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Research Article

Relationship Between Water Surface Elevation and Mean Water Depth for a Partially Wetted Rectangular Grid Cell in Numerical Simulations of Shallow Water Flows with Complex Topography

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Abstract

We propose a relationship between the water surface elevation and the mean water depth for partially wetted rectangular grid cells in the numerical simulation of shallow water flow with complex topography, and apply it to the numerical simulation of shallow water flow to verify the correctness of the relationship. Many natural terrains have complicated surface topography. It is very important to accurately predict the steep-fronted flows that occur after heavy rainfall flash floods or as inundation from dyke breaches. When modeling any terrain, rectangular grid cells are used to facilitate grid generation. In order to achieve a numerical balance of flux gradient and the source terms and to avoid numerical instability, a relationship between the water surface elevation and the mean water depth was derived if wetted and dry parts coexist within a rectangular grid cell. To estimate the momentum flux at the grid cell boundaries, we applied the second-order spatial accuracy Godunov finite volume algorithm with Roe approximation solver using the MUSCL method. Next, these relationships are applied to numerical simulations of three-dimensional shallow water flows with three humps and the validity of the proposed relationship is discussed. The relationship presented in this paper accurately reflects time-varying water regimes and boundaries in flow problems with moving wetting and drying zone interfaces and can be used for numerical simulations of three dimensional shallow water flows with arbitrary topography.

Keywords

Shallow Water, Godunov Finite Volume Method, Algorithm

1. Introduction

Nowadays, numerical simulation studies of shallow water flows have been widely used to predict global flows such as tidal currents, tsunami phenomena and surge waves, and river flooding due to dam breakage, and their practical value has been confirmed [1-3, 6, 8].

One difficulty in numerical simulation of shallow water equations is the problem of achieving a numerical balance between the source term and flux term. Due to disagreement arising from the discretization method between the numerical calculation of the bottom gradient term, one of the source

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terms, and the numerical calculation of the pressure gradient component of the flux term, is not well balanced between these two terms. The error due to this imbalance is present with the time derivative term, causing pseudo-numerical flow, and the accumulation of them during the numerical simulation results in instability during the simulation [4, 11-16, 19, 20].

Studies have been conducted to achieve the numerical balance of the bottom gradient source term and the flux gradient term. Bermudez proposed a new scheme based on the upwind method for unsteady shallow water flow problems to deal with the bottom gradient terms. This method greatly improves the accuracy of the numerical solution, but the main drawback is its complexity, which makes it difficult to use for large-scale flow calculations [9, 17]. Leveque proposed a treatment of bottom gradient terms to balance the source and flux gradient terms, which is suitable for slow steady flow but not sufficient for steady supercritical flows with shock waves [18].

One method that is widely used for balancing the bottom gradient source term and the flux gradient term is to replace the water depth variable h with the water level variable η in the shallow water equation system and to achieve a numerical balance using the formula $h=\eta-z_b$ between the water elevation variable and the bed elevation variable z_b. In this method, η becomes constant if water is still at rest even on the slope bottom, so that the slope of η is zero regardless of the bottom topography, and thus does not cause false flow. However, because this method uses the relationship between water surface elevation and bottom elevation, $\eta=h+z_b$, it is difficult to apply in cases of water inflow or runoff in partially wet areas where wet and dry areas coexist, and once applied, a negative water depth and water elevation are obtained, causing the instability of the simulation.

Begnudelli and Sanders [10] proposed a finite volume algorithm for unstructured grids using a relationship of water surface elevation and average water depth near the wet/dry interface. This method can greatly reduce the computational time because, unlike in dynamic or adaptive quad tree models, the mesh can be fixed without adding more mesh elements during the computation. Also, in the partially wetting region where wet and dry regions coexist, it does not cause pseudo-flow in almost all shallow water flows, as it can provide a numerical balance between source and flux terms. However, if we use unstructured mesh, the convergence of the solution is worse compared to structured mesh, which requires more simulation time.

Other methods use moving computational mesh with wet/dry area interface [5, 7]. These methods are relatively precise and accurate, but computationally expensive and do not suffice to simulate flow in any terrain. The reason is that each time the boundary moves, the mesh must be regenerated, and often the computational nodes must be added during the flooding and drained during the flooding, thus reducing the error of the mesh distortion.

In this paper, we propose a new relationship between water surface elevation and average water depth in partially wetted rectangular cells in a finite volume scheme for a structural mesh, and apply it to simulation of shallow water flow with complex topography to verify its accuracy. This paper is composed of the following 4 sections. In Section 2, basic equations of shallow water flow are presented. In Section 3, relationships between water surface elevation and mean water depth for partially wetted rectangular cells are derived. In Section 4, the simulation results are discussed.

2. Basic Equations of Shallow Water Flow

For shallow water flow, the vertical velocity and acceleration components of the fluid particle are negligible compared to the corresponding components in the horizontal direction. Thus, the assumption that the influence of internal viscous forces on the flow with hydrostatic pressure distribution is negligible is applied to the three-dimensional incompressible N-S equation, and that the integration along the depth (vertical) is carried out, and then the governing equations for shallow water flow are obtained considering the influence of external factors such as the experimentally determined bed friction stress, the wind friction stress acting on the free surface, and the Coriolis force. The integral form of the two-dimensional conservative shallow water equation is described as follows.

$$\frac{\partial}{\partial t}\int_{\Omega} \mathrm{U}dV + \int_{\Omega}\left(\frac{\partial \mathrm{F}}{\partial x}+\frac{\partial \mathrm{G}}{\partial y}\right)dV = \int_{\Omega} \mathrm{S}dV \tag{1}$$

$$\mathrm{U}=\begin{pmatrix} h \\ hu \\ hv \end{pmatrix},\quad F=\begin{pmatrix} hu \\ hu^2+gh^2/2 \\ huv \end{pmatrix},\quad \mathrm{G}=\begin{pmatrix} hv \\ huv \\ hv^2+gh^2/2 \end{pmatrix},\quad \mathrm{S}=\begin{pmatrix} 0 \\ -gh\partial z/\partial x-\tau_{bx} \\ -gh\partial z/\partial y-\tau_{by} \end{pmatrix}$$

$$\tau_{bx}=\frac{gm^2u\sqrt{u^2+v^2}}{h^{1/3}},\ \tau_{by}=\frac{gm^2v\sqrt{u^2+v^2}}{h^{1/3}} \tag{2}$$

Where, h is depth, (u, v) are x and y direction components of the depth-averaged horizontal velocity, ‘g’ is gravitational acceleration, z is bed elevation, τ_b is friction force of bed and ‘m’ is a constant with respect to the bed state.

Applying Green's formula to the system of equations (1), we obtain Eq. (2)

$$\frac{\partial}{\partial t}\int_{\Omega} \mathrm{U} dV + \int_{\Gamma} \mathrm{F}_{\mathrm{n}} d\Gamma = \int_{\Omega} \mathrm{S} dV \tag{3}$$

$$\mathrm{F}_{\mathrm{n}} = \mathrm{F}\cdot n_x + \mathrm{G}\cdot n_y$$

Where Γ is the boundary of the volume Ω and F_{n} is the flow rate across the boundary Γ. Applying the integral expression to a rectangular finite volume, we obtain the following finite volume equation.

$$\mathrm{F}_{\mathrm{n}}^{n} = \frac{1}{2}\left(\mathrm{F}_{\mathrm{n}}\left(\mathrm{U}_L^n\right) + \mathrm{F}_{\mathrm{n}}\left(\mathrm{U}_R^n\right) - \hat{\mathrm{R}}\left|\hat{\Lambda}\right|\Delta\hat{\mathrm{V}}_{RL}^{n}\right) \tag{4}$$

Where, $\hat{\mathrm{R}}$ and $\left|\hat{\Lambda}\right|$ are diagonal matrices consisting of the eigenvector matrix of the Roe mean matrix and the absolute values of the eigenvalues, and $\Delta\hat{\mathrm{V}}_{RL}^{n}$ are characteristic variable difference matrices.

$\mathrm{F}_{\mathrm{n}}\left(\mathrm{U}_L^n\right)$ and $\mathrm{F}_{\mathrm{n}}\left(\mathrm{U}_R^n\right)$ are fluxes calculated using MUSCL reconstructed data on the left and right sides of the cell boundary, respectively, and the label ‘^’ indicates the quantities obtained from the Roe averages by reconstructed data.

3. Relationship Between Water Surface Elevation and Mean Water Depth for Partially Wetted Rectangular Cells

In finite volume schemes of shallow water equations, the water depth is defined as the value at the cell centroid, in terms of the cell mean value of water depth. However, in partially wetted cells, i.e., cells with enough fluid to submerge at least one vertex but not all, the average depth is badly represented by the depth at the centroid. For example, a cell may contain water while the free surface elevation is below the bed elevation of the centroid, z_c. The flow depth *h* of each cell is defined to be the ratio of the fluid volume *V* contained in the cell to the cell area *A*. In fully wet cells $\eta = h + z_c$, but this equality is not true in partially wetted cells.

In case of partially wetted rectangular cells, the relationship between water surface elevation η and mean value of water depth h is considered.

Vertex coordinates of the ith cell is labeled as $(x_i, y_i, z_i)(i = 1,2,3,4)$ and assume that $z_1 \le z_2 \le z_3 < z_4$.

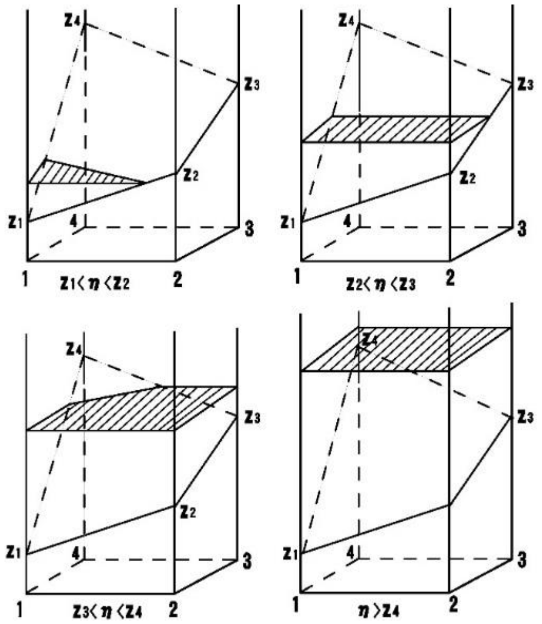

Figure 1. *Configuration of partially wetted rectangular cell.*

As illustrated in Figure 1. If $\eta \le z_1$, fluid is in the cell, so *h*=0. If $z_1 \le \eta \le z_2$, the the relationship between η and *h* is given by

$$h = \frac{V}{A} = \frac{(\eta - z_1)^3}{6(z_2 - z_1)(z_4 - z_1)} \tag{5}$$

If $z_2 \le \eta \le z_3$, the the relationship between η and *h* is given by

$$h = a\eta^2 - b\eta + c \tag{6}$$

$$a = 1/4$$

$$b = \frac{(z_4 - z_1)(5z_2 + z_1) + (z_3 - z_2)(z_2 + 5z_1)}{12(z_3 - z_2)(z_4 - z_1)}$$

$$c = \frac{1}{12}\left(\frac{z_1 z_2 + 2z_1^2}{(z_3 - z_2)} + \frac{z_1 z_2 + 2z_1^2}{(z_4 - z_1)}\right)$$

If $z_2 \le \eta \le z_3$, the the relationship between η and *h* is given by

$$h = \eta - z_c + \frac{(z_4 - \eta)^3}{6(z_4 - z_1)(z_4 - z_3)} \tag{7}$$

If $\eta > z_4$, the the relationship between η and h is given by $h = \eta - z_c$, $z_c = (z_1 + z_2 + z_3 + z_4)/4$.

Using these relations, we can calculate h for a given η, whereas we can calculate η for a given h.

4. Numerical Example

Numerical examples are conducted on rectangular regions with 75 m length and 36 m width surrounded by vertical wall boundaries. The bottom topography in the region has three humps and is expressed as follows.

$$z(x, y) = \max[0, a1, a2, a3]$$

$$a1 = 1 - \frac{1}{8}\sqrt{(x-30)^2 + (y-6)^2}$$

$$a2 = 1 - \frac{1}{8}\sqrt{(x-30)^2 + (y-24)^2}$$

$$a3 = 3 - \frac{3}{10}\sqrt{(x-47.5)^2 + (y-15)^2}$$

At time t= 0, dam is located at x =16m that initially retains still water with surface elevation 1.875 m, Figure 2 shows the computational domain and initial water elevation. In partially wetted cells, the bed friction is neglected to consider only the effect of the relationship between the free surface and depth on the flow. At time t= 0, the vertical water column of 1.87 m water elevation is stopped up to 16 m in the region length, and all other areas are dry. After the initial moment, the flow process was considered as the process of passing through the hump as the water column collapsed and reaching the final steady state.

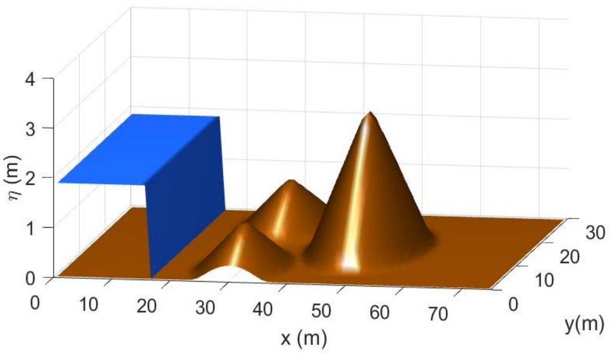

Figure 2. *Computational domain and initial water surface.*

Figure 3-Figure 7 shows the development of the water column with time. After the dam has been destroyed, the large wave starts to inundate the dry zone. By time t =2s, the front has reached the pair of small humps and begins to rise over them. At time *t* =6s, the small humps are entirely submerged, and the wet/dry front has reached the large hump, while water cascades around the side of the hill. At time t = 12 s, the flood water that is passing either side of the large hill starts to flood the lee of the hill. At time t= 55 s, the large wave motion is almost eliminated, the ripple remains, almost steady state is reached until time t=150~200 s, and the humps of the small hills are no longer submerged.

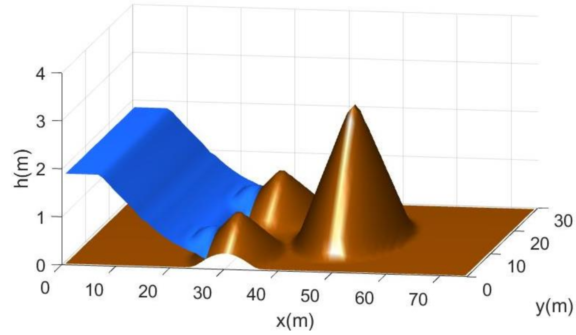

Figure 3. *The development of water column change at time t = 2s.*

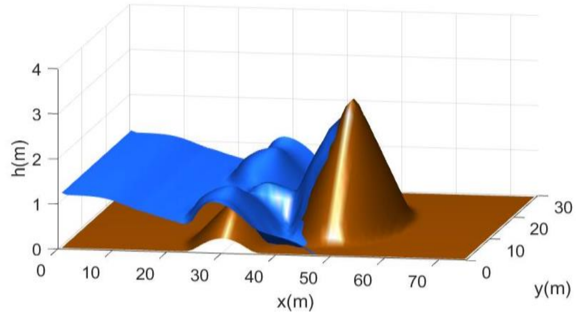

Figure 4. *The development of water column change at time t = 6s.*

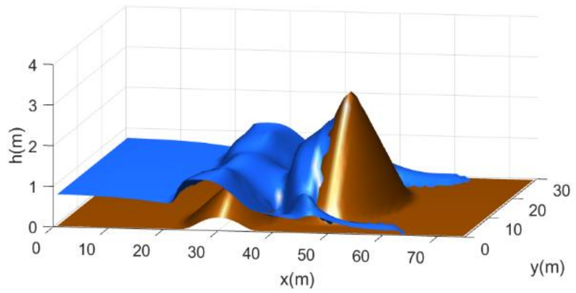

Figure 5. *The development of water column change at time t = 12s.*

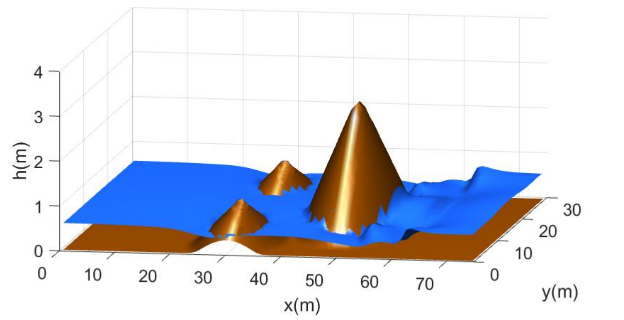

Figure 6*. The development of water column change at time t = 50s.*

Table 1*. The cpu calculation time required for numerical simulations up to the physical time t = 150 s.*

Method	Cpu computation time (s)
Moving grid method	3075
Method of Sanders (2006)	1724
Present method	1258

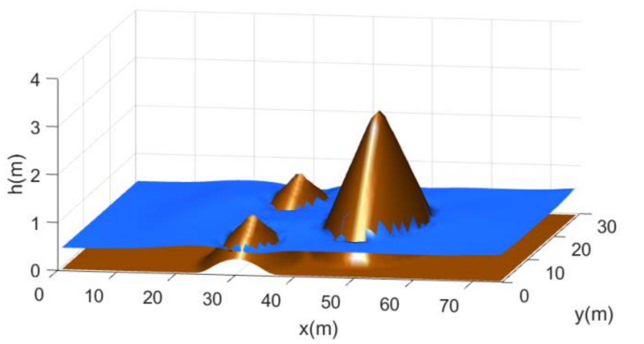

Figure 7*. The development of water column change at time t = 150 s.*

In Figure 8 and Figure 9, the contours of the calculated water depth at different times are plotted, and the reflection and interaction of the waves are clearly visible, and no vibration or disturbance is observed at the wet/dry boundary.

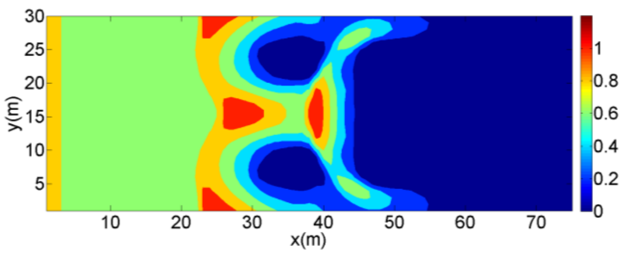

Figure 8*. Concentration changes at time t = 8 s.*

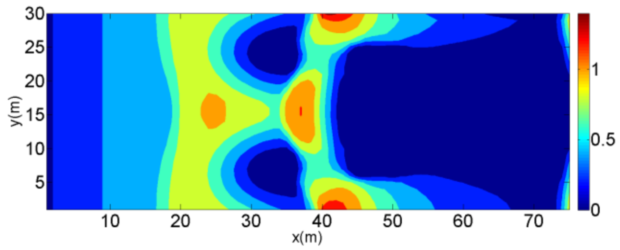

Figure 9*. Concentration changes at time t=12 s.*

Numerical simulation results for complex bottom topography show that our method adequately simulates the flow to dry areas and the complicated processes of drainage and dry area formation, and does not generate pseudo flow. Also, the numerical simulation results are very similar to those reported in literature [2, 3, 7]. Table 1 shows the computational time of using our method, the moving grid method and the adaptive quadtree mesh method for the simulation of shallow water flow with three humps.

5. Conclusions

In the paper, we derived the relationships between the water surface elevation parameter and the mean water depth parameter in a partially wet cell if a rectangular grid is used in numerical simulations by the Godnov-type finite volume method of the two-dimensional shallow water flow equation system. Using these relationships, the dam-break flow simulation over three hills accurately predicts the transport processes of wet/dry fronts and provides stable simulation results for complex bottom topography. This shows the applicability of our method to realistic dam break flow simulation.

Abbreviations

MUSCL Monotone Upwind Scheme for Conservation Laws

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Conflicts of Interest

The authors declare no conflicts of interest.

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