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Research Article

Exponential-Gamma Exponential (EGE) Distribution and Its Statistical Properties

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Abstract

Researchers have developed various models to analyze and assess lifetime data sets across different fields. The need for more applicable and flexible models is highlighted, and statistical distributions emerge as the key to understanding and interpreting these real-world events. However, in many practical situations, these standard distributions do not adequately fit real-life data. As a result, there is a need to develop and modify distributions to increase their flexibility. Statisticians have responded to this need by proposing new families of distributions that extend well-known standard distributions by adding one or more parameters. In this study, a new continuous probability distribution Exponential-Gamma Exponential (EGE) distribution was introduced and investigated. The model was constructed by compounding the exponential baseline distribution with a gamma-generated transformation in order to improve flexibility in modelling complex lifetime and survival data. Many real-world datasets in reliability engineering, biomedical sciences, finance, and hydrology exhibit skewness, heavy tails, and non-monotonic hazard rate behaviours, which are not adequately captured by classical exponential models. The proposed distribution incorporates additional shape parameters that allow greater control over distributional form and hazard behaviour. The statistical properties of the EGE distribution were derived, including the probability density function, cumulative distribution function, survival function, hazard rate function, quantile function, raw moments, mean, variance, coefficient of variation, skewness, kurtosis, moment generating function, characteristic function, Rényi entropy, and maximum likelihood estimation was used to estimate the parameter of the new distribution. The results show that the proposed model is flexible and extends the classical exponential distribution while maintaining analytical tractability. The model can be useful for analyzing lifetime and reliability data, especially in situations involving varying hazard rates and complex data structures.

Keywords

Exponential-Gamma Exponential Distribution, Lifetime Data Modelling, Survival Analysis, Exponential Distribution, Hazard Rate Function

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1. Introduction

In modern statistical modelling, there has been increasing attention on the development of flexible probability distributions capable of accurately describing complex real-world data. Many practical datasets arising from engineering, biomedical studies, finance, hydrology, and reliability analysis often exhibit skewness, heavy tails, and non-monotonic hazard rate behaviour. Classical distributions such as the exponential and gamma distributions, while mathematically tractable, are often inadequate for capturing these complex data structures [9].

Researchers have used transformation and generator-based methods to create a number of generalized distribution families in order to get around these restrictions. By adding more shape characteristics, these methods greatly increase the versatility of modeling survival and longevity data. Due to its capacity to mimic various hazard rate behaviors and enhance statistical fitting performance, gamma-generated and exponentiated families of distributions in particular have gained a lot of attention [14].

Researcher develop new family of continuous distributions called the Rayleigh-Exponential- Gamma distribution by transforming the newly generated continuous T-X family of distribution called the Exponential-Gamma-X distribution using the traditionally existing Rayleigh distribution as a transformer “X” [3]. Several expressions for the new distribution’s theory and properties were explored and obtained; the maximum likelihood estimation approach was used to estimate the distributions' parameters, and finally, simulations studies were conducted to assess the asymptotic behaviour of the estimates. It is essential to understand that most generalised distributions described in the literature were developed using the generalised transformed transformer (T-X) method. This method was proposed by [2]. Also, [1] showed that this generalization approach is beneficial by transforming the Exponential-Gamma distribution developed by [12] to a family of distribution known as theExponential-Gamma-X.

The exponential distribution remains one of the most fundamental models in reliability theory and survival analysis due to its simplicity and memoryless property. However, in real-world systems where failure rates fluctuate over time, its constant hazard rate function restricts its applicability. In order to give more realistic modeling of lifetime phenomena and failure processes, a number of modified and generalized exponential-type distributions have been proposed [16].

Furthermore, gamma-based extensions of classical distributions have garnered a lot of attention lately. These models are very helpful in reliability and degradation research and offer more flexibility in managing skewed and heavy-tailed data patterns. For instance, gamma-based stochastic deterioration models have been effectively used in time-dependent systems and structural reliability issues [15].

Compound and exponentiated gamma distributions are useful for modeling complex survival data with different hazard rate structures, according to [10]. These advancements show how crucial it is to combine generating methods with baseline distributions in order to improve model performance in real-world applications [5-10]. Exponential -Gamma-Rayleigh Distribution was developed and applied to survival data and it outperformed other traditional existing distributions that compared with Exponential -Gamma-Rayleigh Distribution. [6] Also a new Generator for Odd-Generalized Exponential-Gamma Rayleigh using odd techniques method was developed with the statistical properties of the distribution and was applied to a real life data [11].

Inspired by these developments, this work presents a novel distribution called the Exponential-Gamma Exponential (EGE) distribution, which is created by compounding the exponential baseline distribution with a transformation produced by gamma. When modeling lifetime and reliability data, the suggested model offers more flexibility, particularly in situations where hazard rates are irregular or show intricate patterns. Researchers have developed distribution and expressed the statistical properties of the distributions such [4, 5, 7, 8, 11, 13].

The statistical properties of the proposed model are systematically derived, including the probability density function, cumulative distribution function, survival function, hazard rate function, quantile function, moments, and maximum likelihood estimation procedures. These properties show that the proposed model extends the classical exponential distribution while maintaining mathematical tractability.

2. Baseline Exponential Distribution

The exponential distribution, one of the most basic and popular continuous probability distributions in statistical theory and applied probability, serves as the baseline distribution for the suggested model. Because of its memoryless nature and mathematical simplicity, it is frequently used in lifespan modeling, queuing systems, survival studies, and reliability analysis. The rate parameter, a single positive parameter $\lambda>0$ that controls the distribution's behavior and establishes the average waiting time between events, is what distinguishes the exponential distribution. Because of its simplicity, it is an ideal place to start for building more adaptable generalized distributions.

For a random variable X following the exponential distribution, the cumulative distribution function (CDF) is given by

$$G(x) = 1 - e^{-\lambda x}, x > 0 \tag{1}$$

which represents the probability that the random variable takes a value less than or equal to x. This function increases monotonically from 0 to 1 as x increases, reflecting the increasing accumulation of probability mass over time.

The corresponding probability density function (PDF), which describes the instantaneous rate of occurrence of events, is defined as

$$g(x) = \lambda e^{-\lambda x}, x > 0 \tag{2}$$

Because the density function is strictly decreasing, smaller values of X are more likely than bigger ones. Although this drawback encourages its extension in more sophisticated distributional frameworks like the suggested Exponential-Gamma Exponential model, the exponential distribution also has a constant hazard rate, which makes it especially helpful in modeling systems having a constant failure rate.

2.1. Exponential-Gamma Exponential Distribution

In this section, we introduce a new flexible class of probability distributions termed the Exponential-Gamma Exponential (EGE) distribution. This family was created primarily to increase the flexibility of traditional lifespan models by adding a shape parameter via a transformation method based on generators. The model's increased adaptability enables it to accurately represent a variety of distributional shapes, including skewness, heavy tails, and non-monotonic hazard rate behaviors, which are frequently seen in real-world data from environmental studies, survival analysis, reliability engineering, and finance. The cumulative distribution function (CDF) of a baseline exponential distribution, represented by G(x), was transformed to create the EGE family. Through the use of a positive shape parameter $\alpha > 0$ and a scale parameter $\theta > 0$, this transformation creates a gamma-generated structure that controls the distribution's shape and tail behavior.

Let the baseline cumulative distribution function (CDF) be $G(x)$. We define the transformation

$$T(x) = \frac{G(x)}{1-G(x)} \tag{3}$$

For the exponential distribution,

$$G(x) = 1 - e^{-\lambda x}, x > 0 \tag{4}$$

Then,

$$1 - G(x) = e^{-\lambda x} \tag{5}$$

Substituting equations (4) and (5) into equation (3) gives:

$$T(x) = \frac{1-e^{-\lambda x}}{e^{-\lambda x}} \tag{6}$$

Now simplifying equation (4) gives the transformed variable in equation (5):

$$T(x) = \left(1-e^{-\lambda x}\right)e^{\lambda x} = e^{\lambda x} - 1.$$

$$T(x) = e^{\lambda x} - 1, x > 0 \tag{7}$$

Assume that the transformed variable follows a gamma distribution:

$$T \sim \text{Gamma}(\alpha, \theta), \alpha > 0,\ \theta > 0.$$

The probability density function (PDF) of T is

$$f_T(t) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} t^{\alpha-1} e^{-t/\theta}, t > 0 \tag{8}$$

2.2. Derivation of the Cumulative Distribution Function of the Exponential-Gamma-Exponential Distribution

Cumulative distribution function of X is define as

$$F(x) = P(X \le x) = P(T \le T(x)) \tag{9}$$

Substituting $T(x) = e^{\lambda x} - 1$ from equation, we obtain

$$F(x) = P\left(T \le e^{\lambda x} - 1\right) \tag{10}$$

Since T follows a gamma distribution, its CDF is given in terms of the lower incomplete gamma function:

$$F_T(t) = \frac{1}{\Gamma(\alpha)} \gamma\left(\alpha, \frac{t}{\theta}\right) \tag{11}$$

Now substituting $t = e^{\lambda x} - 1$, we have:

$$F(x) = \frac{1}{\Gamma(\alpha)} \gamma\left(\alpha, \frac{e^{\lambda x}-1}{\theta}\right), x > 0 \tag{12}$$

The lower incomplete gamma function is defined as

$$\gamma(a, z) = \int_0^z t^{a-1} e^{-t}\, dt \tag{13}$$

Hence, the cumulative distribution function of the proposed Exponential-Gamma Exponential (EGE) distribution is

$$F(x) = \frac{1}{\Gamma(\alpha)} \gamma\left(\alpha, \frac{e^{\lambda x}-1}{\theta}\right), x > 0, \theta > 0, \alpha > 0, \lambda > 0 \tag{14}$$

Interpretation: Figure 1 demonstrated that the function, which represents the complete probability space, is monotonically non-decreasing, beginning at 0 and progressively approaching 1. The degree of probability concentration is reflected in the curves' steepness; steeper curves show a quicker build-up of probability mass. For example, the distribution is concentrated at lower values of x as the curve corresponding to $\alpha = 3, \theta = 1, \lambda = 2$ rises more quickly. The more progressive curves, on the other hand, show a more scattered distribution in which probability builds up gradually over a larger range of x.

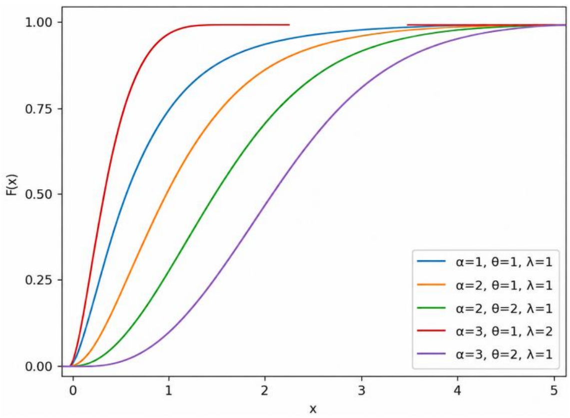

Figure 1*. Cumulative Distribution Function (CDF) of the EGE Distribution.*

2.3. Probability Density Function (PDF) of the EGE Distribution

We now derive the probability density function of the Exponential-Gamma Exponential (EGE) from the previous result. The cumulative distribution function is

$$F(x) = \frac{1}{\Gamma(\alpha)}\gamma\left(\alpha, \frac{e^{\lambda x}-1}{\theta}\right), x > 0. \tag{15}$$

Let

$$u(x) = \frac{e^{\lambda x}-1}{\theta} \tag{16}$$

So the CDF becomes

$$F(x) = \frac{1}{\Gamma(\alpha)}\gamma(\alpha, u(x)) \tag{17}$$

Differentiating the CDF, the PDF is obtained as

$$f(x) = \frac{d}{dx}F(x) \tag{18}$$

Using the known derivative of the lower incomplete gamma function:

$$\frac{d}{dz}\gamma(\alpha, z) = z^{\alpha-1}e^{-z} \tag{19}$$

we apply the chain rule:

$$f(x) = \frac{1}{\Gamma(\alpha)}\cdot\frac{d}{dx}\gamma(\alpha, u(x)) = \frac{1}{\Gamma(\alpha)}\cdot u(x)^{\alpha-1}e^{-u(x)}\cdot\frac{du(x)}{dx} \tag{20}$$

Recall:

$$u(x) = \frac{e^{\lambda x}-1}{\theta} \tag{21}$$

Then

$$\frac{du(x)}{dx} = \frac{\lambda e^{\lambda x}}{\theta} \tag{22}$$

Substitute into the PDF expression

$$f(x) = \frac{1}{\Gamma(\alpha)}\left(\frac{e^{\lambda x}-1}{\theta}\right)^{\alpha-1}\exp\left(-\frac{e^{\lambda x}-1}{\theta}\right)\cdot\frac{\lambda e^{\lambda x}}{\theta} \tag{23}$$

By Simplification

$$f(x) = \frac{\lambda e^{\lambda x}}{\Gamma(\alpha)\theta^{\alpha}}(e^{\lambda x}-1)^{\alpha-1}\exp\exp\left(-\frac{e^{\lambda x}-1}{\theta}\right), x > 0, \alpha > 0, \theta > 0, \lambda > 0 \tag{24}$$

The probability density function of the Exponential-Gamma Exponential (EGE) distribution is

$$f(x;\alpha,\theta,\lambda) = \frac{\lambda e^{\lambda x}}{\Gamma(\alpha)\theta^{\alpha}}(e^{\lambda x}-1)^{\alpha-1}\exp\left(-\frac{e^{\lambda x}-1}{\theta}\right), x > 0, \alpha > 0, \theta > 0, \lambda > 0 \tag{25}$$

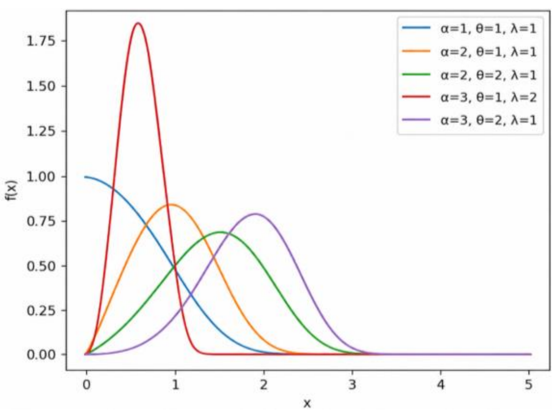

Figure 2*. Probability Density Function (PDF) of the EGE Distribution with Varying Parameters.*

Interpretation: The Exponential-Gamma Exponential (EGE) distribution's Probability Density Function was displayed in Figure 2, illustrating how changes in the parameters α, θ, and λ impact the distribution's shape, spread, and peak. The distribution gradually moves to the right and gets more skewed as α grows, suggesting changes in the failure probabilities' central tendency and dispersion. The density curve's rate of increase and decline is controlled by the scaling factor θ, which affects how quickly the distribution rises to its peak and how slowly it decays toward the tail region.

3. Statistical Properties

The statistical characteristics of the Exponential GammaExponential Distribution are produced in this section, including the first four moments, variance, coefficient of variation, moment generating function, characteristic function, skewness, and kurtosis. A distribution's moments are crucial for characterizing its skewness, kurtosis, central tendency, and dispersion. Specifically, the raw moments offer valuable insights into the general behavior of the random variable.

The r^{th} raw moment for the Exponential-Gamma Exponential (EGE) distribution is obtained as follows. The r^{th} raw moment about the origin is defined as:

$$\mu_r' = E(X^r) = \int_0^\infty x^r f(x)\, dx \tag{26}$$

Substituting the PDF of the EGE distribution:

$$f(x) = \frac{\lambda e^{\lambda x}}{\Gamma(\alpha)\theta^\alpha}(e^{\lambda x}-1)^{\alpha-1}\exp\left(-\frac{e^{\lambda x}-1}{\theta}\right) \tag{27}$$

we obtain:

$$\mu_r' = \frac{\lambda}{\Gamma(\alpha)\theta^\alpha}\int_0^\infty x^r\, e^{\lambda x}(e^{\lambda x}-1)^{\alpha-1}\exp\left(-\frac{e^{\lambda x}-1}{\theta}\right) dx \tag{28}$$

Using the following transformation

$$\text{Let } y = e^{\lambda x} - 1 \tag{29}$$

Then,

$$x = \frac{1}{\lambda}\ln(1+y), dx = \frac{1}{\lambda(1+y)}dy \tag{30}$$

Also note that: $e^{\lambda x} = 1 + y$

Substituting all transformations into the expression for μ_r'

$$\mu_r' = \frac{\lambda}{\Gamma(\alpha)\theta^\alpha}\int_0^\infty \left(\frac{1}{\lambda}\ln(1+y)\right)^r (1+y)y^{\alpha-1}\exp\left(-\frac{y}{\theta}\right)\cdot\frac{1}{\lambda(1+y)}dy \tag{31}$$

Simplifying and collecting like terms gives:

$$\mu_r' = \frac{1}{\lambda^r\Gamma(\alpha)\theta^\alpha}\int_0^\infty [\ln(1+y)]^r y^{\alpha-1}e^{-y/\theta}dy \tag{32}$$

Therefore,

$$\mu_r' = \frac{1}{\lambda^r\Gamma(\alpha)\theta^\alpha}\int_0^\infty [\ln(1+y)]^r y^{\alpha-1}e^{-y/\theta}dy \tag{33}$$

This integral does not admit a closed-form solution in elementary functions and is typically evaluated numerically.

First Four Raw Moments

We now express the first four moments explicitly.

First Moment (Mean)

For $r = 1$

$$\mu_1' = E(X) = \frac{1}{\lambda\Gamma(\alpha)\theta^\alpha}\int_0^\infty \ln(1+y)\; y^{\alpha-1}e^{-y/\theta}dy \tag{34}$$

Second Moment

For $r = 2$

$$\mu_2' = \frac{1}{\lambda^2\Gamma(\alpha)\theta^\alpha}\int_0^\infty [\ln(1+y)]^2 y^{\alpha-1}e^{-y/\theta}dy \tag{35}$$

Third Moment

For $r = 3$

$$\mu_3' = \frac{1}{\lambda^3\Gamma(\alpha)\theta^\alpha}\int_0^\infty [\ln(1+y)]^3 y^{\alpha-1}e^{-y/\theta}dy \tag{36}$$

Fourth Moment

For $r = 4$

$$\mu_4' = \frac{1}{\lambda^4\Gamma(\alpha)\theta^\alpha}\int_0^\infty [\ln(1+y)]^4 y^{\alpha-1}e^{-y/\theta}dy \tag{37}$$

3.1. Mean (First Raw Moment)

The mean of a random variable is the first raw moment about the origin. For the Exponential-Gamma Exponential (EGE) distribution, we start from the general moment expression.

$$E(X) = \mu_1' = \int_0^\infty x f(x)\, dx. \tag{38}$$

Substituting the EGE density:

$$f(x) = \frac{\lambda e^{\lambda x}}{\Gamma(\alpha)\theta^\alpha}(e^{\lambda x}-1)^{\alpha-1}\exp\left(-\frac{e^{\lambda x}-1}{\theta}\right) \tag{39}$$

we obtain:

$$E(X) = \frac{\lambda}{\Gamma(\alpha)\theta^\alpha}\int_0^\infty x\, e^{\lambda x}(e^{\lambda x}-1)^{\alpha-1}\exp\left(-\frac{e^{\lambda x}-1}{\theta}\right) dx \tag{40}$$

Using the following transformations

Let:

$$y = e^{\lambda x} - 1, x = \frac{1}{\lambda}\ln(1+y), dx = \frac{1}{\lambda(1+y)}dy$$

Also: $e^{\lambda x} = 1 + y$

Then we substitute into the expectation integral:

$$E(X) = \frac{\lambda}{\Gamma(\alpha)\theta^\alpha}\int_0^\infty \left(\frac{1}{\lambda}\ln(1+y)\right)(1+y)y^{\alpha-1}e^{-y/\theta}\cdot \frac{1}{\lambda(1+y)}dy \tag{41}$$

By simplification

$$E(X) = \frac{1}{\lambda\Gamma(\alpha)\theta^\alpha}\int_0^\infty \ln(1+y)\; y^{\alpha-1}e^{-y/\theta}dy \tag{42}$$

Standardize the gamma kernel

Let: $y = \theta t \Rightarrow dy = \theta dt$

Substitute:

$$E(X) = \frac{1}{\lambda\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} \ln(1+\theta t)(\theta t)^{\alpha-1} e^{-t} \theta dt \quad (43)$$

By Simplification

$$E(X) = \frac{1}{\lambda\Gamma(\alpha)} \int_0^{\infty} \ln(1+\theta t)\, t^{\alpha-1} e^{-t} dt \quad (44)$$

Expanding the logarithm (series form for final evaluation).

$$\ln(1+\theta t) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (\theta t)^k \quad (45)$$

Substitute:

$$E(X) = \frac{1}{\lambda\Gamma(\alpha)} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\theta^k}{k} \int_0^{\infty} t^{k+\alpha-1} e^{-t} dt \quad (46)$$

Using Gamma identity

$$\int_0^{\infty} t^{m-1} e^{-t} dt = \Gamma(m) \quad (47)$$

So:

$$E(X) = \frac{1}{\lambda} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\theta^k}{k} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \quad (48)$$

3.2. Variance

Variance measures the spread of a distribution around its mean. It is defined as:

$$Var(X) = \mu_2' - (\mu_1')^2 \quad (49)$$

The Second raw moment is given as

$$\mu_2' = E(X^2) = \int_0^{\infty} x^2 f(x)\, dx \quad (50)$$

Substituting the PDF:

$$\mu_2' = \frac{\lambda}{\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} x^2 e^{\lambda x} (e^{\lambda x} - 1)^{\alpha-1} \exp\left(-\frac{e^{\lambda x}-1}{\theta}\right) dx \quad (51)$$

Using: $y = e^{\lambda x} - 1, x = \frac{1}{\lambda}\ln(1+y), dx = \frac{1}{\lambda(1+y)} dy$, we substitute into the integral.

Then

$$\mu_2' = \frac{\lambda}{\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} \left(\frac{1}{\lambda}\ln(1+y)\right)^2 (1+y) y^{\alpha-1} e^{-y/\theta} \cdot \frac{1}{\lambda(1+y)} dy \quad (52)$$

By Simplification

$$\mu_2' = \frac{1}{\lambda^2\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} [\ln(1+y)]^2 y^{\alpha-1} e^{-y/\theta} dy \quad (53)$$

After substitution $y = \theta t$:

$$\mu_2' = \frac{1}{\lambda^2\Gamma(\alpha)} \int_0^{\infty} [\ln(1+\theta t)]^2 t^{\alpha-1} e^{-t} dt \quad (54)$$

Expand square of log

$$[\ln(1+\theta t)]^2 = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^{k+j}}{kj} (\theta t)^{k+j} \quad (55)$$

By Substitution

$$\mu_2' = \frac{1}{\lambda^2\Gamma(\alpha)} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^{k+j}\theta^{k+j}}{kj} \int_0^{\infty} t^{\alpha+k+j-1} e^{-t} dt \quad (56)$$

Apply Gamma function

$$\mu_2' = \frac{1}{\lambda^2} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^{k+j}\theta^{k+j}}{kj} \frac{\Gamma(\alpha+k+j)}{\Gamma(\alpha)} \quad (57)$$

Also

$$\mu_1' = \frac{1}{\lambda} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\theta^k}{k} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \quad (58)$$

$$Var(X) = \mu_2' - (\mu_1')^2 = \frac{1}{\lambda^2\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} [\ln(1+y)]^2 y^{\alpha-1} e^{-\frac{y}{\theta}} dy - \left(\frac{1}{\lambda\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} \ln(1+y)\, y^{\alpha-1} e^{-y/\theta} dy\right)^2$$

$$Var(X) = \mu_2' - (\mu_1')^2 = \frac{1}{\lambda^2} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^{k+j} \theta^{k+j}}{kj} \frac{\Gamma(\alpha+k+j)}{\Gamma(\alpha)} - \left(\frac{1}{\lambda} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \theta^k}{k} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \right)^2 \quad (59)$$

3.3. Moment Generating Function (MGF) of the EGE Distribution

Theorem 3: If X is a continuous random variable distributed as an EGED $(x; \beta, \alpha, \lambda)$, then the moment generating function is given as

The moment generating function (MGF) is defined as:

$$M_X(t) = E(e^{tX})$$

Substitution of the PDF

$$M_X(t) = \int_0^{\infty} e^{tx} f(x)\, dx$$

Substituting the EGE density:

$$M_X(t) = \int_0^{\infty} e^{tx} \frac{\lambda e^{\lambda x}}{\Gamma(\alpha)\theta^{\alpha}} (e^{\lambda x} - 1)^{\alpha-1} \exp\left(-\frac{e^{\lambda x}-1}{\theta}\right) dx \quad (60)$$

$$M_X(t) = \frac{\lambda}{\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} e^{(\lambda+t)x} (e^{\lambda x} - 1)^{\alpha-1} \exp\left(-\frac{e^{\lambda x}-1}{\theta}\right) dx \quad (61)$$

Using the following transformation

Let: $y = e^{\lambda x} - 1 \Rightarrow x = \frac{1}{\lambda} \ln(1+y), dx = \frac{1}{\lambda(1+y)} dy$

Also: $e^{(\lambda+t)x} = (1+y)^{1+t/\lambda}$

Substitute into the integral

$$M_X(t) = \frac{\lambda}{\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} (1+y)^{1+t/\lambda} y^{\alpha-1} e^{-y/\theta} \cdot \frac{1}{\lambda(1+y)} dy \quad (62)$$

By Simplification

$$M_X(t) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} (1+y)^{t/\lambda} y^{\alpha-1} e^{-y/\theta}\, dy \quad (63)$$

Using the generalized binomial expansion:

$$(1+y)^{t/\lambda} = \sum_{k=0}^{\infty} \binom{t/\lambda}{k} y^k \quad (64)$$

Substitute into the integral:

$$M_X(t) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} \sum_{k=0}^{\infty} \binom{t/\lambda}{k} \int_0^{\infty} y^{\alpha+k-1} e^{-y/\theta} dy \quad (65)$$

Using gamma identity:

$$\int_0^{\infty} y^{\alpha+k-1} e^{-y/\theta} dy = \theta^{\alpha+k} \Gamma(\alpha+k)$$

$$M_X(t) = \sum_{k=0}^{\infty} \binom{t/\lambda}{k} \frac{\theta^k \Gamma(\alpha+k)}{\Gamma(\alpha)} \quad (66)$$

3.4. Characteristic Function (CF)

If X is a random variable distributed as an EGED $(x; \beta, \alpha, \lambda)$, then the characteristics function $\phi_x(it)$ is defined as

The characteristic function is defined as:

$$\phi_X(t) = E(e^{itX})$$

Substituting the density function

$$\phi_X(t) = \int_0^{\infty} e^{itx} f(x)\, dx$$

$$\phi_X(t) = \frac{\lambda}{\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} e^{(\lambda+it)x} (e^{\lambda x} - 1)^{\alpha-1} \exp\left(-\frac{e^{\lambda x}-1}{\theta}\right) dx \quad (67)$$

Let: $y = e^{\lambda x} - 1$, Then: $e^{(\lambda+it)x} = (1+y)^{1+it/\lambda}$ and $dx = \frac{1}{\lambda(1+y)} dy$

$$\phi_X(t) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} (1+y)^{it/\lambda} y^{\alpha-1} e^{-y/\theta} dy \quad (68)$$

$$\phi_X(t) = \sum_{k=0}^{\infty} \binom{it/\lambda}{k} \frac{\theta^k \Gamma(\alpha+k)}{\Gamma(\alpha)} \quad (69)$$

3.5. SKEWNESS

Skewness measures the asymmetry of the distribution and is defined as:

$$\gamma_1 = \frac{\mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3}{(\mu_2' - (\mu_1')^2)^{3/2}} \quad (70)$$

Where:

$$\mu_3' = \frac{1}{\lambda^3 \Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} [\ln(1+y)]^3 y^{\alpha-1} e^{-y/\theta} dy \quad (71)$$

$$\mu_3' = \frac{1}{\lambda^3} \sum_{k,j,l \geq 1} \frac{(-1)^{k+j+1} \theta^{k+j+l}}{kjl} \frac{\Gamma(\alpha+k+j+l)}{\Gamma(\alpha)} \quad (72)$$

3.6. Kurtosis

Kurtosis measures the tail heaviness and peakedness of the distribution. It is defined as:

$$\gamma_2 = \frac{\mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2\mu_2' - 3(\mu_1')^4}{(\mu_2' - (\mu_1')^2)^2} \quad (73)$$

Where:

$$\mu_4' = \frac{1}{\lambda^4 \Gamma(\alpha)\theta^\alpha} \int_0^\infty [\ln(1+y)]^4 y^{\alpha-1} e^{-y/\theta} dy \quad (74)$$

3.7. Rényi Entropy

The Rényi entropy of order $(\delta > 0)$, $(\delta \neq 1)$, is

$$I_R(\delta) = \frac{1}{1-\delta} \ln\left(\int_0^\infty f(x)^\delta dx\right) \quad (75)$$

Substituting the density,

$$f(x)^\delta = \left[\frac{\lambda}{\Gamma(\alpha)\theta^\alpha}\right]^\delta e^{\delta\lambda x}(e^{\lambda x}-1)^{\delta(\alpha-1)} \exp\left[-\frac{\delta(e^{\lambda x}-1)}{\theta}\right] \quad (76)$$

Hence,

$$I_R(\delta) = \frac{1}{1-\delta} \ln\left\{\left[\frac{\lambda}{\Gamma(\alpha)\theta^\alpha}\right]^\delta \int_0^\infty e^{\delta\lambda x}(e^{\lambda x} - 1)^{\delta(\alpha-1)} \exp\left[-\frac{\delta(e^{\lambda x}-1)}{\theta}\right] dx\right\}$$

3.8. Quantile Function

The quantile function is used to determine the value of $x$corresponding to a given probability level u, where $U \sim U(0,1)$. It is useful for simulation and statistical inference.

Let

$$F(x) = u.$$

Then,

$$\frac{1}{\Gamma(\alpha)} \gamma\left(\alpha, \frac{e^{\lambda x}-1}{\theta}\right) = u \quad (77)$$

Multiplying both sides by $\Gamma(\alpha)$, we obtain:

$$\gamma\left(\alpha, \frac{e^{\lambda x}-1}{\theta}\right) = u\Gamma(\alpha) \quad (78)$$

Applying the inverse of the lower incomplete gamma function gives:

$$\frac{e^{\lambda x}-1}{\theta} = \gamma^{-1}(\alpha, u\Gamma(\alpha)) \quad (79)$$

Solving for x, we obtain the quantile function:

$$x_u = \frac{1}{\lambda} \ln[1 + \theta\gamma^{-1}(\alpha, u\Gamma(\alpha))] \quad (80)$$

The median of the distribution is obtained by setting $u = 0.5$, i.e., $x_{0.5}$.

3.9. Survival Function

The likelihood that a random variable will surpass a specific value is known as the survival function of a distribution. Because it indicates the likelihood that a system will survive past time x, it is a crucial tool in reliability and survival studies.

For the Exponential-Gamma Exponential (EGE) distribution, the survival function is defined as:

$$S(x) = 1 - F(x) \quad (81)$$

Substituting the cumulative distribution function (CDF), we obtain:

$$S(x) = 1 - \frac{1}{\Gamma(\alpha)} \gamma\left(\alpha, \frac{e^{\lambda x}-1}{\theta}\right) \quad (82)$$

Using the relationship between the lower and upper incomplete gamma functions, the survival function can be equivalently written as:

$$S(x) = \frac{\Gamma\left(\alpha, \frac{e^{\lambda x}-1}{\theta}\right)}{\Gamma(\alpha)} \quad (83)$$

Where $\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt$ is the upper incomplete gamma function.

3.10. Hazard Rate Function

The instantaneous failure rate at time x, given survival up to that point, is described by the hazard rate function. It is frequently used to comprehend failure behavior over time in survival analysis and reliability theory.

The hazard function is defined as:

$$h(x) = \frac{f(x)}{S(x)} \quad (84)$$

Substituting the probability density function (PDF) and survival function of the EGE distribution gives:

$$h(x) = \frac{\lambda e^{\lambda x}(e^{\lambda x}-1)^{\alpha-1} \exp\left(-\frac{e^{\lambda x}-1}{\theta}\right)}{\theta^\alpha \Gamma\left(\alpha, \frac{e^{\lambda x}-1}{\theta}\right)} \quad (85)$$

The hazard function of the EGE distribution is flexible and can take different shapes such as increasing, decreasing, or bathtub-shaped forms depending on the parameter values α, θ, and λ.

The instantaneous failure rate at time x for items that have survived up to that point is represented by the hazard rate function, h(x). Figure 3 illustrates the EGE distribution's considerable flexibility, with hazard curves that, depending on the parameter values, can either be strictly declining or have unimodal (hump-shaped) behavior. While the other curves depicted an early period of rising hazard before gradually dropping, the curve beginning at the highest point correlates to a fast declining failure rate. Because of this feature, the EGE distribution is very useful for simulating different reliability scenarios, including infant mortality or early-stage failure, where the probability of failure varies dynamically over the course of the product's lifetime.

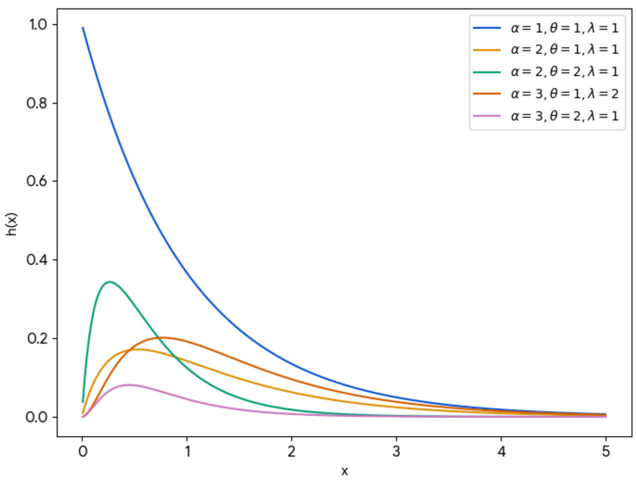

Figure 3*. Hazard Rate Function of the EGE Distribution.*

Maximum Likelihood Estimation

Suppose $x_1, x_2, \ldots, x_n$ is a random sample from the EGE distribution.

The likelihood function is

$$L(\alpha,\theta,\lambda) = \prod_{i=1}^{n} \frac{\lambda e^{\lambda x_i}}{\Gamma(\alpha)\theta^{\alpha}} (e^{\lambda x_i} - 1)^{\alpha-1} \exp\left[-\frac{e^{\lambda x_i - 1}}{\theta}\right] \quad (86)$$

Therefore, the log-likelihood function is

$$\ell = n\ln\lambda + \lambda\sum_{i=1}^{n} x_i - n\ln\Gamma(\alpha) - n\alpha\ln\theta + (\alpha - 1)\sum_{i=1}^{n} \ln(e^{\lambda x_i} - 1) - \frac{1}{\theta}\sum_{i=1}^{n}(e^{\lambda x_i} - 1)$$

The likelihood equations are obtained by differentiating with respect to (α), (θ), and (λ).

Derivative with respect to (α)

$$\frac{\partial \ell}{\partial \alpha} = -n\psi(\alpha) - n\ln\theta + \sum_{i=1}^{n} \ln(e^{\lambda x_i} - 1) = 0, \quad (87)$$

where $\psi(\alpha)$is the digamma function.

Derivative with respect to (θ)

$$\frac{\partial \ell}{\partial \theta} = -\frac{n\alpha}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n}(e^{\lambda x_i} - 1) = 0 \quad (88)$$

Derivative with respect to $(\boldsymbol{\lambda})$

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} x_i + (\alpha - 1)\sum_{i=1}^{n} \frac{x_i e^{\lambda x_i}}{e^{\lambda x_i - 1}} - \frac{1}{\theta}\sum_{i=1}^{n} x_i e^{\lambda x_i} = 0 \quad (89)$$

These equations can be solved numerically using Newton-Raphson or other optimization techniques.

4. Conclusion

In this study, a new flexible probability distribution called the Exponential-Gamma Exponential (EGE) distribution was developed by compounding the exponential baseline distribution with a gamma-generated transformation. The model was motivated by the conventional exponential distribution's inability to handle skewed data, heavy tails, and non-monotonic hazard rate behaviors that are frequently seen in actual lifetime and survival data. The suggested model offers more flexibility while maintaining analytical tractability by adding more form parameters. The Probability density function, Cumulative distribution function, survival and hazard functions, quantile function, raw moments, mean, variance, coefficient of variation, skewness, kurtosis, moment generating function, characteristic function, Rényi entropy, and maximum likelihood estimation were among the statistical characteristics of the distribution that were obtained. The findings demonstrate that the EGE distribution may capture a broad range of data behaviors and generalizes the exponential model.

Some expressions can be efficiently assessed using series expansions and numerical techniques even when they lack closed-form solutions. Lastly, the proposed model is a helpful tool for simulating complicated lifetime and reliability data in hydrology, engineering, biomedical research, and other practical domains.

Author Contributions

Odukoya Elijah Ayooluwa: Methodology,
Ilesanmi Anthony Opeyemi: Conceptualization
Aladejana Ayosunkanmi Emmanuel: Software

Conflicts of Interest

The authors declare no conflicts of interest.

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