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Research Article

The Additive Dhillon-Chen Distribution: Properties and Applications to Failure Time Data

Faisal Muhammad Amiru[1, *], Umar Usman[2], Suleiman Shamsuddeen[3], Umar Muhammad Adamu[1], Badamasi Abba[4]

[1]Department of Mathematical Sciences, Bayero University, Kano, Nigeria
[2]Department of Statistics, Usmanu Danfodio University, Sokoto, Nigeria
[3]Department of Statistics, Federal University Dutsin-ma, Katsina, Nigeria
[4]Department of Mathematics, North-West University, Kano, Nigeria

Abstract

To measure the average lifespan of systems and components, and to analyze lifetime data with a monotonic failure rate, distributions such as Weibull, Exponential, and Gamma are commonly used in reliability and survival studies. However, these distributions are not suitable for datasets with non-monotonic patterns like the bathtub curve. To address this, the Chen distribution, which accommodates increasing or bathtub-shaped failure rates, has been proposed. Yet, this model lacks a scale parameter. This article presents a new four parameter lifetime distribution with bathtub-shaped failure rate called Additive Dhillon-Chen (ADC) distribution. We applied the additive methodology to establish the model, for which the Dhillon distribution was considered as baseline distribution. Some statistical properties such as quartile function, mode, moment and moment generating function, order statistics and asymptotic behavior of the distribution are studied. Parameters of the distribution are estimated using the maximum likelihood estimation method. The ADC distribution is applied to two lifetime dataset and compared with an existing distribution in the literature. Model selection was carried out based on Log-likelihood, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Corrected Akaike Information Criterion (AICc). The results, based on parameter estimation from real-life data, demonstrate that the ADC distribution fits the data well and offers a valuable alternative for modeling datasets with non-monotonic behavior.

Keywords

Reliability, Survival, Bathtub-curve, Additive, Dhillon-Chen Distribution

1. Introduction

In reliability and survival studies, lifetime distributions like Weibull, Exponential, and Gamma are commonly used to gauge the average lifespan of system and device components and analyse data with a monotonic failure rate. However, these models fall short when dealing with data exhibiting non-monotonic patterns, such as the bathtub curve. To address this issue, several alternative approaches have been proposed [17, 23].

*Corresponding author: Faisalameer83@gmail.com (Faisal Muhammad Amiru), fmamiru.mth@buk.edu.ng (Faisal Muhammad Amiru)

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The work of introduced a new two-parameter probability distribution with increasing or bathtub-shaped Failure Rate (FR) function. The c.d.f. of the Chen distribution is given by

$$F(t) = 1 - e^{\alpha(1-e^{t^{\beta}})}, t > 0, \alpha, \beta > 0 \tag{1}$$

It has an increasing FR when $\beta \geq 1$ and a bathtub-shaped when $\beta < 1$. But, the distribution is not flexible because it lacks scale parameter. In view of the above, extended works on Chen distribution have been considered by many researchers to provide more flexibility for describing different form of data sets using different methodologies. For example: A new extension of Chen distribution was proposed by [6]. The shape of the density of the model is decreasing or unimodal according to the value of the parameters, while the shape of the hazard rate is increasing or bathtub-shaped. The authors adopted the method of generalizing a cumulative distribution function of a given distribution introduced by [10] through introduction of additional parameter $\alpha > 0$ given by $F_{\alpha}(t) = [F(t)]^{\alpha}$. [21] consider an extension of the exponentiated Chen distribution based on the quadratic rank transmutation map technique named Transmuted Exponentiated Chen distribution with application to survival data. They estimated the parameters using the method of maximum likelihood and finally, the flexibility of the new distribution was illustrated using strengths of glass fibers data and nicotine in cigarette data [7]. A new continuous probability distribution called Exponentiated Chen distribution was introduced by [8]. Their main focus was on estimation from frequentist point of view and they derived some statistical and reliability characteristics for the model. [12] introduced the Kumaraswamy Exponentiated Chen distribution for modelling a bathtub-shaped hazard rate function. [4] introduced a new censoring scheme named statistical analysis of competing risks model from Marshall-Olkin extended Chen distribution under adaptive progressively interval censoring with random removals. [23] introduced a new lifetime distribution called addictive Chen-Weibull (ACW) distribution and describe its application in reliability modelling. The distribution has an increasing and bathtub-shape failure rates. [22] introduced a three-parameter probability distribution named Chen Exponential distribution with applications to engineering data. The model was developed by introducing additional parameter to Chen distribution. [17] introduced an additive Chen distribution with applications to lifetime data. The model has an increasing, decreasing, and bathtub shape failure rate. The author deployed additive methodology and consider Chen distribution as the baseline distribution and so also, the distribution has excellent flexibility in describing failure rates with non-monotone behavior or with the shape of bathtub curve.

In practice, a considerable number of generalized models have been proposed for modelling lifetime data with non-monotone failure rates (FRs) from various lookouts, particularly reliability engineering [2]. Some of these generalized models includes; Exponentiated additive Weibull distribution by [3], A new weighted Gompertz distribution with applications to reliability data by [5], An extension of Chen's family of survival distribution with bathtub-shape or increasing hazard rate by [6], Chen-Burr XII model as a competing risks model with applications to real-life datasets by [11], on some lifetime distribution with flexible failure rate by [14], A new extension of the Topp-Leone family of models with applications in repairable systems by [15], The alpha power Weibull transformation distribution applied to describe the behavior of electronic devices under voltage stress profile by [18], Beta Sarhan-Zaindin modified Weibull distribution by [19], A new extension of the exponential power distribution with application to lifetime data by [20], Classical and Bayesian estimations of improved Weibull-Weibull distribution for complete and censored failure times data by [25], and On the upper truncated Weibull distribution by [26],. Despite the usefulness of these distributions in reliability engineering, research has shown that a substantial part of these distributions has bathtub failure rate (FR) shapes but lack the FR's relatively constant [2]. According to [13], this phase (useful life) is very vital or may be the most critical phase for reliability modelling, this is because it describes the useful life span of the component or system. Hence, it is paramount important to construct a model(s) that can accurately represent this constant failure rate (CRF) phase.

Combining the FR of two distributions is considered as a very useful technique in obtaining more flexible models with simple FR. In view of this, [24] established the additive Weibull model using the idea of combing the hazard rates of two Weibull distribution. In this paper, we propose a four-parameter lifetime distribution by using additive methodology to combine the failure rates of the Dhillon distribution which was introduced by [9] and the Chen distributions in a serial system, which will be called as the Additive Dhillon-Chen (ADC) distribution. The propose distribution is considered to be very flexible to model data with monotonic and non-monotonic behavior.

The rest of this research paper is organized as follows. In section 2, we define and introduce the new ADC distribution and present its important functional forms. In section 3, we consider some properties of the new distribution such as moment generating function, moments. We also consider the method of maximum likelihood estimator (MLE) for the estimate of the model parameters in section 4. In section 5, we present the flexibility of the ADC distribution using two lifetime datasets and the results are compared with some competitor distributions. Finally, the conclusion of this paper was presented in section 6.

2. The Additive Dhillon-Chen (ADC) Distribution

Let us consider a system with two components arranged and functioning in a series, each component is operating independently at a given time t. The system fails when the first component fails. In view of this, our new model signifies the lifetime of the entire serial system with two components. The first component's lifetime follows a Dhillon distribution with parameters λ and θ, while the second component's lifetime follows a Chen distribution with parameters α and β. The complete system's lifetime is determined by the minimum lifetime of the two components. In other word, let T_1 represent the lifetime of the first component, which follows Dhillon distribution with parameter λ and θ, and let T_2 represent the lifetime of the second component which follows Chen with parameter α and β. If the system lifetime is T, then

T = min (T_1, T_2) has the distribution given by

$$F(t) = 1 - e^{-\ln(\lambda t^{\theta}+1)+\alpha\left(1-e^{t^{\beta}}\right)} \tag{2}$$

for $t, \lambda, \theta, \alpha > 0$ and $\beta \geq 0$.

The probability density function (pdf) of this distribution is given by

$$f(t) = \left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^{\theta}+1} + \alpha\beta t^{\beta-1} e^{t^{\beta}}\right) e^{-\ln(\lambda t^{\theta}+1)+\alpha(1-e^{t^{\beta}})} \tag{3}$$

For random variable T with pdf in (3), we write $T \sim ADC(\lambda, \theta, \alpha, \beta)$.

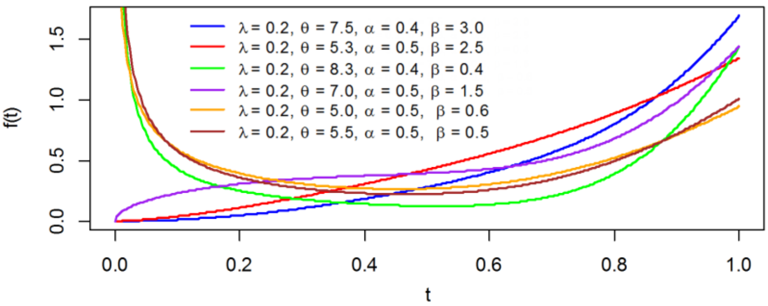

Figure 1. *Plot of the ADC density function for some values of parameter.*

Finally, the failure rate function as well as the survival function of the proposed ADC distribution is given as

$$h(t) = \frac{\lambda\theta t^{\theta-1}}{\lambda t^{\theta}+1} + \alpha\beta t^{\beta-1}\, e^{t^{\beta}} \tag{4}$$

$$S(t) = e^{-\ln(\lambda t^{\theta}+1)+\alpha(1-e^{t^{\beta}})} \tag{5}$$

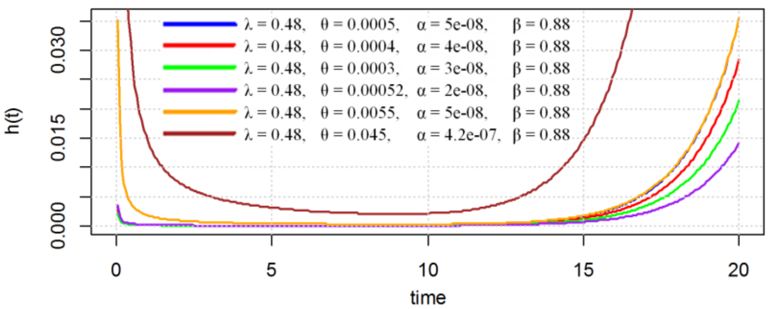

Figure 2. *Plot of the ADC failure rate for some values of parameter.*

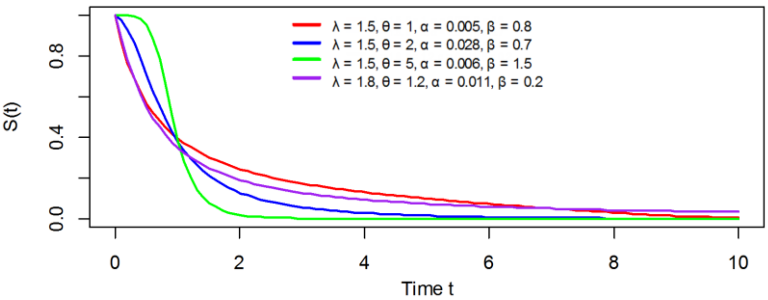

Figure 3. *Plot of the ADC failure rate for some values of parameter.*

3. Properties of ADC Distribution

3.1. Measure of Central Tendency

We shall provide some measures of central tendency of the ADC distribution in this section.

Quartile Function

The quantile function or quantile function is used to obtain random data with which it is possible to carryout pdf simulations. The p^{th} quartile q_p presented by t of ADC based on equation (1) can be calculated as

$$p = \left[1 - e^{-\ln(\lambda q^{\theta}+1)+\alpha(1-e^{q^{\beta}})}\right] \tag{6}$$

The above equation does not have a close form solution, as such, it is necessary to make approximations through the use of numerical methods provided in specialized software's like R-package. We obtained the median of the ADC distribution by setting $q = 0.5$ in equation (6).

Mode

The mode of the ADC distribution is the value(s) of the random variable at which the probability density function (pdf) reaches its maximum. To obtain the mode of the ADC model, we take the first derivative of term in equation (3) and equate to zero.

$$\frac{d\left[\left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^{\theta}+1}+\alpha\beta t^{\beta-1}e^{t^{\beta}}\right)e^{-\ln(\lambda t^{\theta}+1)+\alpha(1-e^{t^{\beta}})}\right]}{dt} = 0 \tag{7}$$

The mode is obtained in equation (8)

$$\left[\frac{[(\lambda t^{\theta}+1)\lambda\theta(\theta-1)t^{\theta-2}-(\lambda\theta t^{\theta-1})(\lambda\theta t^{\theta-1})]}{(\lambda t^{\theta}+1)^2} + \alpha\beta(\beta-1)t^{\beta-2}e^{t^{\beta}} + \alpha\beta^2 t^{2\beta-2}e^{t^{\beta}}\right]\left[\left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^{\theta}+1} + \alpha\beta t^{\beta-1}e^{t^{\beta}}\right)e^{-\ln(\lambda t^{\theta}+1)+\alpha(1-e^{t^{\beta}})}\right] \tag{8}$$

3.2. Moment Generating Function and Moment

If T is a random variable that follows the ADC distribution with parameters λ, θ, α, and β. Then the moment generating function of T defined by $M_t(x) = E[e^{tx}]$ is obtained as:

$$M_t(x) = E[e^{tx}] = \int_0^{\infty} e^{tx} f(t)dt \tag{9}$$

Since by series expansion,

$$e^{tx} = \sum_0^{\infty} \frac{t^i x^i}{i!}$$

Then,

$$M_t(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!} \int_0^{\infty} t^i f(t)dt = 1 + \sum_{i=1}^{\infty} \frac{x^i}{i!} \int_0^{\infty} t^i f(t)dt \tag{10}$$

$$= 1 + \sum_{i=1}^{\infty} \frac{x^i}{i!} E(t^i) = 1 + \sum_{i=1}^{\infty} \frac{x^i}{i!} \mu_i'(t) \tag{11}$$

where

$$\mu_i'(t) = E(t^i) = \int_0^{\infty} t^i e^{-\ln(\lambda t^{\theta}+1)+\alpha(1-e^{t^{\beta}})}dt \tag{12}$$

$$= i\int_0^{\infty} \frac{t^{i-1}}{\lambda t^{\theta}+1} e^{\alpha(1-e^{t^{\beta}})}dt \tag{13}$$

Since $\frac{1}{\lambda t^{\theta}+1} = (\lambda t^{\theta}+1)^{-1} = \sum_{k=0}^{\infty}(-1)^k \lambda^k t^{k\theta}$. Then,

$$\mu_i'(t) = i\sum_{k=0}^{\infty}(-1)^k \lambda^k \int_0^{\infty} t^{k\theta+i-1} e^{\alpha(1-e^{t^{\beta}})}dt \tag{14}$$

Rearranging the exponential terms, we have

$$\mu_i'(t) = ie^{\alpha}\sum_{k=0}^{\infty}(-1)^k \lambda^k \int_0^{\infty} t^{k\theta+i-1} e^{-\alpha e^{t^{\beta}}}dt \tag{15}$$

Hence, the i^{th} moment of T, is given as

$$\mu_i'(t) = \frac{ie^{\alpha}}{\beta}\sum_{k=0}^{\infty}(-1)^k \lambda^k \nabla(m). \tag{16}$$

Consequently, the MGF of T from (16) is therefore derived as

$$M_t(x) = 1 + \sum_{i=1}^{\infty} \frac{x^i}{i!} E(t^i) = 1 + \sum_{i=1}^{\infty} \frac{x^i}{i!} \frac{ie^{\alpha}}{\beta} \sum_{k=0}^{\infty}(-1)^k \lambda^k \nabla(m). \tag{17}$$

3.3. Order Statistics

The order statistics and their moments have great importance in many statistical problems and applications in reliability analysis and life testing. Let $T_1, T_2, \ldots, T_n$ be a random sample from the ADC distribution and $T_{k:n}$ is the k^{th} order statistic of the sample, then the PDF of $T_{k:n}$ is given by

$$f_{k:n}(t) = \frac{1}{B(k, n-k+1)}[F(t)]^{k-1}[1-F(t)]^{n-k} f(t) \tag{18}$$

where

$$B(k, n-k+1) = \frac{\Gamma(k)\Gamma(n-k+1)}{\Gamma(n-k+1+k)} = \frac{\Gamma(k)\Gamma(n-k+1)}{\Gamma(n+1)} = \frac{(k-1)!(n-k)!}{n!}$$

The k^{th} order statistics is given by

$$f_{k:n}(t) = \begin{pmatrix} \frac{n!}{(n-k)!(n-k)!} \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} e^{-i\left\{ln(\lambda t^\theta+1)-\alpha\left(1-e^{t^\beta}\right)\right\}} \\ \left\{e^{-(n-k)\left\{ln(\lambda t^\theta+1)-\alpha\left(1-e^{t^\beta}\right)\right\}}\right\} \left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^\theta+1} + \alpha\beta t^{\beta-1} e^{t^\beta}\right) e^{-\left\{ln(\lambda t^\theta+1)-\alpha\left(1-e^{t^\beta}\right)\right\}} \end{pmatrix} \tag{19}$$

3.4 Asymptotic Behavior

We consider the asymptotic behavior to investigate the behavior of the pdf of the ADC model as $t \to 0$ and as $t \to \infty$.

$$\lim_{t\to 0} f(t) = \lim_{t\to\infty} f(t) = 0 \tag{20}$$

This implies that,

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} \left[\left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^\theta+1} + \alpha\beta t^{\beta-1} e^{t^\beta}\right) e^{-ln(\lambda t^\theta+1)+\alpha(1-e^{t^\beta})}\right] = 0 \tag{21}$$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} \left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^\theta+1} + \alpha\beta t^{\beta-1} e^{t^\beta}\right) \lim_{t\to 0} \left(e^{-ln(\lambda t^\theta+1)+\alpha(1-e^{t^\beta})}\right) = 0 \tag{22}$$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} \left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^\theta + 1} + \alpha\beta t^{\beta-1} e^{t^\beta}\right) X\ 0$$

$$\lim_{t\to 0} f(t) = 0.$$

Similarly,

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \left[\left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^\theta+1} + \alpha\beta t^{\beta-1} e^{t^\beta}\right) e^{-ln(\lambda t^\theta+1)+\alpha(1-e^{t^\beta})}\right] = 0 \tag{23}$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \left(\frac{\lambda\theta t^{\theta-1}}{\lambda t^\theta+1} + \alpha\beta t^{\beta-1} e^{t^\beta}\right) X \lim_{t\to\infty} \left(e^{-ln(\lambda t^\theta+1)+\alpha(1-e^{t^\beta})}\right) = 0 \tag{24}$$

$$\lim_{t\to\infty} f(t) = 0\ X \lim_{t\to\infty} \left(e^{-ln(\lambda t^\theta+1)+\alpha(1-e^{t^\beta})}\right) = 0$$

$$\lim_{t\to\infty} f(t) = 0. \tag{25}$$

Hence, the ADC distribution has a unimodal.

4. Parameter Estimation

Maximum Likelihood Estimator

The Maximum Likelihood Estimation (MLE) is a statistical technique used to estimate the parameters of a given probability distribution or statistical model. It classifies the parameter values that maximize the likelihood function, which measures how fit the model describes the observed data. Let $t_1, t_2, \ldots\ldots, t_n$, be observed values of a random sample drawn from the ADC distribution with parameters vector $\mu = (\lambda, \theta, \alpha, \beta)$. Then, the log-likelihood function of $\mu|t_i, i = 1, 2, \ldots\ldots, n$ is developed from the pdf $f(t)$ in equation (3) as

$$l(\mu) = nlog\left(\frac{\lambda\theta t_i^{\theta-1}}{\lambda t_i^\theta+1} + \alpha\beta t_i^{\beta-1} e^{t_i^\beta}\right) - ln \sum(\lambda t_i^\theta + 1) + n\alpha\left(1 - e^{t_i^\beta}\right) \tag{26}$$

The associated score function for each ADC parameter from (36) are

$$U_\lambda(\mu) = \sum_{i=1}^{n}\left(\frac{\theta t_i^{\theta-1}(\lambda t_i^{\theta}+1)-\lambda\theta t_i^{\theta-1}(t_i^{\theta})}{g(t)(\lambda t_i^{\theta}+1)^2} - \frac{t_i^{\theta}}{\lambda t_i^{\theta}+1}\right), \quad (27)$$

$$U_\theta(\mu) = \sum_{i=1}^{n}\left(\frac{\lambda t_i^{\theta}(\lambda t_i^{\theta}+1)-\lambda\theta t_i^{\theta-1}ln(t_i)t_i^{\theta}}{(\lambda t_i^{\theta}+1)^2 g(t)} - \frac{\lambda t_i^{\theta}ln(t_i)}{\lambda t_i^{\theta}+1}\right), \quad (28)$$

$$U_\alpha(\mu) = \sum_{i=1}^{n}\left(\frac{\beta t_i^{\beta-1}e^{t_i^{\beta}}}{g(t)} + (1-e^{t_i^{\beta}})\right), \quad (29)$$

$$U_\beta(\mu) = \sum_{i=1}^{n}\left(\frac{\alpha(t_i^{\beta-1}e^{t_i^{\beta}}+\beta t_i^{\beta-1}e^{t_i^{\beta}}ln(t_i)}{g(t)} - \alpha e^{t_i^{\beta}}t_i^{\beta}ln(t_i)\right) \quad (30)$$

Considering the nature of the score function $U_\lambda(\mu). U_\theta(\mu), U_\propto(\mu), and\ U_\beta(\mu)$, we recommend computing the estimates $\hat{\mu} = (\hat{\lambda}, \hat{\theta}, \hat{\propto}, \hat{\beta})'$ numerically using statistical software. For this study, we utilized the maxLik package (Henningsen and Toomet, 2011) in R to maximize the likelihood function.

5. Application

This section evaluates the ADC distribution by applying it to real-life data with non-monotonic characteristics and contrasts its performance with other distributions that use an additive approach. The distributions compared are Additive Chen distribution (AddC), Additive Chen Weibull distribution (ACW) and Additive Perks Weibull (APW) distribution. Table 1 present the FR functions for each distribution.

The parameters were estimated through MLE in Rstudio using Maxlik Library, the gradient and Hessian of each of the distributions are considered.

Table 1. *Failure functions of some Bathtub Shape Distributions.*

Model	h(t)
ADC	$\frac{\lambda\theta t^{\theta-1}}{\lambda t^{\theta}+1} + \alpha\beta t^{\beta-1}\, e^{t^{\beta}}$
AddC	$\alpha\beta t^{\beta-1}e^{t^{\beta}} + \lambda\theta t^{(\theta-1)}e^{t^{\theta}}$
ACW	$\alpha\beta t^{\beta-1}e^{t^{\beta}} + \lambda\theta t^{\theta-1}$
APW	$\frac{\alpha\lambda e^{\lambda t}}{1+\alpha e^{\lambda t}} + \theta\beta t^{\beta} - 1$

5.1. Reliability Analysis for Lifetime Data of 50 Devices

In this study, we determining the behavior of the failure times of 50 devices. The data was reported by [1] and was used by [16, 17, 23]. The data is known to have a bathtub-shaped FR function.

Table 2. *Aarset Data of 50 Devices.*

0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	89	82	82	83
84	84	84	85	85	85	85	85	86	86

Table 3. *Descriptive statistics for the Aarset data.*

N	Min.	1st Qu.	Median	Mean	Sd.	3rd Qu.	Max.
50	0.10	13.5	48.5	45.69	32.84	81.25	86.0

We consider the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Corrected Akaike Information Criterion (AICc) for model selection. The model with smaller values is considered to be the best model.

Table 4. *Estimated values, standard errors in bracket and summary values of the fitted models fitted.*

Model	Lambda	Theta	Alpha	Beta	Loglik	AIC	BIC	AICc
ADC	-0.55640 (0.01701)	-0.25385 (0.01327)	0.07325 (0.05181)	0.24405 (0.05319)	-188.54	385.08	383.88	377.96
AddC	2.08×10^{-17} (1.1×10^{-22})	0.822 (0.0165)	0.05977 (0.00059)	0.249 (0.00711)	-203.05	414.12	421.76	406.88
ACW	0.01118 (0.00039)	86.231 (2.414)	0.04215 (1.0×10^{-5})	0.278 (0.0224)	-205.35	418.71	426.36	411.58
APW	0.443 (0.01921)	0.05320 (0.04221)	7.16×10^{-17} (1.1×10^{-19})	0.688 (0.01025)	-212.87	433.75	441.44	426.62

Table 4 above displays the MLEs of the models' parameters as well as the value of log-likelihood, AIC, BIC, and AICc statistic. It is observed that the values of AIC, BIC, AICc of the ADC model appear to be the smallest among the competitive models, hence the ADC distribution appears to be a very competitive for the dataset.

5.2. Analysis on Failure and Running Times of 30 Devices

Table 5 present data of failure and running times of 30 devices reported by [16] and was studied extensively by several authors among which include [17, 23].

Table 5. *Meeker and Escober Data of 30 devices.*

2	10	13	23	23	28	30	65	80	88
106	143	147	173	181	212	245	247	261	266
275	293	300	300	300	300	300	300	300	300

Table 6. *Descriptive statistics for the Meeker and Escober data of 30 devices*

N	Min.	1st Qu.	Median	Mean	Sd.	3rd Qu.	Max.
30	2.00	68.75	196.50	177.03	114.99	298.25	300.00

Table 7. *Estimated values, standard errors in bracket and summary values of the fitted models.*

Estimated Parameters								
Model	**Lambda**	**Theta**	**Alpha**	**Beta**	**Loglik**	**AIC**	**BIC**	**AICc**
ADC	-1.1078 (0.0146)	-0.1481 (0.0189)	0.0004 (2.22×10^{-4})	0.3615 (0.0141)	-114.73	237.47	235.37	238.60
AddC	0.407 (0.01146)	8.419×10^{-2} (0.001654)	3.042×10^{-11} (5.958×10^{-13})	0.561 (0.00711)	-147.89	303.77	311.42	301.68
ACW	0.0033 (0.000017)	259.427 (14.558)	0.01518 (0.00210)	0.260 (0.00301)	-151.34	310.67	316.28	308.58
APW	0.08802 (0.002114)	0.00111 (0.00933)	5.14×10^{-12} (8.06×10^{-8})	0.807 (0.172)	-167.91	343.82	349.42	341.73

Table 7 above displays the MLEs of the models' parameters as well as the value of log-likelihood, AIC, BIC, and AICc statistic. It is observed that the ADC distribution has the least values of AIC, BIC, AICc and hence considered to be the best model.

6. Conclusions

In this article, we developed a new four-parameter lifetime distribution with application to two real reliability data sets which are well known from existing literature. This distribution combines the failure rates of the Dhillon and Chen distributions using an additive methodology, accommodating both monotonic and non-monotonic behavior. The study explores the distribution's properties and applies it to real-life datasets, evaluating its goodness-of-fit through various information criteria. The results, based on parameter estimation from the real-life data, indicate that the ADC distribution fits the data well. Consequently, it is concluded that the ADC distribution offers a valuable alternative for modeling datasets with both monotonic and non-monotonic behavior.

Abbreviations

AddC	Additive Chen
ADC	Additive Dhillon-Chen
AIC	Akaike Information Criterion
AICc	Corrected Akaike Information Criterion
ACW	Additive Chen Weibull
APW	Additive Perks Weibull
BIC	Bayesian Information Criterion
CDF	Cumulative Distribution Function
CFR	Constant Failure Rate
FR	Failure Rate
MGF	Moment Generating Function
MLE	Maximum Likelihood Estimation
PDF	Probability Density Function

Author Contributions

Faisal Muhammad Amiru: Conceptualization, Data curation, Formal Analysis, Methodology, Resources, Software, Writing – original draft

Umar Usman: Supervision, Validation, Visualization, Writing – review & editing

Suleiman Shamsuddeen: Supervision, Validation, Visualization, Writing – review & editing

Umar Muhammad Adamu: Data curation, Methodology, Resources, Software

Badamasi Abba: Validation, Visualization, Writing – review & editing

Conflicts of Interest

The authors declare no conflicts of interest.

Appendix

Let $g(t) = \lambda\theta t^{\theta-1}$ and $q(t) = \lambda t^{\theta} + 1$

This implies that, $g'(t) = \lambda\theta(\theta-1)t^{\theta-2}$ and $q'(t) = \lambda\theta t^{\theta-1}$

By apply quotient rule, we have

$$\frac{q(t)g'(t)-g(t)q'(t)}{q(t)^2} = \frac{[(\lambda t^{\theta}+1)\lambda\theta(\theta-1)t^{\theta-2}-(\lambda\theta t^{\theta-1})(\lambda\theta t^{\theta-1})]}{(\lambda t^{\theta}+1)^2}$$

Also,

$$\frac{d\left(\alpha\beta t^{\beta-1}e^{t^{\beta}}\right)}{dt} = \alpha\beta(\beta-1)t^{\beta-2}e^{t^{\beta}} + \alpha\beta^2 t^{2\beta-2}e^{t^{\beta}}.$$

Now, let $m = \alpha e^{t^{\beta}}$, implies $t = \left[\log\left(\frac{m}{\alpha}\right)\right]^{1/\beta}$ and $dt = \frac{1}{\beta m}\left[\log\left(\frac{m}{\alpha}\right)\right]^{\frac{1}{\beta}-1}$, then

$$\mu_i'(t) = ie^{\alpha}\sum_{k=0}^{\infty}(-1)^k\lambda^k\int_{\alpha}^{\infty}\left[\log\left(\frac{m}{\alpha}\right)\right]^{\frac{k\theta+i-1}{\beta}} e^{-m} \times \left[\log\left(\frac{m}{\alpha}\right)\right]^{\frac{1}{\beta}-1}\frac{dm}{\beta m} \tag{A-1}$$

$$= \frac{ie^{\alpha}}{\beta}\sum_{k=0}^{\infty}(-1)^k\lambda^k\int_{\alpha}^{\infty} m^{-1}\left[\log\left(\frac{m}{\alpha}\right)\right]^{\frac{k\theta+i}{\beta}-1} e^{-m}dm$$

$$\mu_i'(t) = \frac{ie^{\alpha}}{\beta}\sum_{k=0}^{\infty}(-1)^k\lambda^k\,\nabla(m) \tag{A-2}$$

Where the integral $\nabla(m) = \int_{\alpha}^{\infty} m^{-1}\left[\log\left(\frac{m}{\alpha}\right)\right]^{\frac{k\theta+i}{\beta}-1} e^{-m}dm$ is intricated and can thus be evaluated numerically.

$$f_{k:n}(t) = \frac{1}{\left\{\frac{(k-1)!(n-k)!}{n!}\right\}}[F(t)]^{k-1}[1-F(t)]^{n-k}f(t) = \frac{n!}{(n-k)!(n-k)!}[F(t)]^{k-1}[1-F(t)]^{n-k}f(t)$$

Where $F(t)$ and $f(t)$ are given in equations (2) and (3) respectively

$$[F(t)]^{k-1} = \left[1 - e^{-\left\{ln\left(\lambda t^{\theta}+1\right)-\alpha\left(1-e^{t^{\beta}}\right)\right\}}\right]^{k-1} = \sum_{i=0}^{k-1}(-1)^i \binom{k-1}{i} e^{-i\left\{ln\left(\lambda t^{\theta}+1\right)-\alpha\left(1-e^{t^{\beta}}\right)\right\}}$$

Similarly,

$$[1-F(t)]^{n-k} = \left[e^{-\left\{ln\left(\lambda t^{\theta}+1\right)-\alpha\left(1-e^{t^{\beta}}\right)\right\}}\right]^{n-k} = e^{-(n-k)\left\{ln\left(\lambda t^{\theta}+1\right)-\alpha\left(1-e^{t^{\beta}}\right)\right\}}$$

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