ (or \emptyset) $\Leftrightarrow I_c \neq A$ (or $\{0\}$). The proof is completed.

Let us formulate one result from the work [10].

Proposition 3.2. *A real C^* -algebra R is an injective if and only if the C^* -algebra $R + iR$ is an injective.*

Now let's prove the following result.

Proposition 3.3. *Let R be a real C^* -algebra. Then a real C^* -algebra B is injective envelope of R if and only if $B + iB$ is injective envelope C^* -algebra of $A = R + iR$.*

Proof. Let $B + iB$ be an injective envelope of $R + iR$. By Proposition 3.2 B is an injective real C^* -algebra. If S is an injective real C^* -algebra with $R \subset S$, then $A = R + iR \subset S + iS$ and by Proposition 3.2 $S + iS$ also is an injective, hence $B + iB \subset S + iS$. Therefore $B \subset S$ and B is injective envelope real C^* -algebra of R .

Conversely, let B be an injective envelope real C^* -algebra of R . Then $A = R + iR \subset B + iB$ and by Proposition 3.2 $B + iB$ is an injective C^* -algebra. It's easy to see that $B + iB$ is injective envelope of A . The proof is completed.

Hence, using the result of [3] we obtain the following corollary.

Corollary 1. *Any real C^* -algebra has a unique injective envelope real C^* -algebra.*

Proposition 3.4. *Let R be a real C^* -algebra, α be a $*$ -automorphism of R . Then α is outer $*$ -automorphism of R if and only if $\tilde{\alpha}$ is outer $*$ -automorphism of $A = R + iR$.*

Proof. If α is an inner $*$ -automorphism of R , there is an unitary $u \in R$ such that $\alpha(x) = Adu(x) = uxu^*, \forall x \in R$. Hence, we obtain

$$\begin{aligned}\tilde{\alpha}(x + iy) &= \alpha(x) + i\alpha(y) = uxu^* + iuyu^* = \\ &= u(x + iy)u^* = Adu(x + iy),\end{aligned}$$

i.e., α is also an inner $*$ -automorphism of A . Conversely, let $\tilde{\alpha}$ is an inner, i.e. $\tilde{\alpha}(x + iy) = Adv(x + iy)$, where $v \in A$ is an unitary. Since $\tilde{\alpha}(R) \subset R$, then by Corollary 3.1 from [11], there exists an unitary $u \in R$ such that $\tilde{\alpha} = Adu$. Therefore $\alpha = Adu$, i.e., α is also an inner. The proof is completed.

Now we will prove the main result of the section.

Theorem 3.2. *Let R be real C^* -algebra, α be a $*$ -automorphism of R such that R is α -simple. Let B be the injective envelope real C^* -algebra of R . If α is an outer $*$ -automorphism of R , then α has a unique extension to an outer $*$ -automorphism of B .*

Proof. By Proposition 3.1 C^* -algebra $A = R + iR$ is $\tilde{\alpha}$ -simple. By Proposition 3.4 $*$ -automorphism $\tilde{\alpha}$ also is an outer $*$ -automorphism. By Proposition 3.3 C^* -algebra $B_c = B + iB$ is the injective envelope of A . Then by [1, Theorem 3.6] $*$ -automorphism $\tilde{\alpha}$ has a unique extension $\tilde{\tilde{\alpha}}$ to outer $*$ -automorphism of B_c . It is obvious that the restriction of $\tilde{\tilde{\alpha}}$ on R coincides with α , i.e. $\tilde{\tilde{\alpha}}|_R = \alpha$. Then it directly shows that $\tilde{\alpha} = \tilde{\tilde{\alpha}}|_B$ is a unique extension of α to an outer $*$ -automorphism of B . The proof is completed.

2.3. Injective Envelope of Real Simple C^* -Algebras

To motivate the next definitions, suppose A is a $*$ -ring with unity, and let w be a partial isometry in A . If $e = w^*w$, it results from $w = ww^*w$ that $wy = 0$ iff $ey = 0$ iff $(1 - e)y = y$ iff $y \in (1 - e)A$, thus the elements that right-annihilate w form a principal right ideal generated by a projection. If S is a nonempty subset of A , we write

$$R(S) = \{x \in A : sx = 0, \forall s \in S\}$$

and call $R(S)$ the *right-annihilator* of S . Similarly, the set $L(S) = \{x \in A : xs = 0, \forall s \in S\}$ denotes the *left-annihilator* of S .

A *Baer $*$ -ring* is a $*$ -ring A such that, for every nonempty subset S of A , $R(S) = gA$ for a suitable projection g . It follows that

$L(S) = (R(S^*))^* = (hA)^* = Ah$ for a suitable projection h . A real (resp. complex) *AW $*$ -algebra* is a real (resp. complex) C^* -algebra that is a Baer $*$ -ring (for more details see [12-14]). An AW $*$ -algebra A is called a *factor* if the center of A is trivial. It is known that, every W $*$ -algebra is an AW $*$ -algebra. The converse of it was shown to be false by J.Dixmier, who showed that exist commutative AW $*$ -algebras that cannot be represented ($*$ -isomorphically) as W $*$ -algebras on any Hilbert space.

The following interesting result is true.

Theorem 4.1. *Let R be a real C^* -algebra with the unit and let B its injective envelope. If R is a simple algebra, then B is also simple, in which case B is a real AW $*$ -factor.*

Proof. By Proposition 3.3 C^* -algebra $B + iB$ is an injective envelope of $A = R + iR$. By Theorem 3.1, since R is a simple algebra, then A is also simple. By Proposition 4.15 from [3], $B + iB$ is also simple and $B + iB$ is a (complex) AW $*$ -factor. Then B is also simple and by Proposition 4.3.1 from [4], algebra B is a real AW $*$ -factor. The proof is completed.

Now, by analogy with Hamana's work, we give an example of an injective non W $*$ -, AW $*$ -factor of type III [3].

Consider the Calkin algebra: $A = B(H)/K(H)$, where H is an infinite-dimensional separable Hilbert space and $K(H)$ is an algebra of all compact operators on H . Example 5.1 from [3] shows that if B is an injective envelope of A , then B

is an injective AW*-factor of type III, which is not a W*-algebra. Let us recall here that the Calkin algebra A is not an AW*-algebra [15].

Now let's look at the real analogue of this example. Let H_r is an infinite-dimensional separable real Hilbert space. Then since $B(H) = B(H_r) + iB(H_r)$ and $K(H) = K(H_r) + iK(H_r)$, where $H = H_r + iH_r$. It's easy to see that $A_r = B(H_r)/K(H_r)$ is a real C*-algebra and we have $A_r + iA_r = A$. Moreover, if B_r is an injective real envelope of A_r , then $B_r + iB_r = B$. By analogy [15], since B is not a W*-algebra, then B_r is also not real W*-algebra.

Thus A_r is a real C*-algebra that is not real AW*-algebra and the injective envelope B_r of A_r is an injective real AW*-factor of type III, which is not a real W*-algebra.

It is known that for each number $\lambda \in [0, 1]$ the class of injective (complex) W*-factors of type III_λ is unique, i.e. any two injective III_λ -factors are isomorphic. From the Hamana's example we get the following interesting result [3, Example 5.1].

Corollary 2. Up to isomorphism the class of injective (complex) AW-factors of type III_λ ($0 \leq \lambda \leq 1$) is not unique, i.e. there are at least two isomorphism classes of injective (complex) AW*-factors of type III_λ .*

In the real case: up to isomorphism there exist exactly two injective real W*-factors of type III_λ ($0 < \lambda \leq 1$) and up to isomorphism there exists a unique injective real W*-factor of type III_1 . For the case of real injective type III_0 factor we can construct a countable number of pairwise non isomorphic real injective factors of type III_0 , with isomorphic enveloping (complex) W*-factors (see [4, 5]).

Thus, from the above example, in the real case we can state the following

Corollary 3. Up to isomorphism, the class of of injective real AW-factors of type III is at least one larger than the class injective real W*-factors of type III.*

3. Materials and Methods

To obtain the results, methods of operator algebras were used, as well as the method of transition to the enveloping von Neumann algebra.

4. Results

Outer *-automorphism of a real C*-algebra can be uniquely extended to an injective envelope of real C*-algebra is proved. It is proven that if a real C*-algebra is a simple, then its injective envelope is also simple, and it is a real AW*-factor. The example of a real C*-algebra that is not real AW*-algebra and the injective envelope is a real AW*-factor of type III, which is not a real W*-algebra is constructed. This leads to the interesting result that up to isomorphism, the class of injective real (resp. complex) AW*-factors of type III is at least one larger than the class injective real (resp. complex) W*-factors

of type III.

5. Discussion

It is known that the class of injective complex von Neumann factors of type III is unique, i.e. any two injective III_λ -factors are isomorphic. But, as shown in this paper the class of injective real (resp. complex) AW*-factors of type III is at least one larger than the class injective real (resp. complex) W*-factors of type III.

6. Conclusions

Recently, interest in the study of AW*-algebras has increased. The results obtained here will undoubtedly be of interest to specialists in the field of operator algebra theory. It is proven that if a real C*-algebra is a simple, then its injective envelope is also simple, and it is a real AW*-factor. The example of a real C*-algebra that is not real AW*-algebra and the injective envelope is a real AW*-factor of type III, which is not a real W*-algebra is constructed. It is known that the class of injective complex von Neumann factors of type III is unique, i.e. any two injective III_λ -factors are isomorphic. But, as shown in this paper the class of injective real (resp. complex) AW*-factors of type III is at least one larger than the class injective real (resp. complex) W*-factors of type III.

Abbreviations

$B(H)$	Algebra of all Bounded Linear Operators Acting on a Complex Hilbert Space H
$M_n(A)$	Algebra of all $n \times n$ Matrices over A
property E	An Extensional Property
H	An Infinite-dimensional Separable Hilbert Space
$K(H)$	An Algebra of all Compact Operators on H

Author Contributions

Abdugafur Rakhimov: Conceptualization, Supervision, Writing - original draft, Data curation, Investigation, Validation, Formal Analysis, Methodology, Writing - review & editing

Laylo Ramazonova: Investigation, Methodology, Writing - original draft, Validation, Writing - review & editing, Formal Analysis

Conflicts of Interest

The authors declare no conflicts of interest.

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Research Field

Abdugafur Rakhimov: Functional analysis, Theory of operator algebras, Theory of unbounded operators, Topology, Fuzzy topology

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