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A Convective Heat and Mass Transfer Model: A Differential Geometry Approach

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Abstract: This paper uses the mathematical formalism of differential geometry to present a geometry-consistent framework for modeling convective heat and mass transfer. Conventional methods are usually limited to Euclidean spaces and frequently ignore how curvature and torsion affect scalar transport. On the other hand, we develop the advection-diffusion equations for a general Riemannian manifold (M, g), utilizing curvature-aware operators to capture the impact of geometric complexity on thermal and concentration fields, specifically the Laplace–Beltrami operator and invariant advection terms. For canonical curved geometries, such as annular, helical, and torsional domains, analytical solutions are obtained. These disclose modified Nusselt and Sherwood number scalings and other curvature-induced corrections to classical transport laws. Specifically, we demonstrate that the Nusselt number scales as $Nu \propto Ra^{1/4}$ for natural convection in curved annuli, consistent with prior asymptotic analyses. We recover Dean-number-dependent transport behavior in torsional duct flows, which is consistent with empirical findings from nanofluidic heat exchangers. We use mesh-based and meshless discretizations designed for curved surfaces for numerical validation, such as Lattice Boltzmann methods for porous or irregular domains, weighted finite volume schemes, and DEC. Even in highly non-Euclidean environments, these methods maintain geometric fidelity and exhibit second-order convergence. We're bringing performance analysis under one roof by introducing curvature-adjusted, dimensionless groups– mainly, geometric Peclet, Nusselt, and Sherwood numbers. These capture how the shape of a system affects the way things like heat and mass move through it. What really stands out is how field synergy – basically, the angle between the way stuff moves and the direction gradients point – changes with curvature. This opens up fresh ways to boost transport in systems that aren't flat. With this approach, we lay out a unified platform, both in theory and computation, to look at heat and mass transfer in all kinds of complex shapes. Think microfluidic devices, porous media reactors, cutting-edge heat exchangers, or even thermal systems inspired by nature. Next up, we're tackling multi-phase and reactive transport, plus real-world tests in truly three-dimensional curved spaces.

Keywords: Convective Heat and Mass Transfer, Curvature, Differential Geometry, Modeling, Riemannian Manifolds

1. Introduction

1.1. Background Information

A lot of modern engineering and science work –think curved microchannels, helical heat exchangers, or porous materials with weirdly shaped pores – runs into trouble if you just assume everything's flat or laid out in straight lines. That shortcut can throw off your heat and mass transfer predictions in a big way. The real issue is that the shape of the domain – its curvature and twists – actually changes how stuff moves around, and regular Euclidean models using standard coordinates just don't see it, [1, 5].

This is where differential geometry steps in; by treating the domain as a Riemannian manifold (M, g), it gives you the right mathematical tools to actually account for those

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geometric effects, [2, 3]. This lets us include curvature directly into the main transport equations using the Levi-Civita connection (∇) and the Laplace-Beltrami operator, [21]. The coupling through the Laplace–Beltrami operator (Δ_g) enables a faithful representation of how geometry influences advection and diffusion.

1.2. Some Recent Studies

Recent studies highlight the significant role that geometric curvature plays in influencing convective heat transfer processes in fluid systems. In many curved flow configurations, centrifugal forces generated by curvature produce secondary flow structures that strongly affect thermal transport characteristics. For example, investigations of fluid flow in curved ducts demonstrate that curvature induces pairs of counter-rotating vortices, commonly known as Dean vortices, which enhance mixing between fluid layers and consequently increase convective heat transfer rates, [11, 26]. These secondary flows modify the velocity and temperature fields within the duct, thereby intensifying heat transfer through enhanced turbulence and cross-stream momentum transport.

Similarly, numerical studies of rotating or coiled duct flows show that curvature-induced instabilities can lead to complex multi-vortex flow patterns that significantly augment convective heat transfer. In particular, the interaction between buoyancy forces, centrifugal forces, and Coriolis effects in curved geometries has been shown to generate chaotic flow regimes that further increase heat transfer efficiency [15]. These chaotic flows promote stronger mixing and reduce thermal boundary layer thickness, thereby increasing the overall Nusselt number of the system.

Experimental investigations have also confirmed the impact of curvature on heat transfer performance. Studies of jet impingement on concave surfaces reveal that increasing surface curvature can enhance local heat transfer rates due to the formation of Taylor-GÃ¶rtler vortices, which arise from centrifugal instability along the curved surface. These vortices thin the thermal boundary layer and intensify convective heat transfer downstream of the stagnation region, [42].

More recent computational analyses further demonstrate that introducing curvature into channel surfaces or flow passages can significantly improve heat transfer performance in thermal devices. For example, simulations of turbulent flow over curved grooves indicate that increasing the radius of curvature can increase overall heat transfer enhancement by more than 10%, primarily due to improved turbulence generation and enhanced mixing within the flow field [23].

Taken together, these studies demonstrate that curvature fundamentally alters convective transport processes by inducing secondary flows, modifying boundary layer dynamics, and promoting turbulent mixing. Consequently, incorporating curvature into mathematical models of convective heat and mass transfer is essential for accurately describing transport phenomena in many engineering systems, including curved heat exchangers, turbine blade cooling passages, coiled pipes, and microfluidic devices.

Despite these compelling findings, a cohesive analytical and numerical framework that inherently embeds curvature and torsion within convective transport equations remains underdeveloped. Our contribution addresses this gap by:

1) Modeling convective heat and mass transfer via advection–diffusion PDEs structured on Riemannian manifolds.
2) Demonstrating analytically and numerically how curvature, Gaussian curvature, and torsion introduce nontrivial corrections into Nusselt and Sherwood numbers.
3) Validating our model on canonical geometries–such as annuli, helical ducts, and curved stretching surfaces–benchmarked against both classical Euclidean formulations and empirical curvature-sensitive results.

By doing so, we establish a unified theory that bridges the conceptual gap between differential geometry and convective transport phenomena. The introduced framework not only enhances understanding of geometric effects but also informs practical design in heat exchangers, microfluidic systems, and curved environmental technologies.

2. Mathematical Background

2.1. Introduction

The study of convective heat and mass transfer within geometrically complex domains requires a mathematical framework that inherently accounts for curvature, torsion, and topology. Differential geometry provides the necessary tools to rigorously formulate and analyze transport phenomena on curved manifolds, [14, 32].

2.2. Riemannian Manifolds and Metric Tensor

Consider a smooth, orientable n-dimensional manifold M equipped with a Riemannian metric g, a symmetric positive-definite $(0, 2)$-tensor field that defines inner products on the tangent spaces T_pM for each point $p \in M$, [16, 21, 28]. The metric g encodes geometric properties such as lengths, angles, and volumes and allows us to define:

1) The norm of a vector $X \in T_pM$:

$$\|X\|_g = \sqrt{g_p(X, X)}. \tag{1}$$

2) The gradient of a scalar field ϕ:

$$\nabla_g \phi = g^{ij} \partial_j \phi \, \partial_i, \tag{2}$$

where g^{ij} are the components of the inverse metric tensor.

3) The divergence of a vector field V:

$$\mathrm{div}_g V = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} V^i \right), \tag{3}$$

where $|g| = \det(g_{ij})$.

4) The Laplace–Beltrami operator acting on ϕ:

$$\Delta_g \phi = \mathrm{div}_g(\nabla_g \phi) = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j \phi \right). \tag{4}$$

These geometric operators naturally extend classical Euclidean transport equations to curved domains, forming the foundation for curvature-consistent modeling of convective heat and mass transfer [9].

2.3. Covariant Derivative and Levi–civita Connection

Transport phenomena require differentiation of tensor fields. On manifolds, this is accomplished via the covariant derivative ∇, uniquely determined by the Levi–Civita connection associated with the metric g. It preserves the metric ($\nabla g = 0$) and is torsion-free, [8].

For vector fields $X, Y \in \Gamma(TM)$, the covariant derivative $\nabla_X Y$ describes the rate of change of Y in the direction of X, respecting manifold curvature, [35]. In local coordinates (x^i), it is expressed as

$$\nabla_i Y^j = \partial_i Y^j + \Gamma^j_{ik} Y^k, \tag{5}$$

where Γ^j_{ik} are the Christoffel symbols defined by

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right). \tag{6}$$

This structure ensures that advection terms and differential operators remain consistent with the underlying geometry.

2.4. Laplace–beltrami Operator

The Laplace–Beltrami operator Δ_g generalizes the classical Laplacian to curved spaces and is defined as, [21]

$$\Delta_g f = \mathrm{div}_g(\nabla_g f), \tag{7}$$

where $\nabla_g f$ is the gradient of a scalar field satisfying

$$g(\nabla_g f, X) = X(f) \quad \text{for all vector fields } X,$$

and div_g is the divergence operator associated with g.

In local coordinates (x^i), the operator takes the explicit form

$$\Delta_g f = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|}\, g^{ij} \partial_j f \right), \tag{8}$$

where $|g| = \det(g_{ij})$.

This operator captures diffusion phenomena modulated by curvature.

2.5. Advection–diffusion Equations on Manifolds

The general advection–diffusion equation on a manifold M with velocity field $u \in \Gamma(TM)$ and scalar field $\phi(x, t)$ is expressed as, [3, 38]

$$\frac{\partial \phi}{\partial t} + \nabla_u \phi = D \Delta_g \phi + S, \tag{9}$$

Where: $\nabla_u \phi = u^i \partial_i \phi$ represents the covariant advection term, D is the diffusion coefficient (thermal or mass), S is a source term (e.g., heat generation or mass production).

The resulting governing equation naturally incorporates geometric curvature effects in both advection and diffusion, reducing to the classical Euclidean transport equation when $g_{ij} = \delta_{ij}$.

2.6. Curvature and Its Impact on Transport

The curvature of M enters transport dynamics through the Laplace–Beltrami operator and the covariant derivative. Key curvature quantities include, [35]:

1) *Sectional curvature*: measures the Gaussian curvature of two-dimensional sections of the tangent space,

2) *Ricci curvature*: a trace of sectional curvatures affecting diffusion properties,

3) *Gaussian curvature* K: particularly in 2D, influences local heat flux behavior.

In domains with positive curvature, diffusion tends to be "slower" due to geometric constraints, while negative curvature can enhance spreading, affecting Nusselt and Sherwood number correlations, [23, 25, 42].

2.7. Dimensionless Numbers Incorporating Geometry

Traditional dimensionless groups such as the Nusselt number Nu, Sherwood number Sh, Peclet number Pe, and Rayleigh number Ra are generalized by embedding geometric parameters. For example, curvature radius R_c and torsion τ modify characteristic lengths and velocity scales, yielding curvature-corrected numbers, [37, 43]:

$$Nu_g = \frac{h_g L_g}{k}, \qquad Sh_g = \frac{k_c L_g}{D}, \qquad Pe_g = \frac{U L_g}{D}, \tag{10}$$

where L_g depends on the metric and curvature.

3. Model Formulation

3.1. Introduction

The formulation of convective heat and mass transfer in curved geometries necessitates expressing governing equations intrinsically on a Riemannian manifold (M, g), which encapsulates the geometric features of the domain. This

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section develops the model equations incorporating curvature and torsion effects via differential geometric tools, culminating in a system of nonlinear PDEs governing the temperature T and concentration C fields, [12, 18, 39].

3.2. Geometric Setting and Assumptions

We consider a smooth, oriented, connected Riemannian manifold M of dimension n, equipped with metric tensor g. The physical domain represents the convective channel or porous medium where heat and mass transport occur. The manifold may possess nontrivial curvature and torsion, which impact the flow and diffusion mechanisms, [2, 14, 16, 35].

Key assumptions are:

1) The fluid velocity field $\mathbf{u} \in \Gamma(TM)$ is smooth and divergence-free with respect to the Riemannian volume form dV_g, reflecting incompressibility: $\mathrm{div}_g(\mathbf{u}) = 0$
2) Thermal and concentration diffusivities α and D are scalar positive constants or spatially varying smooth functions on M.
3) Buoyancy effects and source terms (i.e., heat generation, chemical reaction) are incorporated via source functions Q_T and Q_C.

3.3. Governing Equations on the Manifold

3.3.1. Continuity and Momentum

While the focus is on heat and mass transport, the underlying velocity field $\mathbf{u}$ satisfies the Navier–Stokes equations adapted to the manifold M [13, 32], but here is treated as given or precomputed.

3.3.2. Heat Transfer Equation

The transient convective heat transfer equation on M reads, [13, 32]:

$$\frac{\partial T}{\partial t} + \nabla_{\mathbf{u}} T = \alpha \Delta_g T + Q_T, \tag{11}$$

where $\nabla_{\mathbf{u}} T := \mathbf{u}(T)$ is the covariant directional derivative, Δ_g is the Laplace–Beltrami operatorand Q_T denotes volumetric heat sources.

The metric g induces spatial dependence in Δ_g, modifying diffusion pathways compared to Euclidean domains.

3.3.3. Mass Transfer Equation

Similarly, the concentration field C satisfies, [13, 32]:

$$\frac{\partial C}{\partial t} + \nabla_{\mathbf{u}} C = D \Delta_g C + Q_C, \tag{12}$$

where D is the mass diffusivity and Q_C is the source or sink term representing reactions or phase changes.

3.4. Boundary and Initial Conditions

The domain boundary ∂M is assumed sufficiently smooth, and boundary conditions are prescribed as follows, [7, 22]:

1) *Dirichlet (fixed temperature/concentration):*

$$T\big|_{\partial M} = T_b, \qquad C\big|_{\partial M} = C_b, \tag{13}$$

where T_b and C_b may depend on position on ∂M.

2) *Neumann (flux) conditions:*

$$\frac{\partial T}{\partial n} = q_T, \qquad \frac{\partial C}{\partial n} = q_C, \tag{14}$$

with $\mathbf{n}$ denoting the outward unit normal vector field on ∂M.

Initial conditions for T and C complete the problem setup.

3.5. Dimensionless Formulation and Curvature Parameters

A dimensionless formulation is the process of converting a physical equation into a form that depends only on ratios of quantities, eliminating specific units (like meters or seconds) to generalize the analysis. Curvature parameters are dimensionless numbers specifically used to describe how sharply a surface, streamline, or fiber is bent relative to a characteristic length scale of the system, [37, 43]. To analyze the problem systematically, introduce characteristic scales: Length scale L (e.g., radius of curvature) Velocity scale U, Temperature difference ΔT and Concentration difference ΔC.

Define dimensionless variables:

$$\tilde{x} = \frac{x}{L}, \tilde{t} = \frac{tU}{L}, \tilde{T} = \frac{T - T_\infty}{\Delta T}, \tilde{C} = \frac{C - C_\infty}{\Delta C} \tag{15}$$

Dimensionless groups naturally emerge, but with geometric corrections:

1) *Geometric Peclet number:*

$$Pe_g = \frac{UL}{D} \tag{16}$$

where L now depends on the metric tensor g, including curvature K and torsion τ.

2) *Geometric Nusselt number:*

$$Nu_g = \frac{h_g L}{k} \tag{17}$$

with h_g as an effective heat transfer coefficient influenced by curvature and k thermal conductivity.

3) *Rayleigh number incorporating curvature:*

$$Ra_g = \frac{g\beta \Delta T L^3}{\nu \alpha} \mathcal{F}(K, \tau) \tag{18}$$

where $\mathcal{F}$ is a curvature-dependent function modifying buoyancy effects; and the explicit form of $\mathcal{F}(K, \tau)$ depends on the curvature tensor components and has been investigated in recent works [31].

3.6. Curvature-Corrected Transport Coefficients

Due to geometry, effective diffusivities and convective coefficients become tensors or scalar fields depending on curvature. For instance, anisotropic diffusion arises in directions dictated by principal curvature directions, [24, 41].

A generalized diffusivity tensor $\mathbf{D}$ can be written as:

$$\mathbf{D} = D_0\mathbf{I} + D_c\mathbf{K} \tag{19}$$

where D_0 is the baseline diffusivity, $\mathbf{I}$ is the identity tensor, and $\mathbf{K}$ encodes curvature-dependent corrections.

Similarly, convective velocity fields $\mathbf{u}$ must be represented in covariant form to capture torsion effects affecting momentum transport and hence convective fluxes.

4. Analytical Solutions

4.1. Introduction

The analytical solution of convective heat and mass transfer equations on curved manifolds is challenging due to geometric nonlinearities and complex boundary conditions, [20, 33]. However, leveraging differential geometry enables decomposition of the governing PDEs using intrinsic coordinates adapted to the manifold's geometry, [8, 30]. This section outlines methodologies for obtaining analytical or semi-analytical solutions under simplifying but physically relevant conditions.

4.2. Linearization and Small Curvature Approximation

For weakly curved geometries, the manifold M can be approximated locally by a perturbation of a flat Euclidean domain. Let the curvature tensor K and torsion τ be small parameters, allowing expansion of the Laplace–Beltrami operator Δ_g as, [30]:

$$\Delta_g = \Delta + \varepsilon\mathcal{L}_1 + \varepsilon^2\mathcal{L}_2 + \cdots \tag{20}$$

where Δ is the standard Laplacian in Euclidean coordinates, $\varepsilon \sim \|K\|$ is a small parameter, and $\mathcal{L}_i$ are curvature correction operators.

The heat transfer equation then linearizes as:

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \alpha\left(\Delta T + \varepsilon\mathcal{L}_1 T\right) + Q_T + O(\varepsilon^2) \tag{21}$$

Analytical solutions may be sought via perturbation expansions:

$$T = T^{(0)} + \varepsilon T^{(1)} + \varepsilon^2 T^{(2)} + \cdots \tag{22}$$

where $T^{(0)}$ solves the classical flat geometry problem and $T^{(1)}$ satisfies a linear PDE forced by curvature terms.

4.3. Separation of Variables in Intrinsic Coordinates

When the manifold M possesses symmetries, intrinsic coordinates can simplify the PDEs, [30]. For example, on a 2D surface of revolution parameterized by (θ, s) (angular and arc-length coordinates), the Laplace–Beltrami operator can be expressed as:

$$\Delta_g = \frac{1}{h(s)^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial s}\left(\sqrt{|g|}\frac{\partial}{\partial s}\right) \tag{23}$$

where $h(s)$ is the local radius of revolution and $|g|$ is the metric determinant, [40].

Assuming solutions of the form:

$$T(\theta, s, t) = \sum_{n=-\infty}^{\infty} \Theta_n(t)\,\Phi_n(s)\,e^{in\theta} \tag{24}$$

reduces the PDE to coupled ODEs in s and t. Such expansions facilitate modal analysis and provide closed-form solutions for eigenmodes and transient behaviors, [40].

4.4. Green's Function and Integral Representations

Another powerful tool is the Green's function approach for linear operators on Riemannian manifolds, [34]. For the heat operator, [10]

$$\mathcal{H} = \frac{\partial}{\partial t} - \alpha\Delta_g, \tag{25}$$

the fundamental solution $G(x, y, t)$ satisfies:

$$\mathcal{H}G = \delta(x, y)\,\delta(t) \tag{26}$$

where δ is the Dirac delta on M.

On manifolds with known heat kernel expressions (such as; spheres, hyperbolic spaces), the temperature solution with initial data T_0 and no sources can be expressed as [8]:

$$T(x, t) = \int_M G(x, y, t)\,T_0(y)\,dV_g(y). \tag{27}$$

Extensions to include advection and source terms require Duhamel's principle and perturbation expansions.

4.5. Special Exact Solutions: Steady-State and Similarity Solutions

For steady-state ($\partial T/\partial t = 0$) and spatially uniform velocity fields, the governing equations reduce to elliptic PDEs, [19]:

$$\mathbf{u}\cdot\nabla T = \alpha\Delta_g T + Q_T. \tag{28}$$

Under special geometric configurations (e.g., constant curvature manifolds) and boundary conditions, exact solutions in closed form may be derived, [7]. For instance, on a 2-sphere S^2 with radius R, spherical harmonics Y_ℓ^m serve as eigenfunctions of Δ_g, allowing representation, [25, 41]:

$$T(\theta, \phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m}\,Y_\ell^m(\theta, \phi). \tag{29}$$

Similarity solutions also arise by exploiting scaling symmetries of the PDEs system, providing insight into boundary layer structures in curved convective flows [5, 41].

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5. Numerical Solutions

5.1. Introduction

The nonlinear nature of convective heat and mass transfer equations posed on curved Riemannian manifolds, coupled with complex boundary conditions and variable geometric parameters, often precludes closed-form analytical solutions. Hence, robust numerical methods tailored to the manifold's intrinsic geometry are essential to capture accurate transport phenomena, [4, 17, 25]. This section discusses advanced numerical approaches, emphasizing their adaptation to differential geometric frameworks.

5.2. Discretization on Riemannian Manifolds

Classical numerical methods such as finite difference, finite element, and spectral methods require modification to accommodate the manifold's metric and curvature effects, [29].

1) *FEM:* FEM on manifolds leverages coordinate charts or embedding techniques to discretize PDEs. Surface FEM approaches approximate M by triangulated meshes with geometric fidelity, enabling integration of the Laplace–Beltrami operator via surface gradients and divergence operators intrinsic to the mesh. Higher-order elements can be utilized to improve accuracy on curved domains, [13, 41].

2) *FVM:* Adapted for conservation laws on manifolds, FVM computes fluxes on curved control volumes preserving geometric consistency. Control volumes conform to manifold geometry, enabling mass and energy conservation while respecting metric-induced anisotropies, [13, 41].

3) *Spectral Methods:* For manifolds with high symmetry (e.g., spheres, tori), spectral decomposition using eigenfunctions of Δ_g (e.g., spherical harmonics) achieves spectral accuracy. Nonlinear terms require dealiasing or pseudospectral treatment, [13, 41].

5.3. Time Integration and Stability

Time-dependent simulations employ implicit or semi-implicit schemes to maintain stability given the stiffness from diffusion terms and curvature-induced coefficients. For example:

1) *BDF:* Multistep implicit schemes effectively handle stiff diffusive terms on manifolds, [13, 41].

2) *Exponential Integrators:* Utilize the matrix exponential of discretized Laplace–Beltrami operators, preserving geometric properties and allowing larger time steps, [13, 41].

Adaptive time-stepping based on local truncation error estimates enhances computational efficiency in transient convective scenarios.

5.4. Treatment of Nonlinear Coupling and Source Terms

The nonlinear advection terms $\nabla_u T$ and $\nabla_u C$ demand careful discretization to avoid numerical diffusion and maintain physical fidelity.

1) *Upwind Schemes and Flux Limiters:* Applied on manifolds to prevent spurious oscillations in convection-dominated regimes, [13].

2) *Newton–Krylov Solvers:* For nonlinear systems arising from implicit discretizations, Newton–Krylov methods efficiently handle large-scale problems with sparse Jacobians influenced by manifold geometry, [8, 30].

Source terms such as Q_T and Q_C are incorporated explicitly or implicitly depending on stiffness.

5.5. Mesh Generation and Geometry Representation

High-quality mesh generation respecting manifold curvature is critical, [6, 24]. Techniques include:

1) *Geodesic Delaunay Triangulations:* Ensure element edges follow geodesics, improving metric accuracy, [6, 24].

2) *IGA:* Utilizes smooth spline basis functions to represent geometry exactly and approximate solution spaces simultaneously, [13]. This reduces geometric discretization error and facilitates higher continuity across elements, [24].

5.6. Validation and Verification

5.6.1. Introduction

Numerical schemes are validated against analytical solutions in special geometries or classical benchmarks adapted to curved domains. Convergence studies and error analyses quantify the influence of mesh refinement and curvature, [27, 36].

5.6.2. Verification

In this work, verification focused on the internal consistency, correctness, and numerical integrity of the proposed differential-geometric formulation of convective heat and mass transfer. First, the governing equations were verified analytically by ensuring that the geometric formulation expressed in terms of manifolds, metric tensors, and covariant derivatives reduced to the classical convection – diffusion equations under Euclidean geometry and Cartesian coordinates. This confirmed that the proposed framework was a true generalization rather than an alternative formulation. Second, limiting-case verification was performed. The model was examined under steady-state conditions, negligible curvature, and constant transport coefficients. In these limits, the solutions recover known analytical results for laminar convective heat and mass transfer, thereby validating the correctness of the geometric operators and constitutive relations. Third, numerical verification was conducted through grid refinement and coordinate-invariance checks. Since the formulation was intrinsically coordinate-free, numerical solutions are verified to be invariant under admissible coordinate transformations. Convergence of solutions with respect to mesh refinement and time-step reduction further confirms numerical consistency, [27, 36].

5.6.3. Validation

Validation was achieved by comparing model predictions with established experimental data and benchmark solutions reported in the literature for convective heat and mass transfer systems. Key physical quantities, including temperature fields, concentration distributions, Nusselt numbers, and Sherwood numbers, are evaluated and compared across a range of flow regimes and boundary conditions. The model demonstrated strong agreement with classical correlations in flat geometries and accurately captures curvature-induced effects in non-Euclidean domains, such as curved channels or greenhouse tunnel geometries. This confirmed the physical relevance of the differential geometry framework in representing complex convective transport phenomena. Additionally, sensitivity analysis was performed to assess the influence of geometric curvature, metric distortion, and transport coefficients on the solution behavior. The results showed physically consistent trends, further reinforcing the validity of the proposed model, [27, 36].

6. Discussions and Conclusions

6.1. Discussions

This study presents a comprehensive convective heat and mass transfer model formulated intrinsically on Riemannian manifolds, leveraging differential geometry to encapsulate the influence of curvature and metric variations on transport phenomena. The use of the Laplace–Beltrami operator and covariant derivatives ensures that the governing equations maintain geometric consistency, a critical advancement over classical Cartesian approaches, especially in systems with complex curved boundaries or interfaces.

Our analytical and numerical results reveal several key insights. First, curvature significantly modulates both diffusion and convection mechanisms. Positive curvature tends to enhance local concentration gradients due to geodesic contraction effects, whereas negative curvature distributes gradients more broadly. This geometric sensitivity has implications for designing microfluidic devices or biological membranes where surface morphology governs transport efficiency. Second, the coupling between heat and mass transfer is accentuated by curvature-dependent coupling coefficients, reflecting realistic thermodiffusion or Soret effects on manifolds. Traditional flat-domain models overlook these subtle interdependencies, potentially leading to erroneous predictions in scenarios such as catalytic surfaces or porous media embedded in curved geometries. Third, the differential geometry-based numerical framework successfully handles the complex interplay of nonlinear advection, anisotropic diffusion, and source terms without sacrificing stability or accuracy. The incorporation of advanced mesh generation and isogeometric analysis techniques ensures faithful geometry representation, crucial for high-fidelity simulations. These computational strategies open avenues for extending the model to time-dependent and multi-physics problems, such as reactive transport on evolving manifolds.

Despite these advances, challenges remain. The increased computational cost associated with manifold discretization and curvature-dependent operators demands further optimization, potentially via adaptive mesh refinement or machine learning-based surrogate models. Moreover, experimental validation of manifold effects on convective heat and mass transfer is limited, necessitating interdisciplinary efforts combining geometry, fluid mechanics, and material science.

6.2. Conclusions

This paper establishes a novel convective heat and mass transfer model intrinsically grounded in differential geometry, offering a robust theoretical and computational framework for transport phenomena on curved domains. Key conclusions include:

1) The intrinsic geometric formulation using Riemannian metrics and the Laplace–Beltrami operator provides an elegant and physically consistent generalization of classical transport equations to curved manifolds.

2) Curvature exerts a pronounced influence on the local and global behavior of heat and mass transfer, impacting gradient formation, flux distributions, and coupling effects.

3) Advanced numerical methods adapted to manifold geometries, including finite element methods and isogeometric analysis, enable accurate and stable solutions of the nonlinear PDEs system.

4) The model's flexibility supports extension to time-dependent and multiphysics problems, with potential applications ranging from microfluidics and biological transport to porous media and material interfaces.

Future work should focus on integrating dynamic manifold evolution, experimental validation, and computational acceleration to fully harness the capabilities of the differential geometry approach for convective heat and mass transfer.

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Abbreviations

BDF	Backward Differentiation Formulas
DEC	Discrete Exterior Calculus
FEM	Finite Element Methods
FVM	Finite Volume Methods
IGA	Isogeometric Analysis
ODEs	Ordinary Differential Equations
PDEs	Partial Differential Equations

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Author Contributions

Kande Dickson Kinyua: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Writing – original draft

Karimi Kennedy John: Resources, Software, Supervision, Validation, Visualization, Writing – review & editing

Conflicts of Interest

The authors declare no conflicts of interest.

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