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Generating New Lifetime Distributions Using Parsimonious Transformation: Properties and Applications

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Abstract: In this paper, we propose a new parsimonious transformation for obtaining lifetime distributions, and as special cases, we obtain two new lifetime distributions using exponential and Weibull distributions as baselines in the transformation. We study the mathematical properties of the transformation, and for the two new lifetime distributions, we obtain survival functions, hazard functions, moments, moment-generating functions, mean deviation, Rényi entropy, and quantile function. We estimate the parameters of the new lifetime distributions using the maximum likelihood (ML) estimation method, and the Monte Carlo simulations are used to assess the consistency of the ML estimators of the parameters. The proposed new lifetime distributions provide a better fit in terms of Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) scores in comparison to the baseline distributions and other competing models, based on two real datasets, namely the exceedance of the flood peaks of the Wheaton River, and the failure times of 50 items.

Keywords: Lifetime Distribution, Parsimonious Transformation, Exponential Distribution, Weibull Distribution

1. Introduction

The study of the duration of life of organisms, systems, or devices is of major importance in the biological and engineering sciences [8]. Modeling lifetime distribution is a fundamental aspect of reliability theory and survival analysis, where understanding the failure mechanisms of systems and organisms plays an important role [8]. Recently, many authors have introduced and studied various transformations to obtain new lifetime models using existing distributions as baselines. Most of these transformations are obtained by adding one or more parameters to the baseline distribution. For instance, Sule et al. [16] introduced the Topp Leone Kumaraswamy-G transformation and showed its application using cancer data; Sule et al. [17] introduced a new five-parameter distribution and showed its application using biomedical datasets; Marshall and Olkin [10] introduced the Marshall-Olkin transformation to generate new lifetime models; Gupta et al. [3] studied the exp-G (exponentiated-G) family (also known as the Lehmann family) of distributions.

The added parameter(s) can improve the flexibility of the model, but it complicates parameter estimation and incurs penalties in model selection criteria such as Akaike Information Criterion (AIC) [1] and Bayesian Information Criterion (BIC) [15]. To overcome this difficulty, researchers developed new transformations without adding any extra parameter(s) to the baseline distribution. Such transformations are referred to as parsimonious transformations, and the resulting distribution as parsimonious distribution [4]. Recently, parsimonious modeling of lifetime distributions has gained renewed importance. Kumar et al. [6] proposed DUS transformation and showed its application to bladder cancer patient data using the exponential distribution as a baseline. In [7], Kumer et al. introduced another transformation known as SS transformation using the sine function and demonstrated its applications using cancer data. Kavya and Manoharan [4] proposed another parsimonious transformation, known as the KM transformation, and by applying it to the exponential and the Weibull distributions, they introduced new lifetime distributions referred to as the KME and KMW distributions. The KME and KMW distributions outperformed several other competing distributions for the exceedance of flood peak and bladder cancer datasets, respectively [4].

Both the DUS [6] and KM [11] transformations are obtained by compounding the baseline df with an exponential function, while the SS [7] transformation is obtained by compounding the df of the baseline with a sine function. Parsimonious transformations based on other trigonometric functions have also been studied in the literature (see [4]). However, parsimonious transformation based on compounding the baseline df with the logarithmic function does not seem to be studied in the literature. The logarithmic function, is a fundamental mathematical function with many valuable properties such as monotonicity, continuity, and differentiability.

In this paper, we introduce a new parsimonious transformation (referred to as logarithmic transformation) by compounding the baseline distribution with a logarithmic function. Let $G(x)$ be the distribution function (df) of a given baseline distribution. The transformed df $F(x)$ is defined as

$$F(x) = \frac{1}{1-\ln(e-1)} \ln\left(1 + \frac{G(x)}{e-1}\right) \tag{1}$$

where $\ln$ denotes the natural logarithm. If $G(x)$ is a continuous distribution with probability density function (pdf) $g(x)$, then the pdf $f(x)$ of the transformed distribution is

$$f(x) = \frac{g(x)}{(1-\ln(e-1))\,(e-1+G(x))}. \tag{2}$$

The hazard function of the distribution with df (1) is obtained as

$$h(x) = \frac{g(x)}{[1-\ln(e-1+G(x))]\,(e-1+G(x))}. \tag{3}$$

In the sequel, as special cases, we obtain two new lifetime distributions by taking $G(x)$ equal to the dfs of exponential and Weibull distributions in (1); we refer to these new distributions as LTE and LTW, respectively.

The hazard function, which describes the instantaneous failure rate at any given time, is a key characteristic used to determine the suitability of a probability distribution for modeling lifetime data. From Fig. 1(b), we observe that the hazard rate of the LTE distribution can be non-increasing, making it suitable for modeling datasets that have a non-increasing hazard rate. The hazard function of the LTW distribution can be both increasing and decreasing for different choices of the parameters (see Fig. 2(b)). Furthermore, we observe that the proposed LTE distribution outperforms the exponential, KME, and DUS$_E$ [6] distributions in terms of AIC and BIC when fitted to the exceedance of the flood peaks of the Wheaton River dataset. The proposed LTW distribution outperforms the Weibull, KME, DUSE$_E$, and GDUSE [11] distributions in terms of AIC and BIC when fitted to the failure times of 50 component dataset.

In Section 2, we investigate the properties of our proposed transformation for an arbitrary baseline distribution. Lemma 2.1 shows that the transformed df defined in (1) can be expressed as an infinite linear combination of exp-G dfstribution functions. Lemma 2.2 establishes that if the moment-generating function (m.g.f.) of the baseline distribution exists, then the m.g.f. of the transformed distribution with df (1) also exists.

In Section 3.1 and Section 3.2, we introduce two new lifetime models, namely, LTE and LTW distributions, respectively. We obtain several mathematical properties of LTE and LTW distributions, such as survival functions, hazard functions, moments, m.g.f., mean deviation, entropy, and quantile functions. The parameters are estimated using the ML method.

In Section 4.1, we conduct simulation studies to assess the consistency of the parameters involved in the LTE and LTW distributions. The simulation results are reported in Tables 1 and 2 of the Appendix. In Section 4.2, we fit LTE, LTW, exponential, and Weibull distributions and other competing models such as KME, DUS$_E$, and GDUSE to two datasets, viz., the exceedance of the flood peaks of the Wheaton River, and failure times of 50 components using the maximum likelihood estimation method. We assess the goodness-of-fit of each model using the Kolmogorov-Smirnov (KS) test. In Table 5 - 6 (Appendix), we report the ML estimates and other goodness-of-fit measures for each of the fitted models. From Table 5, we observe that the LTE distribution outperforms all the fitted distributions in terms of AIC and BIC scores for the exceedance of the flood peaks dataset. From Table 6, we observe that the LTW outperforms all the other fitted models in terms of AIC and BIC scores for the failure times of component dataset.

2. Properties of the Logarithmic Transformation

In this section, we study mathematical properties of the logarithmic transformation (1) for any given baseline dfstribution function G.

Lemma 2.1. The logarithmic transformation in equation (1) can be represented as

$$F(x) = \sum_{n=1}^{\infty} c_n H_n(x) \tag{4}$$

where c_n is given by (5) and $H_n(x)$ is the df of the exp-G distribution with power parameter n.

Proof Using the series expansion of $\ln(1+x)$ for $x < 1$, equation (1) can be written as

$$F(x) = \frac{1}{1-\ln(e-1)} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{G(x)^n}{n(e-1)^n}.$$

Now, taking

$$c_n = \frac{(-1)^{n+1}}{n(1-\ln(e-1))(e-1)^n} \tag{5}$$

and $H_n(x) = G(x)^n$ we get (4).

Lemma 2.2. If the m.g.f. of the baseline distribution exists, then the m.g.f. of the transformed distribution (1) also exists.

Proof Let Y be a random variable with pdf $g(y)$ and X be the random variable of the transformed distribution with pdf $f(x)$ in equation (2).

$$M_X(t) = \int_{\mathbb{R}} e^{tx} f(x) dx. \tag{6}$$

Using (2), we have

$$M_X(t) = \int_{\mathbb{R}} \frac{e^{tx} g(x)}{(1-\ln(e-1))(e-1+G(x))} dx,$$

since

$$\frac{g(x)}{(1-\ln(e-1))(e-1+G(x))} < \frac{g(x)}{(1-\ln(e-1))},$$

therefore, we have

$$M_X(t) < \int_{\mathbb{R}} \frac{e^{tx} g(x)}{(1-\ln(e-1))} dx$$

which implies

$$M_X(t) < \frac{M_Y(t)}{1-\ln(e-1)}.$$

This shows that if the m.g.f. of Y exists, then the m.g.f. of X also exists.

3. Properties of LTE and LTW Distribution

3.1. LTE Distribution

The exponential distribution has been widely used in reliability and survival analysis because of its simplicity and analytical tractability [8]. The exponential distribution's hazard function is always constant, but this need not be the case in real-life situations. So we try to obtain a new distribution with a non-constant hazard function by using the exponential distribution as the baseline distribution in our newly proposed transformation. We take the df of the exponential distribution $G(x) = 1 - e^{-\lambda x}$, $x > 0$, $\lambda > 0$ in Equation (1), to get a new lifetime distribution. We call this the LT-Exponential ($LTE(\lambda)$) distribution. The df and pdf are respectively defined as follows:

$$F(x) = \frac{1}{1-\ln(e-1)} \ln\left(\frac{e-e^{-\lambda x}}{e-1}\right), \quad x > 0, \quad \lambda > 0, \tag{7}$$

and

$$f(x) = \frac{\lambda e^{-\lambda x}}{[1-\ln(e-1)]\left(e-e^{-\lambda x}\right)}, \quad x > 0, \quad \lambda > 0. \tag{8}$$

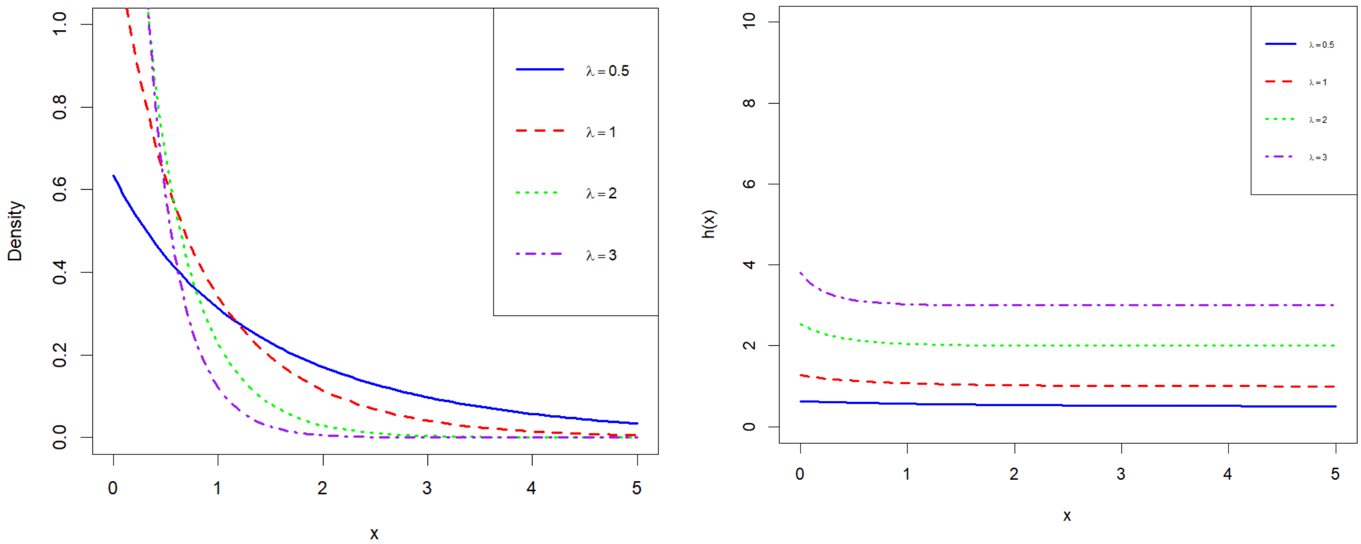
The survival and hazard functions of the $LTE(\lambda)$ are respectively given by

$$S(x) = \frac{1-\ln\left(e-e^{-\lambda x}\right)}{1-\ln(e-1)}, \tag{9}$$

and

$$h(x) = \frac{\lambda e^{-\lambda x}}{\left[1-\ln\left(e-e^{-\lambda x}\right)\right]\left(e-e^{-\lambda x}\right)}. \tag{10}$$

To better understand the possible shapes of the pdf and hazard function of the $LTE(\lambda)$, we plot these graphs for different values of λ in Figs. 1(a) and 1(b). From Fig. 1(b) we see that the hazard function of the $LTE(\lambda)$ distribution is non-increasing.



(a) Plot of the pdf of the $LTE(\lambda)$ for different values of λ

(b) Plot of the hazard function of the $LTE(\lambda)$ for different values of λ

Figure 1. *Plots of the pdf and hazard function of $LTE(\lambda)$ for different values of λ.*

We now discuss the mathematical properties of the LTE distribution.

3.1.1. Moments

Theorem 3.1. Let $X \sim LTE(\lambda)$, $\lambda > 0$. Then, for any $r \in \mathbb{N}$, the rth moment of X exists and can be expressed as

$$E(X^r) = \frac{\lambda}{[1 - \ln(e-1)]} \sum_{m=0}^{\infty} \frac{r!}{e^{m+1}(\lambda + \lambda m)^{r+1}}.$$

Proof We begin by examining the existence of the moments X. For any $r \in \mathbb{N}$, we have

$$\begin{aligned} E(X^r) &\leq \frac{1}{[1 - \ln(e-1)]} \int_0^{\infty} \lambda x^r e^{-\lambda x} dx \\ &\leq \frac{r!}{\lambda^r [1 - \ln(e-1)]}. \end{aligned}$$

Thus, the rth moment of X exists for any $r \in \mathbb{N}$, and it can be expressed as follows:

$$\begin{aligned} E(X^r) &= \frac{\lambda}{[1 - \ln(e-1)]} \int_0^{\infty} \frac{x^r e^{-\lambda x}}{(e - e^{-\lambda x})} dx \\ &= \frac{\lambda}{[1 - \ln(e-1)]\, e} \int_0^{\infty} \frac{x^r e^{-\lambda x}}{\left(1 - e^{-(\lambda x + 1)}\right)} dx. \end{aligned}$$

Expanding $1/\left(1 - e^{-(\lambda x+1)}\right)$, we get

$$\begin{aligned} E(X^r) &= \frac{\lambda}{[1 - \ln(e-1)]\, e} \int_0^{\infty} x^r e^{-\lambda x} \sum_{m=0}^{\infty} e^{-(\lambda m x + m)} dx \\ &= \frac{\lambda}{[1 - \ln(e-1)]\, e} \sum_{m=0}^{\infty} e^{-m} \int_0^{\infty} x^r e^{-(\lambda + \lambda m)x} dx, \end{aligned}$$

which reduces to

$$E(X^r) = \frac{\lambda}{[1 - \ln(e-1)]} \sum_{m=0}^{\infty} \frac{r!}{e^{m+1}(\lambda + \lambda m)^{r+1}}.$$

3.1.2. Moment Generating Function

Theorem 3.2. Let $X \sim LTE(\lambda)$, $\lambda > 0$. Then, the m.g.f. of X exists for any $\lambda > 0$, and it can be expressed as

$$M_X(t) = \frac{\lambda}{[1 - \ln(e-1)]} \sum_{m=0}^{\infty} \frac{1}{e^{m+1}(\lambda m + \lambda - t)}, \quad t < \lambda.$$

Proof Since the m.g.f. of the exponential distribution exists for any $\lambda > 0$, therefore by Lemma 2.2 the m.g.f. of $LTE(\lambda)$

also exists. For $t < \lambda$, the m.g.f. of X can be expressed as

$$\begin{aligned} M_X(t) &= \frac{\lambda}{[1-\ln(e-1)]}\int_0^\infty \frac{e^{tx}e^{-\lambda x}}{(e-e^{-\lambda x})}dx \\ &= \frac{\lambda}{[1-\ln(e-1)]\,e}\int_0^\infty e^{tx}e^{-\lambda x}\sum_{m=0}^{\infty} e^{-(\lambda m x+m)}dx \\ &= \frac{\lambda}{[1-\ln(e-1)]}\sum_{m=0}^{\infty}\frac{1}{e^{m+1}}\int_0^\infty e^{-(\lambda m+\lambda-t)x}dx \\ &= \frac{\lambda}{[1-\ln(e-1)]}\sum_{m=0}^{\infty}\frac{1}{e^{m+1}(\lambda m+\lambda-t)} \end{aligned}$$

3.1.3. Mean Deviation

The mean deviation about the mean is defined as

$$\begin{aligned} E(|X-\mu|) &= \int_0^\infty |x-\mu| f(x)dx \\ &= \int_0^\mu (\mu-x)f(x)dx + \int_\mu^\infty (x-\mu)f(x)dx \end{aligned}$$

where μ is the mean. Simplifying, we obtain

$$E(|X-\mu|) = 2\mu F(\mu) - 2\mu + 2\int_\mu^\infty x f(x)dx.$$

Now, for the $LTE(\lambda)$ with pdf (3.1) we compute the following integral

$$\begin{aligned} \int_\mu^\infty x f(x) &= \frac{\lambda}{[1-\ln(e-1)]}\int_\mu^\infty \frac{xe^{-\lambda x}}{(e-e^{-\lambda x})}dx \\ &= \frac{\lambda}{[1-\ln(e-1)]}\sum_{m=0}^{\infty}\frac{1}{e^{m+1}}\int_\mu^\infty xe^{-(\lambda+\lambda m)x}dx. \end{aligned}$$

The complementary incomplete gamma function is defined as $\Gamma(n,x) = \int_x^\infty x^{n-1}e^{-x}$dx, which can also be expressed as $(n-1)!e^{-x}\sum_{k=0}^{n-1}\frac{x^k}{k!}$. Using this result in the above expression, we obtain the mean deviation about the mean of the LTE distribution

$$E(|X-\mu|) = \frac{2\lambda}{[1-\ln(e-1)]}\sum_{m=0}^{\infty}\frac{e^{-(\lambda+\lambda m)\mu}(1+(\lambda+\lambda m)\mu)}{e^{m+1}(\lambda+\lambda m)^2} + 2\mu F(\mu) - 2\mu.$$

3.1.4. Quantile Function

The quantile function of $LTE(\lambda)$ is obtained by inverting (7) as follows

$$Q(p) = -\frac{1}{\lambda}\left[1+\ln\left\{1-\left(\frac{e}{e-1}\right)^{p-1}\right\}\right], \quad 0 \le q \le 1. \tag{11}$$

3.1.5. Rényi Entropy

The Rényi entropy of a random variable X represents a measure of the variation of the uncertainty [13]. The Rényi entropy is defined by

$$I_\gamma(X) = \frac{1}{1-\gamma}\ln\left[\int_{-\infty}^{\infty}(f_X(x))^\gamma dx\right], \ \gamma > 0 \text{ and } \gamma \ne 1. \tag{12}$$

Thus, the Rényi entropy of $X \sim LTE(\lambda)$ is given by

$$I_\lambda(X) = \frac{1}{1-\lambda}\ln\left[\left(\frac{1}{1-\ln(e-1)}\right)^\gamma \int_0^\infty \frac{\lambda^\gamma e^{-\gamma\lambda x}}{(e-e^{-\lambda x})^\gamma}dx\right].$$

Setting $z = e^{-\lambda x}$, the above equation reduces to

$$I_\lambda(X) = \frac{1}{1-\lambda}\ln\left[\left(\frac{1}{1-\ln(e-1)}\right)^\gamma \int_0^1 \frac{\lambda^{\gamma-1}t^{\gamma-1}}{(e-t)^\gamma}dt\right].$$

Now, for $0 < \gamma < 1$, we have

$$I_\lambda(X) = \frac{1}{1-\lambda}\ln\left[(1-\ln(e-1))^{-\gamma}\lambda^{\gamma-1}\Gamma(\gamma)\Gamma(1-\gamma)\right].$$

3.1.6. ML Estimation of the Parameter of LTE Distribution

Let $X_1,\dots,X_n$ be independently and identically distributed (i.i.d.) random variables from the $LTE(\lambda)$ where $\lambda > 0$. The log-likelihood function associated with the random sample is given by

$$L(\lambda) = -n\ln[1-\ln(e-1)] + n\ln(\lambda) \tag{13}$$

$$-\lambda\sum_{i=1}^{n}X_i - \sum_{i=1}^{n}\ln\left(e-e^{-\lambda X_i}\right). \tag{14}$$

It can be checked that the log-likelihood function satisfies the regularity conditions on the parameter space

$$\Omega = \{\lambda \in \mathbb{R} | \lambda > 0\}.$$

Partial derivative of the log-likelihood function with respect to the parameter λ is

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \frac{x_i e^{-\lambda x_i}}{\left(e - e^{-\lambda x_i}\right)}.$$

The estimating equation is obtained by equating the above partial derivative to zero. Since a closed-form solution of the estimating equation cannot be obtained, we maximize the log-likelihood function (13) using the optim function in R.

3.2. LTW Distribution

The Weibull distribution is an important distribution in reliability theory and survival analysis [8]. In this section, we introduce a new distribution, which we refer to as the LTW (LT-Weibull) distribution, by using the df $G(x) = 1 - e^{-(\beta x)^{\alpha}}, x > 0, \alpha, \beta > 0$ of the Weibull distribution in (1). The df and pdf of the $LTW(\alpha, \beta)$ distribution are obtained, respectively, as

$$F(x) = \frac{1}{1 - \ln(e-1)} \ln\left(\frac{e - e^{-(\beta x)^{\alpha}}}{e-1}\right), \quad x > 0, \qquad (15)$$

and

$$f(x) = \frac{\alpha\beta(\beta x)^{\alpha-1} e^{-(\beta x)^{\alpha}}}{(1 - \ln(e-1))\left(e - e^{-(\beta x)^{\alpha}}\right)}, \quad x > 0, \qquad (16)$$

where $\alpha > 0, \beta > 0$.

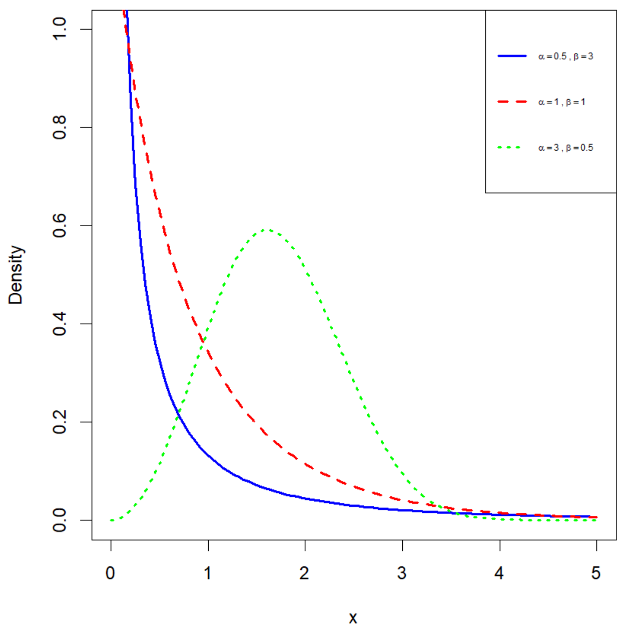
The survival and hazard functions of the $LTW(\alpha, \beta)$ are rsepectively given by

$$S(x) = \frac{\ln\left(e - e^{-(\beta x)^{\alpha}}\right)}{1 - \ln(e-1)}, \qquad (17)$$

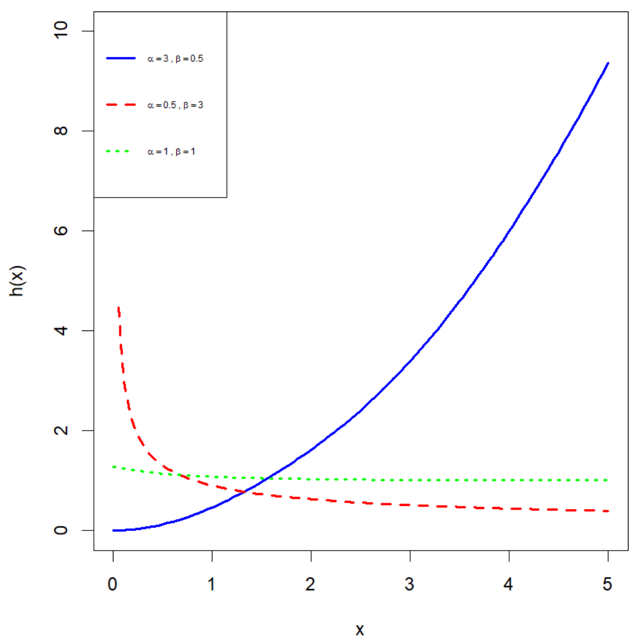
and

$$h(x) = \frac{\alpha\beta(\beta x)^{\alpha-1} e^{-(\beta x)^{\alpha}}}{\left[1 - \ln\left(e - e^{-(\beta x)^{\alpha}}\right)\right]\left(e - e^{-(\beta x)^{\alpha}}\right)}. \qquad (18)$$

The plots of the pdf and hazard functions of the $LTW(\alpha, \beta)$, for different values of α and β, are presented in Figs. 2(a) and 2(b). From Fig. 2(b) we see that the hazard function of the $LTW(\alpha, \beta)$ distribution can be both increasing and decreasing depending on the values of the parameters.



(a) PDF of $LTW(\alpha, \beta)$



(b) Hazard function of $LTW(\alpha, \beta)$

Figure 2. *Comparison of the pdf and hazard function of $LTW(\alpha, \beta)$ for different values of α and β.*

We now discuss mathematical properties of LTW distribution.

3.2.1. Moments

Theorem 3.3. Let $X \sim LTW(\alpha, \beta)$, $\alpha, \beta > 0$. Then, for any $r \in \mathbb{N}$, rth moment of X exist and can be expressed as

$$E(X^r) = \frac{1}{(1 - \ln(e-1))\,\beta^r} \sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{r}{\alpha} - 1\right)}{e^{m+1}(m+1)^{\frac{r}{\alpha}}}.$$

Proof The rth moment of X exist for any $r \in \mathbb{N}$ and it can be expressed as follows

$$\begin{aligned}
E(X^r) &= \frac{\alpha\beta^\alpha}{[1-\ln(e-1)]\,e}\int_0^\infty \frac{x^{r+\alpha-1}e^{-(\beta x)^\alpha}}{\left(1-e^{-(\beta x)^\alpha-1}\right)}dx \\
&= \frac{\alpha\beta^\alpha}{[1-\ln(e-1)]}\sum_{m=0}^{\infty}\frac{1}{e^{m+1}}\int_0^\infty x^{r+\alpha-1}e^{-(m+1)(\beta x)^\alpha}dx \\
&= \frac{1}{(1-\ln(e-1))\,\beta^r}\sum_{m=0}^{\infty}\frac{1}{e^{m+1}(m+1)^{\frac{r}{\alpha}}}\int_0^\infty x^{\frac{r}{\alpha}}e^{-x}dx \\
&= \frac{1}{(1-\ln(e-1))\,\beta^r}\sum_{m=0}^{\infty}\frac{\Gamma\left(\frac{r}{\alpha}-1\right)}{e^{m+1}(m+1)^{\frac{r}{\alpha}}}.
\end{aligned}$$

3.2.2. Moment Generating Function

Theorem 3.4. Let $X \sim LTW(\alpha,\beta)$, $\alpha,\beta > 0$. Then, the m.g.f. function of X exists and can be expressed as

$$M_X(t) = \frac{\beta^{\alpha-1}}{[1-\ln(e-1)]}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{t^n\Gamma\left(\frac{n+1}{\alpha}\right)}{\beta^n e^{m+1} n!(m+1)^{\frac{n+1}{\alpha}}}.$$

Proof Since the m.g.f. of the Weibull distribution exists for any $\alpha,\beta > 0$, therefore by Lemma 2.2, the moment generating function of $LTW(\alpha,\beta)$ also exists. The m.g.f. of X can be expressed as

$$\begin{aligned}
M_X(t) &= \frac{\alpha\beta^\alpha}{[1-\ln(e-1)]\,e}\int_0^\infty \frac{e^{tx}e^{-(\beta x)^\alpha}}{\left(1-e^{-(\beta x)^\alpha-1}\right)}dx \\
&= \frac{\alpha\beta^\alpha}{[1-\ln(e-1)]}\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{t^n}{e^{m+1}n!}\int_0^\infty x^n e^{-(m+1)(\beta x)^\alpha}dx \\
&= \frac{\beta^{\alpha-1}}{[1-\ln(e-1)]}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{t^n\Gamma\left(\frac{n+1}{\alpha}\right)}{\beta^n e^{m+1} n!(m+1)^{\frac{n+1}{\alpha}}}.
\end{aligned}$$

3.2.3. Quantile Function

By inverting (15), we obtain the quantile function $LTW(\alpha,\beta)$ as follows:

$$Q(u) = \frac{1}{\beta}\left[-1-\ln\left\{1-\left(\frac{e}{e-1}\right)^{u-1}\right\}\right]^{\frac{1}{\alpha}}, \quad 0 \le u \le 1. \tag{19}$$

3.2.4. ML Estimation of the Parameters of the LTW Distribution

Let $X_1,\ldots,X_n$ be i.i.d. random variables following the $LTW(\alpha,\beta)$ distribution with parameters $\alpha,\beta > 0$. The log-likelihood function associated with the random sample is given by

$$\begin{aligned}
L(\alpha,\beta) &= -n\ln[1-\ln(e-1)] + n\ln(\alpha\beta) - \sum_{i=1}^{n}(\beta X_i)^\alpha \\
&\quad + (\alpha+1)\sum_{i=1}^{n}\ln(\beta X_i) - \sum_{i=1}^{n}\ln\left(e-e^{-(\beta X_i)^\alpha}\right).
\end{aligned} \tag{20}$$

It can be checked that the log-likelihood function satisfies the regularity conditions on the parameter space

$$\Omega = \{(\alpha,\beta)\in\mathbb{R}^2|\alpha,\beta > 0\}.$$

Partial derivatives of the log-likelihood function with respect to the parameters α and β are respectively

$$\begin{aligned}
\frac{\partial \ln L}{\partial\alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n}\ln(\beta X_i) - \sum_{i=1}^{n}(\beta X_i)^\alpha\ln(\beta X_i) \\
&\quad - \sum_{i=1}^{n}\frac{(\beta X_i)^\alpha\ln(\beta X_i)e^{-(\beta X_i)^\alpha}}{\left(e-e^{-(\beta X_i)^\alpha}\right)}
\end{aligned}$$

and

$$\frac{\partial \ln L}{\partial \beta} = \frac{n}{\beta} + \frac{n(\alpha+1)}{\beta} - \alpha \sum_{i=1}^{n} X_i (\beta X_i)^{\alpha-1} - \sum_{i=1}^{n} \frac{X_i(\beta X_i)^{\alpha-1} e^{-(\beta X_i)^\alpha}}{\left(e - e^{-(\beta X_i)^\alpha}\right)}.$$

The estimating equations are obtained by equating the above partial derivative to zero. Since the closed-form solution of the estimating equations cannot be obtained, we maximize the log-likelihood function (20) using the optim function in R.

4. Simulation Study and Real Data Analysis

4.1. Simulation Study

In this section, we use the Monte Carlo simulation method to evaluate the performance of the maximum likelihood (ML) estimators for the parameter λ in the LTE distribution. Specifically, we employ the quantile function (11) to generate $N = 10,000$ random samples of varying sizes ($n = 50, 100, 500, 700, 1000$) for different values of the true parameter λ (See Table 1).

In Table 1, we present the mean ML estimates ($\widehat{\lambda}$) of the parameter along with the corresponding root mean squared errors (RMSE), which are defined as follows:

$$RMSE(\hat{\lambda}) = \sqrt{E\left(\hat{\lambda} - \lambda\right)^2}. \tag{21}$$

Remark 4.1. The results in Table 1 indicate that the mean estimates of the parameter approach the true parameter value as the sample size n increases. Additionally, the RMSE decreases toward zero with increasing sample size, suggesting strong consistency of the ML estimator for λ.

We apply a similar simulation approach to examine the performance of the ML estimators for the parameters of the $LTW(\alpha, \beta)$ distribution. Here, we utilize the quantile function of $LTW(\alpha, \beta)$, given by equation (19), to generate $N = 10,000$ random samples of different sizes ($n = 50, 100, 500, 700, 1000$) for various values of the true parameters (See Table 2).

Remark 4.2. The results in Table 2 demonstrate that the mean estimates of the parameters converge to their true values as the sample size n increases. Furthermore, the RMSE for each parameter decreases toward zero with increasing sample size, further supporting the strong consistency of the ML estimators for α and β.

4.2. Real Data Analysis

We use the ML estimation method to fit the models to the following two datasets. Kolmogorov-Smirnov (KS) test is used to assess the adequacy of each model's fit. We retain the null hypothesis, indicating that the model adequately fits the data, if the p-value of the KS test exceeds the predetermined significance level of 5 percent. The best-fitted model is selected from among those that adequately fit a specific dataset, based on the lowest AIC and BIC scores. The datasets and their corresponding analyses are as follows:

1. *Dataset I*: This dataset consists of 72 exceedances of flood peaks (in m^3/s) recorded from 1958 to 1984 for the Wheaton River near Carcross in Yukon Territory, Canada. This data is obtained from [2] and is presented in Table 3. Kavya and Manohoran [4] used this dataset in their study and demonstrated that the KME distribution outperforms several other competing models.
   We fit the LTE, exponential, KME, and DUS_E distributions to Dataset I. In Table 5, we report the ML estimates, log-likelihood (LL), AIC, BIC values, and the p-value from the KS test for the fitted distributions. From Table 5, we observe that the all the models adequately fit Dataset I when assessed using the KS test at 5 percent significance level. Among the fitted distributions, the LTE distribution has the lowest AIC and BIC scores. Thus, the LTE distribution is the best-fitted model for Dataset I.
2. *Dataset II*: The second dataset consists of the failure times of 50 components (per 1000 hours). This dataset was obtained from Murthy et al. [12]. Table 4 present this dataset. Sengweni et al. [14] used this dataset in their study.
   We fit Dataset II to the LTW, Weibull, KME, DUS_E, and GDUSE distributions. In Table 6, we report the ML estimates, LL, AIC, BIC values, and the p-value from the KS test for the fitted distributions. From Table 6, we observe that all the models (except DUS_E) adequately fits Dataset II when assessed using the KS test at 5 percent significance level. Among the fitted distributions, the LTW distribution has the lowest AIC and BIC scores. Thus, the LTW distribution is the best-fitted model for Dataset II.

5. Conclusion

In this paper, we introduced a new parsimonious transformation, referred to as the logarithmic transformation, and as special cases, two new lifetime models are obtained, and their mathematical properties are studied. The hazard functions of both the new distributions are seen to be monotonic. We used the ML estimation method to estimate the parameters in the two new lifetime models, and simulations are performed to assess the consistency of ML estimators. Finally, we demonstrated that the two new lifetime models outperform several other competing models when fitted to two real datasets.

Abbreviations

MLE Maximum Likelihood Estimation
M.G.F Moment Generating Function

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Conflicts of Interest

The authors declare that they have no conflict of interest.

Appendix

Table 1. *Simulation results for $LTE(\lambda)$ distribution.*

Parameter	n	$\widehat{\lambda}$	RMSE($\widehat{\lambda}$)	Parameter	n	$\widehat{\lambda}$	RMSE($\widehat{\lambda}$)
	50	0.5114	0.0793		50	1.5317	0.2353
	100	0.5059	0.0545		100	1.5197	0.1639
$\lambda = 0.5$	500	0.5013	0.0241	$\lambda = 1.5$	500	1.5035	0.0712
	700	0.5007	0.0202		700	1.5027	0.0607
	1000	0.5004	0.0167		1000	1.5019	0.0501
	50	1.0199	0.1569		50	2.0467	0.3194
	100	1.0131	0.1095		100	2.0209	0.2185
$\lambda = 1$	500	1.0018	0.0475	$\lambda = 2$	500	2.0028	0.0961
	700	1.0011	0.0405		700	2.0029	0.0801
	1000	1.0014	0.0339		1000	2.0022	0.0674

Table 2. *Simulation results for $LTW(\alpha, \beta)$ distribution.*

Parameters	n	$\widehat{\alpha}$	$\widehat{\beta}$	RMSE($\widehat{\alpha}$)	RMSE($\widehat{\beta}$)
	50	1.5406	0.5065	0.1791	0.0532
	100	1.5202	0.5033	0.1193	0.0368
$\alpha = 1.5, \beta = 0.5$	500	1.5038	0.5008	0.0516	0.0164
	700	1.5024	0.5004	0.0437	0.0135
	1000	1.5019	0.5002	0.0362	0.0114
	50	1.0281	1.0234	0.1164	0.1625
	100	1.0138	1.0110	0.0811	0.1115
$\alpha = 1, \beta = 1$	500	1.0030	1.0025	0.0344	0.0486
	700	1.0018	1.0016	0.0292	0.0407
	1000	1.0010	0.1569	0.0242	0.0345

Table 3. *Dataset I.*

1.7	2.2	14.4	1.1	0.4	20.6	5.3	0.7	1.9	13.0
12.0	9.3	1.4	18.7	8.5	25.5	11.6	14.1	22.1	1.1
2.5	14.4	1.7	37.6	0.6	2.2	39.0	0.3	15.0	11.0
7.3	22.9	1.7	0.1	1.1	0.6	9.0	1.7	7.0	20.1
0.4	2.8	14.1	9.9	10.4	10.7	30.0	3.6	5.6	30.8
13.3	4.2	25.5	3.4	11.9	21.5	27.6	36.4	2.7	64.0
1.5	2.5	27.4	1.0	27.1	20.2	16.8	5.3	9.7	27.5
2.5	27.0								

Table 4. *Dataset II.*

0.036	0.148	0.590	3.076	6.816	0.058	0.183	0.618	3.147	7.896
0.061	0.192	0.645	3.625	7.904	0.074	0.254	0.961	3.704	8.022
0.078	0.262	1.228	3.931	9.337	0.086	0.379	1.600	4.073	10.940
0.102	0.381	2.006	4.393	11.020	0.103	0.538	2.054	4.534	13.880
0.114	0.570	2.804	4.893	14.730	0.116	0.574	3.058	6.274	15.080

Table 5. *Parameter estimates and several goodness-of-fit criteria for Dataset I.*

Distribution	ML Estimates	LL	AIC	BIC	p-value(KS-Test)
LTE	$\hat{\lambda} = 0.0736$	-251.7429	505.4858	507.7624	0.2271
Exp(λ)	$\hat{\lambda} = 0.0819$	-252.1280	506.2559	508.5326	0.1083
KME	$\hat{\lambda} = 0.0632$	-252.0125	506.0250	508.3017	0.3443
DUS$_E(\lambda)$	$\hat{\lambda} = 1$	-254.4682	510.9364	513.2130	0.3677

Table 6. *Parameter estimates and several goodness-of-fit criteria for Dataset II.*

Distribution	ML Estimates	LL	AIC	BIC	p-value(KS-Test)
LTW	$\hat{\alpha} = 0.6941, \hat{\beta} = 0.3282$	-102.2207	208.4414	212.2654	0.4173
Weibull	$\hat{\alpha} = 0.6613, \hat{\beta} = 0.3951$	-102.3643	208.7286	212.5527	0.3645
KME(λ)	$\hat{\lambda} = 0.2454$	-107.7518	217.5037	219.4157	0.0119
DUS$_E$ (λ)	$\hat{\lambda} = 0.3505$	-115.5035	233.0070	234.9190	< 0.05
GDUSE(λ, α)	$\hat{\lambda} = 0.2146, \hat{\alpha} = 0.4239$	-103.3417	210.6834	214.5075	0.1570

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