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The Modified Kies-Weibull Distribution: A Flexible Model for Survival Analysis

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Abstract: This paper introduces the Modified Kies-Weibull (MKW) distribution, a novel and flexible probability model that generalizes the Weibull distribution to better accommodate various hazard rate structures. The MKW distribution is derived by incorporating the Weibull distribution into the Modified Kies Generalized (MKi-G) family, enhancing its adaptability for reliability analysis and survival modeling. Key statistical properties, including the cumulative distribution function, probability density function, moments, and order statistics, are derived. Three estimation methods: (i) Maximum Likelihood Estimation (ML), (ii) Maximum Product Spacing (MPS), and (iii) Least Squares (LS) are examined and compared through simulation studies. The results demonstrate that LS estimation outperforms ML and MPS, particularly in small samples, exhibiting lower bias and greater stability. Furthermore, the empirical application of the MKW distribution to a bladder cancer remission dataset reveals superior model fit compared to existing Weibull-based models, as confirmed by information criteria and goodness-of-fit tests. The MKW distribution proves to be an effective tool for modeling lifetime data, offering enhanced flexibility for applications in medicine, engineering, and reliability studies.

Keywords: Modified Kies-Weibull Distribution, Survival Analysis, Parameter Estimation, Reliability Modeling, Hazard Rate, Goodness-of-fit

1. Introduction

Probability distributions play a crucial role in modeling real-world phenomena, particularly in fields such as reliability analysis, lifetime data modeling, and survival analysis. Among the various probability distributions available, the Weibull distribution has emerged as one of the most widely used due to its inherent flexibility in capturing a broad range of failure rate behaviors. The probability density function (PDF) of the Weibull distribution is given by:

$$g(x;\alpha,\beta) = \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}, \quad x>0,\, \alpha>0,\, \beta>0$$

where α and β represent the scale and shape parameters, respectively. The corresponding cumulative distribution function (CDF) is:

$$G(x;\alpha,\beta) = 1-e^{-\alpha x^{\beta}}, \quad x>0,\, \alpha>0,\, \beta>0 \qquad (1)$$

Despite its widespread applicability, the Weibull distribution does have certain limitations, particularly in modeling datasets with more complex or varying hazard rate structures. To address these limitations, researchers have sought to introduce modifications and generalizations to enhance the flexibility of the Weibull model. For a comprehensive overview of the modified Weibull model, interested readers may refer to Lai et al. (2023) [1].

To further improve upon these limitations, Al-Babtain et al. [8] introduced a new generalized family of distributions known as the Modified Kies Generalized (MKi-G) family. The details about the Kies distribution can be found in Kumar and Dharmaja (2013, 2017) [2, 3]. Shakhatreh and Al-Babtain (2021) applies the modified Kies distribution to reliability data, which is often used in engineering contexts, highlighting its flexibility [4]. Kumar, D., & Nassar, M. (2017) provide a generalized modified Kies distribution and explores its properties and estimation techniques [5]. The moments and estimation of the reduced Kies distribution based on progressive type-II right-censored order statistics are discussed in Dey, Nassar, and Kumar (2019) [6]. Ferreira and Cordeiro (2024) develop the MKi-G family, derive its properties, and demonstrate its flexibility in modeling complex data [7]. The CDF of the MKi-G family is expressed as:

$$F(x;\gamma,\psi) = 1 - \exp\left\{-\left[\frac{G(x;\psi)}{1-G(x;\psi)}\right]^{\gamma}\right\}, \quad x>0,\, \gamma>0 \tag{2}$$

where $G(x;\psi)$ represents the baseline CDF with parameter vector ψ. The corresponding probability density function (PDF) and hazard rate function (HRF) of the MKi-G family are given by:

$$f(x;\gamma,\psi) = \frac{\gamma g(x;\psi)[G(x;\psi)]^{\gamma-1}}{[1-G(x;\psi)]^{\gamma+1}} \exp\left\{-\left[\frac{G(x;\psi)}{1-G(x;\psi)}\right]^{\gamma}\right\} \quad \text{for} \quad x>0,\, \gamma>0 \tag{3}$$

$$h(x;\gamma,\psi) = \frac{\gamma g(x;\psi)[G(x;\psi)]^{\gamma-1}}{[1-G(x;\psi)]^{\gamma+1}}, \quad x>0,\, \gamma>0$$

This paper introduces the Modified Kies-Weibull (MKW) distribution, a new extension within the MKi-G family. By combining the Weibull model with the MKi-G family, the MKW distribution offers enhanced flexibility in capturing a diverse range of hazard rate and probability density behaviors, making it suitable for various reliability and survival analysis applications.

The remainder of this paper is organized as follows: Section 2 introduces the MKW distribution and its formulation. Section 3 presents its key statistical properties, including moments and order statistics. Section 4 discusses several parameter estimation methods. In Section 5, a simulation study compares these estimation methods. Section 6 demonstrates the application of the MKW distribution to a real dataset, highlighting its practical utility. Finally, Section 7 concludes the paper and suggests directions for future research.

2. The Modified Kies-Weibull (MKW) Distribution

In this section, we introduce the Modified Kies-Weibull (MKW) distribution, which is derived by incorporating the Weibull distribution into the Modified Kies Generalized (MKi-G) family. This formulation enhances the flexibility of the standard Weibull model, enabling it to capture a wider range of behaviors in the probability density and hazard rate functions.

2.1. Cumulative and Probability Density Functions

The cumulative distribution function (CDF) of the MKW distribution is obtained by substituting the CDF of the Weibull distribution, given by Equation (1), into the general form of the MKi-G family in Equation (2). This leads to the following CDF.

$$F(x;\alpha,\beta,\gamma) = 1 - e^{-(e^{\alpha x^{\beta}}-1)^{\gamma}}, \quad x>0,\, \alpha,\beta,\gamma>0 \tag{4}$$

Similarly, the probability density function (PDF) of the MKW distribution can be derived by inserting Equations (1) and (1) into Equation (3), resulting in:

$$f(x;\alpha,\beta,\gamma) = \gamma\alpha\beta x^{\beta-1} e^{\gamma\alpha x^{\beta}-(e^{\alpha x^{\beta}}-1)^{\gamma}} (1-e^{-\alpha x^{\beta}})^{\gamma-1} \qquad x>0,\, \alpha,\beta,\gamma>0 \tag{5}$$

The distribution is characterized by the parameters α, β, and γ. The parameter α controls the scaling or stretching of the distribution, while β and γ determine the shape of the distribution. Symbolically, we denote the MKW distribution as $X \sim \text{MKW}(\gamma,\alpha,\beta)$. Notably, by setting $\beta = 2$ in Equation (1), the MKW model reduces to the modified Kies Rayleigh distribution.

Figure 1 displays plots of the PDF of the MKW distribution for various parameter values. From these plots, we observe that the PDF can exhibit symmetric, unimodal, right-skewed, and left-skewed behaviors, depending on the selected values of the parameters.

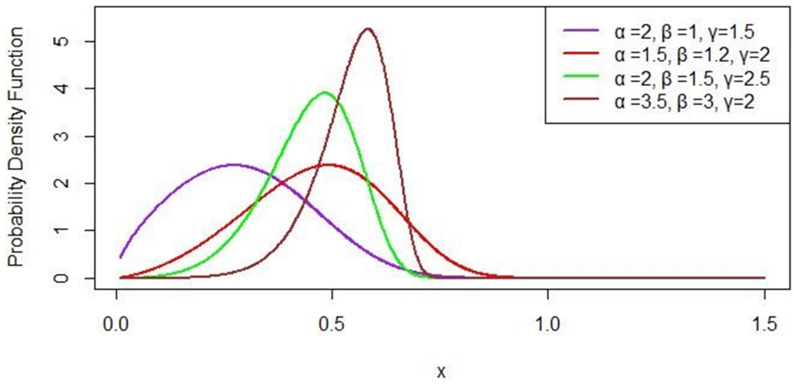

Figure 1. *Density curves of the MKW distribution for selected values of the parameters.*

2.2. Linear Presentation

The CDF of the MKW distribution, as given by Equation (4), can also be expressed as a series expansion:

$$F(x;\alpha,\beta,\gamma) = 1 - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(e^{\alpha x^\beta} - 1)^{\gamma n} \tag{6}$$

We can also write

$$\begin{aligned} F(x;\alpha,\beta,\gamma) &= 1 - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} e^{\gamma n \alpha x^\beta}(1 - e^{-\alpha x^\beta})^{\gamma n} \\ &= 1 - \sum_{k,n=0}^{\infty} \frac{(-1)^{n+k}(\gamma n)_k}{n!k!} e^{-[\alpha(k-\gamma n)x^\beta]} \end{aligned} \tag{7}$$

where $(\gamma n)_k = (\gamma n)(\gamma n - 1)\cdots(\gamma n - k + 1)$.

Differentiating Equation (7), we obtain the PDF as:

$$f(x;\alpha,\beta,\gamma) = \alpha(k-\gamma n)\beta x^{\beta-1} \sum_{k,n=0}^{\infty} \frac{(-1)^{n+k}(\gamma n)_k}{n!k!} \times e^{-[\alpha(k-\gamma n)x^\beta]}$$

which can be rewritten as:

$$f(x;\alpha,\beta,\gamma) = \sum_{k,n=0}^{\infty} \phi_{k,n} g_{\alpha(k-\gamma n),\beta}(x) \tag{8}$$

where $g(\cdot)$ is the Weibull density function with scale parameter $\alpha(k-\gamma n)$ and shape parameter β.

2.3. Reliability Analysis

In reliability analysis, the survival function (SF) and hazard function (HF) are essential for understanding the behavior of a system over time. For the MKW distribution, the survival function is given by:

$$S(x;\alpha,\beta,\gamma) = e^{-(e^{\alpha x^\beta}-1)^\gamma}, \quad x > 0,\ \alpha,\beta,\gamma > 0$$

The hazard function (HF) is expressed as:

$$h(x;\alpha,\beta,\gamma) = \gamma\alpha\beta x^{\beta-1} e^{\gamma\alpha x^\beta}(1 - e^{-\alpha x^\beta})^{\gamma-1} \quad x > 0,\ \alpha,\beta,\gamma > 0$$

The cumulative hazard function (CHF) of the MKW distribution is given by:

$$H(x;\alpha,\beta,\gamma) = -\ln S(x) = -\ln\left(e^{-(e^{\alpha x^\beta}-1)^\gamma}\right) = (e^{\alpha x^\beta} - 1)^\gamma.$$

Figure 2 presents plots of the hazard function for the MKW distribution with different parameter values. The plot illustrates how the hazard function can accommodate increasing, decreasing, or constant failure rates, highlighting the remarkable flexibility of the MKW distribution.

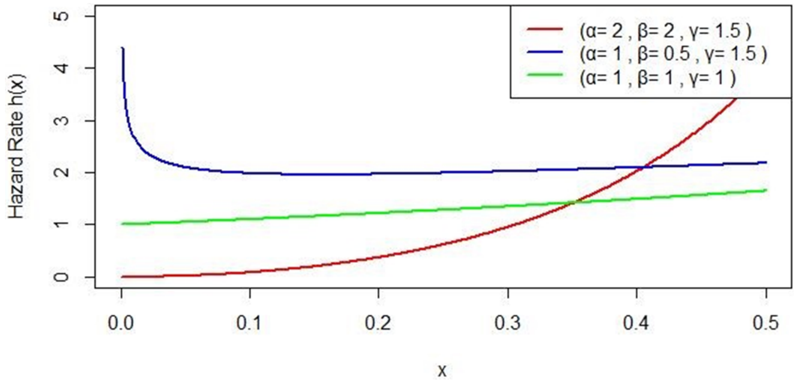

Figure 2. *Hazard rate of the MKW distribution with different parameters.*

3. Properties

In this section, the mathematical properties of the MKW distribution are derived, including quantiles, moments, and order statistics.

3.1. Quantiles

The p-th quantile of the MKW distribution, found as the solution of $F(x_p) = p$, is:

$$Q(p) = \left[\frac{\log\left[(-\log(1-p))^{1/\gamma}+1\right]}{\alpha}\right]^{1/\beta}, \quad 0 < p < 1 \tag{9}$$

The first, second, and third quartiles of the MKW distribution are obtained by setting $p = 0.25$, $p = 0.5$, and $p = 0.75$, respectively, in Eq. (9).

3.2. Skewness and Kurtosis

The Bowley skewness (BS) [9] and Moor's kurtosis (MK) [10] are given as follows:

$$\text{BS} = \frac{Q(0.75) - 2Q(0.5) + Q(0.25)}{Q(0.75) - Q(0.25)},$$

and

$$\text{MK} = \frac{Q(0.875) - Q(0.625) + Q(0.375) - Q(0.125)}{Q(0.75) - Q(0.25)},$$

where $Q(p)$ is the quantile function of the three parameters MKW(γ, α, β).

3.3. Generating Function

The moment generating function is derived as:

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_0^\infty e^{tx} f(x)\, dx \tag{10}$$

Using the series representation $e^{tx} = \sum_{m=0}^{\infty} \frac{(tx)^m}{m!}$, Eq. (10) can be rewritten as:

$$
\begin{aligned}
M_X(t) &= \int_0^\infty \sum_{m=0}^{\infty} \frac{(tx)^m}{m!} f(x)\,dx = \sum_{m=0}^{\infty} \frac{t^m}{m!} \int_0^\infty x^m f(x)\,dx \\
&= \sum_{m=0}^{\infty} \sum_{k,n=0}^{\infty} \frac{(-1)^{n+k}(\gamma n)_k}{n!k!} \frac{\alpha(k-\gamma n)\beta t^m}{m!} \times \int_0^\infty x^{m+\beta-1} e^{-\alpha(k-\gamma n)x^\beta}\,dx
\end{aligned} \tag{11}
$$

Using the identity:

$$
\int_0^\infty x^{p-1} e^{-\lambda x^q}\,dx = \frac{\Gamma(p/q)}{q\lambda^{p/q}},
$$

Eq. (11) becomes:

$$
M_X(t) = \sum_{m=0}^{\infty} \sum_{k,n=0}^{\infty} \frac{(-1)^{n+k}(\gamma n)_k}{n!k!} \frac{\Gamma\left(\frac{m+\beta}{\beta}\right)}{m![\alpha(k-\gamma n)\beta]^{m/\beta}} t^m
$$

3.4. Distribution of Order Statistics

Let $X_1, X_2, \ldots, X_n$ be an i.i.d. random sample of size n from the MKW distribution, with CDF and PDF given by Eqs. (4) and (5), respectively. Let $\{X_{(1)}, X_{(2)}, \ldots, X_{(n)}\}$ be the corresponding order statistics. The PDF of the r-th order statistic, $X_{(r)}$, is given by:

$$
f_{X_{(r)}}(x) = \frac{1}{B(r, n-r+1)} f(x) \left[F(x)\right]^{r-1} \left[1 - F(x)\right]^{n-r} \tag{12}
$$

where $B(\cdot,\cdot)$ is the Beta function. Using binomial expansion in Eq. (12), we get:

$$
f_{X_{(r)}}(x) = \frac{f(x)}{B(r, n-r+1)} \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} \left[F(x)\right]^{j+r-1}
$$

After applying Eqs. (7) and (8), the PDF of the r-th order statistic becomes:

$$
\begin{aligned}
f_{X_{(r)}}(x) = \frac{1}{B(r, n-r+1)} \alpha(k-\gamma n)\beta x^{\beta-1} \times \sum_{j=0}^{n-r} \sum_{k,n=0}^{\infty} \frac{(-1)^j \binom{n-r}{j} (-1)^{n+k}(\gamma n)_k}{n!k!} e^{-\alpha(k-\gamma n)x^\beta} \\
\times \left[1 - \sum_{k,n=0}^{\infty} \frac{(-1)^{n+k}(\gamma n)_k}{n!k!} e^{-\alpha(k-\gamma n)x^\beta}\right]^{j+r-1}
\end{aligned}
$$

4. Method of Estimation

4.1. Maximum Likelihood Estimation

Let the observation be $\{x_1, x_2, \ldots, x_n\}$. The likelihood function is

$$
\begin{aligned}
L(\alpha, \beta, \gamma \mid x_1, x_2, \ldots, x_n) &= \prod_{i=1}^{n} \gamma\alpha\beta x_i^{\beta-1} e^{\gamma\alpha x_i^\beta - (e^{\alpha x_i^\beta} - 1)^\gamma} \times \left(1 - e^{-\alpha x_i^\beta}\right)^{\gamma-1} \\
&= (\gamma\alpha\beta)^n \prod_{i=1}^{n} x_i^{\beta-1} e^{\gamma\alpha x_i^\beta - (e^{\alpha x_i^\beta} - 1)^\gamma} \left(1 - e^{-\alpha x_i^\beta}\right)^{\gamma-1}
\end{aligned}
$$

Since products of terms can be cumbersome to compute, the log-likelihood is often used instead:

$$l(\alpha, \beta, \gamma \mid x_1, x_2, \ldots, x_n) = n \log(\alpha\beta\gamma) + (\beta - 1) \times \sum_{i=1}^{n} \log(x_i) + \gamma\alpha \sum_{i=1}^{n} x_i^{\beta} - \sum_{i=1}^{n} \left(e^{\alpha x_i^{\beta}} - 1\right)^{\gamma} + (\gamma - 1) \sum_{i=1}^{n} \log\left(1 - e^{-\alpha x_i^{\beta}}\right)$$

The maximum likelihood (ML) estimators of parameters (α, β, γ) can be obtained by maximizing the above log-likelihood function. This can be done by solving the following normal equations simultaneously:

$$\frac{n}{\alpha} + \gamma \sum_{i=1}^{n} x_i^{\beta} - \gamma \sum_{i=1}^{n} \left(e^{\alpha x_i^{\beta}} - 1\right)^{\gamma - 1} x_i^{\beta} e^{\alpha x_i^{\beta}} + (\gamma - 1) \sum_{i=1}^{n} \frac{x_i^{\beta} e^{-\alpha x_i^{\beta}}}{1 - e^{-\alpha x_i^{\beta}}} = 0$$

$$\frac{n}{\beta} + \sum_{i=1}^{n} \log(x_i) + \gamma\alpha \sum_{i=1}^{n} x_i^{\beta} \log(x_i) - \gamma \sum_{i=1}^{n} \left(e^{\alpha x_i^{\beta}} - 1\right)^{\gamma - 1} \times e^{\alpha x_i^{\beta}} \alpha x_i^{\beta} \log(x_i) - (\gamma - 1) \sum_{i=1}^{n} \frac{\alpha x_i^{\beta} \log(x_i) e^{-\alpha x_i^{\beta}}}{1 - e^{-\alpha x_i^{\beta}}} = 0$$

$$\frac{n}{\gamma} + \alpha \sum_{i=1}^{n} x_i^{\beta} - \sum_{i=1}^{n} \left(e^{\alpha x_i^{\beta}} - 1\right)^{\gamma} \log\left(e^{\alpha x_i^{\beta}} - 1\right) + \sum_{i=1}^{n} \log\left(1 - e^{-\alpha x_i^{\beta}}\right) = 0$$

4.2. Maximum Product of Spacing Estimation

Let $x_{(1:n)}, x_{(2:n)}, \ldots, x_{(n:n)}$ be the order statistics of a random sample of size n from the MKW distribution, resulting in

$$0 \equiv F(x_{(0:n)}, \theta) \leq F(x_{(1:n)}, \theta) \leq \cdots \leq F(x_{(n+1:n)}, \theta) \equiv 1.$$

The spacings are defined as follows:

$$D_1 = F(x_{(1:n)}, \theta), \quad D_{(n+1)} = 1 - F(x_{(n:n)}, \theta),$$

and

$$D_i = F(x_{(i:n)}, \theta) - F(x_{(i-1:n)}, \theta), \quad i = 2, 3, \ldots, n.$$

The maximum product of spacing (MPS) method is to choose θ which maximizes the geometric mean of the spacings, i.e.

$$G = \left(\prod_{i=1}^{n+1} D_i\right)^{\frac{1}{n+1}} \quad \text{or equivalently,} \quad S = \log G.$$

Cheng and Amin [11] examined that maximizing S as a method of parameter estimation is as efficient as ML estimation.

The CDF of the MKW distribution is given by equation (4), and the spacings are defined as:

$$D_1 = 1 - e^{-(e^{\alpha x_{(1:n)}^{\beta}} - 1)^{\gamma}},$$

$$D_{(n+1)} = e^{-(e^{\alpha x_{(n:n)}^{\beta}} - 1)^{\gamma}},$$

and in general,

$$D_i = e^{-(e^{\alpha x_{(i-1:n)}^{\beta}} - 1)^{\gamma}} - e^{-(e^{\alpha x_{(i:n)}^{\beta}} - 1)^{\gamma}}.$$

Therefore,

$$S(x; \alpha, \beta, \gamma) = \log G = \frac{1}{n+1} \sum_{i=1}^{n+1} \log(D_i)$$

Under the MPS method, $\theta = (\alpha, \beta, \gamma)$ is chosen which maximizes the above expression.

4.3. Least-Squares Estimation

Let $x_{(1:n)}, x_{(2:n)}, \ldots, x_{(n:n)}$ be the order statistics of a random sample of size n from the MKW distribution. The least-squares (LS) estimators of the unknown parameters α, β, and γ can be obtained by minimizing:

$$\sum_{i=1}^{n}\left[F(x_{(i:n)};\alpha,\beta,\gamma)-\frac{i}{n+1}\right]^2=\sum_{i=1}^{n}\left[1-e^{-(e^{\alpha x_{(i:n)}^{\beta}}-1)^{\gamma}}-\frac{i}{n+1}\right]^2.$$

Consequently, least-squares estimates of the parameters can be obtained by solving the following equations:

$$\sum_{i=1}^{n}\left[F(x_{(i:n)};\alpha,\beta,\gamma)-\frac{i}{n+1}\right]\frac{\partial F(x_{(i:n)};\alpha,\beta,\gamma)}{\partial\alpha}=0$$

$$\sum_{i=1}^{n}\left[F(x_{(i:n)};\alpha,\beta,\gamma)-\frac{i}{n+1}\right]\frac{\partial F(x_{(i:n)};\alpha,\beta,\gamma)}{\partial\beta}=0$$

$$\sum_{i=1}^{n}\left[F(x_{(i:n)};\alpha,\beta,\gamma)-\frac{i}{n+1}\right]\frac{\partial F(x_{(i:n)};\alpha,\beta,\gamma)}{\partial\gamma}=0$$

5. Simulation Study

This section evaluates the performance of three parameter estimation methods: *Maximum Likelihood Estimation (ML)*, *Maximum Product of Spacing (MPS)*, and *Least Squares (LS)* for the *Modified Kies-Weibull (MKW) distribution* using a simulation study.

5.1. Simulation Setup

1. *Data Generation*: Random samples of sizes $30, 70$, and 100 are generated from the MKW distribution using the acceptance-rejection algorithm, with parameter values:

$$\gamma=(1,2),\quad \alpha=(1.5,2.5),\quad \beta=(1.2,2.0).$$

2. *Estimation Methods*: The parameters are estimated using ML, MPS, and LS estimators.
3. *Evaluation Metrics*: The accuracy of estimators is measured using *Absolute Bias (AB)* and *Root Mean Squared Error (RMSE)* over $N = 1000$ replications:

$$AB(\widehat{\theta})=\frac{1}{N}\sum_{i=1}^{N}\left|\widehat{\theta}_i-\theta\right|$$

$$RMSE(\widehat{\theta})=\sqrt{\frac{1}{N}\sum_{i=1}^{N}(\widehat{\theta}_i-\theta)^2}$$

where $\widehat{\theta}$ is an estimator of the parameter $\theta=(\gamma,\alpha,\beta)$.

5.2. Results and Discussion

The simulation results are presented in Tables 1 and 2 for different settings.

1. *Effect of Sample Size*: Both AB and RMSE decrease as sample size increases, confirming the asymptotic unbiasedness of all estimators.
2. *Estimator Performance*:
   (a) LS consistently achieves the lowest bias and RMSE, making it the most accurate and stable method.
   (b) ML and MPS show higher bias and variability, particularly in small samples.
3. *Shape Parameter γ Estimation*:
   (a) ML and MPS tend to overestimate γ significantly, especially for small samples.
   (b) LS provides the most stable and reliable estimates for all parameters.
4. *Overall Ranking*: LS > MPS > ML in terms of accuracy and efficiency.

Table 1. *Absolute Biases (AB) and Root Mean Squared Errors (RMSEs) (in parentheses) of Modified Kies-Weibull distribution with parameters $\alpha = 2, \beta = 1, \gamma = 1.5$ and $\alpha = 1.5, \beta = 1.2, \gamma = 2$*

Sample size (n)	Method	($\alpha = 2, \beta = 1, \gamma = 1.5$)			($\alpha = 1.5, \beta = 1.2, \gamma = 2$)		
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$
30	ML	646.7929	3.3455	38.7966	88.0745	3.153	92.6788
		(12481.9443)	(4.7063)	(140.7324)	(905.9236)	(5.0622)	(230.9317)
	MPS	543.6722	2.7702	25.8403	49.3575	2.1898	72.7211
		(8372.2135)	(4.3694)	(96.7681)	(559.6321)	(4.0062)	(197.7159)
	LS	1.9947	1.4261	1.5713	1.4869	1.7376	1.9229
		(1.9247)	(1.5252)	(3.8522)	(1.4873)	(2.0499)	(1.9392)
70	ML	17.0471	3.1135	12.7246	4.0119	2.2081	54.0954
		(299.6278)	(4.3128)	(57.2553)	(9.9915)	(3.4934)	(141.3113)
	MPS	5.4384	2.803	10.0376	2.415	1.5906	34.6803
		(24.4573)	(4.2421)	(46.7122)	(7.9291)	(2.7592)	(92.3552)
	LS	1.9912	1.3493	1.4384	1.4092	1.5764	2.0297
		(-1.9914)	(-1.4272)	(-1.4395)	(-1.0946)	(-1.6976)	(-1.6579)
100	ML	4.0663	3.2382	11.2188	2.5515	2.1585	42.5672
		(13.1316)	(4.429)	(60.5051)	(5.6058)	(3.5281)	(118.0308)
	MPS	2.7179	2.9046	7.0839	1.6839	1.6026	26.3903
		(8.9166)	(4.2781)	(35.252)	(3.8049)	(2.823)	(75.5805)
	LS	1.9902	1.3128	1.4896	1.0914	1.61129	1.9371
		(1.0943)	(1.0785)	(1.1251)	(1.0416)	(1.6072)	(1.6381)

Table 2. *Absolute Biases (AB) and Root Mean Squared Errors (RMSEs) (in parentheses) of Modified Kies-Weibull distribution with parameters $\alpha = 2, \beta = 1, \gamma = 1.5$ and $\alpha = 1.5, \beta = 1.2, \gamma = 2$.*

Sample size (n)	Method	($\alpha = 3.5, \beta = 3, \gamma = 2$)			($\alpha = 2, \beta = 1.5, \gamma = 2.5$)		
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$
30	ML	10000	5	91.8813	918813	4.5741	97.8446
		(2500)	(10)	(19.533)	(195331)	(7.2719)	(2.5376)
	MPS	800	4.5	135.448	135448	3.2134	74.9865
		(200)	(9)	(257.47)	(257475)	(5.6476)	(96.33)
	LS	3.4858	3.82787	2.06632	1.9923	2.1759	2.5806
		(3.4861)	(4.2834)	(4.9293)	(1.9924)	(2.6516)	(4.4101)
70	ML	6092.95	4.7717	105.692	2196.47	2.9759	73.8589
		(1573.9)	(8.0648)	(226.12)	(66526)	(4.5026)	(193.48)
	MPS	464.278	4.0184	80.0529	453.697	3.0268	42.3047
		(100.12)	(7.2881)	(169.54)	(13505)	(4.67729)	(125.992)
	LS	3.4006	3.6694	1.9243	1.9889	2.0513	2.4602
		(3.4207)	(3.8063)	(1.9263)	(1.9891)	(2.1996)	(2.8035)
100	ML	34.7895	4.3558	103.453	27.211	2.7157	61.5827
		(159)	(7.4909)	(207)	(113.76)	(4.0786)	(168.06)
	MPS	17.6385	3.7914	72.9889	12.4487	2.9536	35.2496
		(95.095)	(6.778)	(156.702)	(51.801)	(4.5601)	(98.7)
	LS	3.0923	3.5696	1.9133	1.9857	1.9769	2.4346
		(3.0424)	(3.6855)	(1.9035)	(1.9862)	(2.1061)	(2.4356)

6. Real Data Application

To evaluate the practical performance of the Modified Kies-Weibull (MKW) distribution, we apply it to a real-world clinical dataset that records the remission durations (in months) of 118 bladder cancer patients, originally published by Lee and Wang (2003)[12]. The dataset consists of observed remission times, which exhibit significant variability, making it an ideal candidate for testing the flexibility of the MKW distribution. The dataset includes the following remission times:

Table 3. *Data of remission times in months for bladder cancer patients.*

4.5	32.15	3.88	13.8	19.13	4.87	5.85	14.24	5.71	7.09
7.87	7.59	20.28	3.02	46.12	4.51	5.17	2.83	9.22	1.05
0.2	8.37	3.82	9.47	36.66	14.77	26.31	79.05	10.06	8.53
4.98	11.98	2.62	4.26	5.06	1.76	0.9	11.25	16.62	4.4
21.73	10.34	12.07	34.26	10.66	6.97	2.07	0.51	12.03	0.08
17.12	2.64	1.4	12.63	43.01	14.76	2.75	7.66	0.81	1.19
7.32	4.18	3.36	8.66	1.26	13.29	1.46	14.83	6.76	23.63
5.62	3.25	18.1	7.62	7.63	17.14	25.74	3.52	2.87	15.96
17.36	9.74	3.31	7.28	1.35	0.4	2.26	4.33	9.02	22.69
6.94	2.54	11.79	2.46	7.26	5.34	3.48	8.26	6.93	4.23
3.7	0.5	10.75	6.54	3.64	13.11	8.65	3.57	5.09	7.39
11.64	2.09	2.23	6.25	7.93	4.34	25.82	12.02		

To assess the effectiveness of the MKW model, we compare it against several well-established probability distributions used in survival analysis:

1. Weibull (W) distribution (Weibull, 1951)[13]
2. Exponentiated Exponential-Weibull (EE-W) distribution (Dawlah Al-Sulami, 2020)[14]
3. Modified Weibull (MW) distribution (Lai et al., 2003)[1]
4. Inverse Weibull (IW) distribution (Akgül et al., 2016)[15]

6.1. Parameter Estimation and Model Comparison

The parameters of each distribution were estimated using the Maximum Likelihood Estimation (MLE) method. Table 4 presents the maximum likelihood estimates (MLEs) and their corresponding standard errors (SEs) for each model. Notably, the MKW distribution exhibits smaller standard errors compared to alternative models, indicating greater precision and reliability in its parameter estimates.

Table 4. *Maximum likelihood estimates of fitted distributions for data of remission time in month of bladder cancer patients.*

Distributions	$\widehat{\alpha}$	$\widehat{\lambda}$	$\widehat{\beta}$	$\widehat{\gamma}$
W	10.0192 (0.9315)	-	1.0462 (0.0709)	-
EE-W	0.2658 (2.1218)	0.6801 (7.6341)	2.2812 (1.0190)	0.7075 (0.1516)
MW	0.6122 (0.0666)	-	0.0537 (0.0468)	0.1397 (0.1221)
IW	2.4029 (0.2254)	-	0.7302 (0.0431)	-
MKW	0.0717 (0.0177)	-	1.2142 (0.1037)	0.0109 (0.0041)

To determine the best-fitting model, we evaluate the distributions using widely recognized model selection criteria:

1. Akaike Information Criterion (AIC)
2. Consistent Akaike Information Criterion (CAIC)
3. Bayesian Information Criterion (BIC)
4. Hannan-Quinn Information Criterion (HQIC)

Additionally, we conduct goodness-of-fit tests, including:

1. Anderson-Darling (AD) test
2. Cramér-von Mises (CVM) test
3. Kolmogorov-Smirnov (KS) test, along with its corresponding p-value

Table 5 summarizes the results of these statistical measures.

Table 5. *Goodness of fit criteria and tests for fitted distributions.*

Distribution	AIC	CAIC	BIC	HQIC	-log(likelihood)	AD	CVM	KS (p-value)
W	778.747	778.852	784.289	780.997	387.373	0.53170	0.08644	0.0593 (0.7995)
EE-W	778.462	778.816	789.544	782.962	385.231	0.25999	0.04079	0.0428 (0.9818)
MW	781.756	781.966	790.068	785.131	387.878	0.58723	0.09566	0.0618 (0.7575)
IW	839.035	839.139	844.576	841.285	417.517	4.61809	0.77540	0.1471 (0.0120)
MKW	776.297	776.507	784.609	779.671	385.148	0.20813	0.03391	0.0392 (0.9934)

6.2. Findings and Interpretation

The results in Table 4 clearly demonstrate that the MKW distribution outperforms all competing models. Specifically, the MKW model achieves the lowest AIC, BIC, CAIC, and HQIC values, indicating the best balance between goodness-of-fit and model complexity. Furthermore, the smallest AD, CVM, and KS statistics, along with high p-values, confirm that the MKW distribution provides the most suitable fit for the bladder cancer remission data.

To further validate this conclusion, Figure 3 presents a graphical comparison of the observed and fitted cumulative distribution functions (CDFs) for all models. The MKW distribution exhibits the closest agreement with the empirical data, reinforcing its superior flexibility in capturing survival trends.

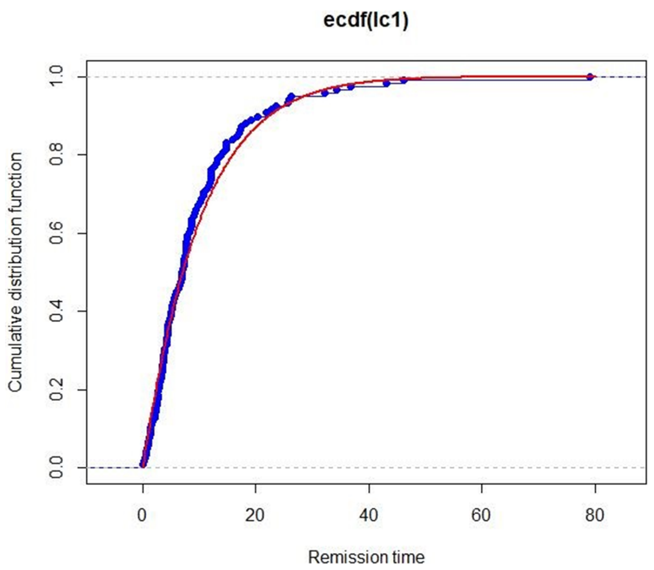

Figure 3. Graphical comparison of observed and fitted cumulative distribution functions of remission time for bladder cancer patients

6.3. Practical Implications

The findings of this study highlight the practical significance of the MKW distribution in survival analysis. Compared to traditional Weibull-based models, the MKW distribution offers:

1. Enhanced flexibility in capturing varying hazard rate behaviors (increasing, decreasing, constant).
2. Greater accuracy in modeling real-world lifetime data, as demonstrated by its superior fit to cancer remission times.
3. Reliable parameter estimation, with lower standard errors and reduced estimation bias.

Given these advantages, the MKW distribution can be a valuable tool for medical research, reliability engineering, and risk assessment. Future studies may explore its application to other survival datasets, such as disease progression, failure-time analysis, or reliability modeling in engineering systems.

7. Conclusion

This paper introduced the Modified Kies-Weibull (MKW) distribution, a new probability model that extends the flexibility of the traditional Weibull distribution. By incorporating the Weibull model into the Modified Kies Generalized (MKi-G) family, the MKW distribution is capable of capturing a wide range of probability density and hazard rate functions, making it suitable for various reliability and survival analysis applications.

Theoretical properties of the MKW distribution, including moments, quantiles, skewness, kurtosis, and order statistics, were derived, providing a strong mathematical foundation for its use. Three estimation methods: (i) Maximum Likelihood (ML), (ii) Maximum Product Spacing (MPS), and (iii) Least Squares (LS) were applied and evaluated through eetxtensive simulation studies. The results demonstrated that LS estimation consistently outperformed ML and MPS, particularly for small sample sizes, due to its lower bias and superior accuracy.

To assess its practical applicability, the MKW distribution was fitted to a bladder cancer remission dataset and compared against several established models, including the Weibull, Exponentiated Exponential Weibull (EE-W), Modified Weibull (MW), and Inverse Weibull (IW) distributions. The MKW model provided the best fit, as indicated by its superior performance in AIC, BIC, CAIC, HQIC, and goodness-of-fit tests (AD, CVM, KS). These findings reinforce its potential as a powerful model for lifetime and survival data analysis.

In conclusion, the MKW distribution is a highly flexible and robust model, capable of accurately describing diverse datasets with varying failure rate structures. Future research could explore Bayesian estimation techniques, extensions of the MKW model, or its applications in additional fields such as finance, engineering, and epidemiology.

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The first author was responsible for conducting the research, performing the theoretical derivation, conducting the simulation study, analyzing the data, and drafting the initial manuscript. The second author critically reviewed the manuscript and contributed to revising the final article. Both authors have read and approved the final version of the manuscript.

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Conflicts of Interest

The authors declare no conflicts of interest.

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