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Research Article

Ergodicity of Maps on the Two-Dimensional Torus

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Abstract

This paper considers the ergodicity of maps on the two-dimensional torus, focusing on transformations where invariant real-valued functions are constant. The study considers both additive and multiplicative transformations, providing a detailed analysis of the conditions required for a map to be classified as ergodic. The investigation is grounded in the theory of dynamical systems and leverages mathematical tools such as orthonormal double sequences in Hilbert spaces and Fourier series to establish necessary and sufficient conditions for ergodicity. By connecting these conditions to the Lebesgue measure, the research outlines the fundamental properties of ergodic transformations on the torus. A key aspect of this study is its treatment of invariant functions and their role in defining ergodic behavior. Invariant functions, which remain unchanged under the dynamics of a given transformation, are examined in depth to understand how their constancy relates to the overall system. The analysis also highlights the interplay between additive and multiplicative transformations and their impact on the ergodic properties of the system. The results of this work not only provide a robust framework for understanding the dynamics of transformations on the two-dimensional torus but also have implications for higher-dimensional systems. This contribution is particularly relevant for studying complex systems in ergodic theory, where the behavior of transformations under the Lebesgue measure often serves as a foundation for further exploration. By addressing these fundamental aspects, the paper lays the groundwork for extending these concepts to more intricate and multidimensional settings in mathematical and applied research.

Keywords

Ergodic Theory, Two-Dimensional Torus, Invariant Functions, Fourier Series, Lebesgue Measure

1. Introduction

Ergodic theory (ET) is a dynamic branch of mathematics that investigates the long-term behavior of systems preserving a measure. Its roots trace back to Ludwig Boltzmann in the 19th century, who introduced the term "ergodic" in the context of statistical mechanics to describe the motion of gas particles. The term itself is derived from the Greek words *ergon* (work) and *odos* (path) [1]. Over time, ET has evolved into a versatile framework applied in probability theory, statistical physics, functional analysis, and dynamical systems [2].

At the foundation of ET lies the concept of transformations that preserve a measure. A transformation is deemed ergodic if, when iterated, the system cannot be decomposed into simpler invariant subsystems [3]. A foundational result in this area is the Poincaré Recurrence Theorem, which asserts that almost every point in a measurable subset of the phase space will eventually return to that subset [4]. This theorem provided early insights into the deep connections between geometry, dynamics, and measure theory, setting the

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stage for further developments [5].

A central aspect of ET is its focus on invariant functions—functions that remain constant under the action of a transformation. For a transformation to be ergodic, any invariant real-valued function must be constant almost everywhere. This property is pivotal for understanding the statistical behavior of systems over time and underpins applications in areas such as statistical physics, number theory, and chaos theory [6, 7].

The study of ergodic transformations on the torus has attracted considerable attention due to the torus's intrinsic simplicity and structural richness. Often visualized as a doughnut-shaped surface, the torus serves as a model for periodic and quasi-periodic systems [8]. While transformations on the one-dimensional torus are well-explored, with established conditions for ergodicity [9], extending these analyses to the two-dimensional torus introduces additional challenges. These include proving ergodicity and analyzing invariant functions in higher-dimensional settings.

Motivated by these complexities, this study builds on foundational concepts in ET to explore the ergodic properties of maps on the two-dimensional torus. In particular, it examines cases where every invariant real-valued function is constant, using FS as a central analytical tool. Fourier analysis facilitates the decomposition of functions into orthonormal sequences in Hilbert spaces, enabling systematic comparison of Fourier coefficients to demonstrate ergodicity [10].

This research also underscores the interplay between ergodic transformations, Lebesgue measures (LM), and invariant functions. By leveraging these connections, the study establishes robust conditions under which maps on the two-dimensional torus exhibit ergodic behavior [11]. The findings not only extend classical results in ET but also contribute to understanding higher-dimensional systems, where ergodicity plays a critical role in analyzing dynamical systems [12].

In summary, this work provides a systematic approach to proving ergodicity using FS and invariant measures. The implications extend beyond the two-dimensional torus, offering a foundation for analyzing higher-dimensional transformations and advancing the broader study of ergodic properties in dynamical systems [6].

2. Remakes on Special Torus

Originally, skew-product transformations were considered in the context of the torus. This concept can be traced back to [13], who demonstrated that a skew-product transformation $(x,y)\mapsto(x+\alpha, y+\phi(x))$ on the torus $(\mathbb{T}^2)$, where (α) is irrational, has the same spectral type as the cross-product of the shift transformation on an infinite torus. However, while the induced operators in $(L^2(\mathbb{T}^2))$ are unitarily equivalent, the transformations are not isomorphic.

In the context of [14], the transformation $T(x,y)=(x+\alpha, y)$ is given for α irrational, where y is a measurable function. The condition for non-ergodicity in this case is that there exists an integer n and a measurable map g such that $g(T^n(x,y))=g(x,y)$ almost everywhere. This condition provides a basis for analyzing non-ergodicity in skew-product systems.

The study of skew-product transformations extends naturally to infinite products. For instance, one can consider the map $T(x_1,x_2,\ldots)=(x_2,x_3,\ldots,f(x_1))$, where f represents the skewing function. These transformations are closely linked to representations of groups and serve as valuable tools for understanding the structure of complex dynamical systems [15, 13].

Clotet in [16] examined horocycle flows and showed that they admit measurable reparameterizations that yield irrational flows on the torus. However, such flows are not metrically weakly mixing. Nicol, in [2] established that certain skew-product transformations, such as those involving irrational rotations, are uniquely ergodic for almost every initial point on the torus.

Furthermore, translations on the torus and irrational rotations of the circle play a significant role in proving fundamental theorems related to ergodicity. These results underscore the deep connections between skew-product transformations, ergodic behavior, and the broader framework of dynamical systems.

2.1. Properties of Measure-preserving Dynamical Systems and Topological Groups

(i) Let $X=(0,1)$ and let B be the σ-algebra of all Lebesgue measurable subsets of X. Suppose that μ is $XT = X+\alpha \bmod 1$ where α is irrational. Here X is locally compact (but not compact), however is Hausdorff subspace of the real line $\mathbb{R}$ [17].

(ii) Let G be the circle group, that is, the topological group of all complex numbers Z with absolute value 1 equipped with the normalized Haar measure V on B ($B=$ the ο-algebra of all Borel subset of G) then, the map $T(z)=az\forall z\in G$, where $a\in G$ is not a root of unity [18].

2.2. Proposition in Some Cases on Two-torus [9, 15]

Let $f,g\in L^2(X,B,\mu)$, then $f=g$ holds, and μ is almost everywhere, if and only if, their Fourier coefficients are equal i.e.

$$C_n(f)=C_n(g)\quad \forall n\in\mathbb{Z} \tag{1}$$

Proof:

Proving Ergodicity using FS.

Suppose $f \in L^2(X, B, \mu)$, then we associate with f the FS.

$$f = \sum_{n\to\infty}^{\infty} C_n(f) e^{2\pi inx} \tag{2}$$

where,

$$C_n(f) = \int_0^1 f(x) e^{-2\pi inx} du \tag{3}$$

If we let,

$$S_n(x) = \sum_{l=-n}^{n} C_l(f) e^{2\pi ilx} \tag{4}$$

Then $\|S_n - f\|_2 \to 0$ as $n \to \infty$.

If T is measure preserving transformation it follows that

$$\left.\begin{aligned} \|S_n oT - f oT\|_2 &= \left(\int |S_n oT - f oT|^2\right)^{\frac{1}{2}} \\ &= \left(\int \left(|S - f|_n\right)^2 oTd\mu\right)^{\frac{1}{2}} \\ &= \left(\int \left(|S_n - f|\right)^2 d\mu\right)^{\frac{1}{2}} \\ &= \|S_n - f\|_2 \to 0 \end{aligned}\right\} \tag{5}$$

As $n \to \infty$, it follows that the $\lim_{n\to\infty} S_n oT$ is the possible sum of the form $e^{2\pi inx}$, which gave FS of $f oT$.

In particular if we take the FS for $f(x)$ and evaluate at $T(x)$, then we obtain the FS $f(Tx)$. i.e

$$f oT = f \tag{6}$$

almost everywhere. This prove help us to compare the relationship between Fourier coefficients and show that f is constant [15].

3. Additive Theorem on Two Torus

Let α be irrational and let T be the mapping of the two dimensional tours (T^2) in mod 1 given by the formula.

$$T(x, y) \to (x+\alpha, y+x), \tag{7}$$

which is uniquely ergodic with invariant function in lebsegue measure. [5].

Proof:

Consider an irrational rotation $x \to x+\alpha$ on T as the base and

$$\phi(x) = x \tag{8}$$

We only need to verify that T is ergodic on T^2 with respect to the lebsegue measure.

Let $E \subset T^2$ be some Borel set with

$$M^2\left(E \Delta T^{-1} E\right) = 0 \tag{9}$$

Consider the FS decomposition,

$$\chi_E \in L^2\left(T^2, M^2\right),$$

$$\chi_E = \sum_{(k,l)} C_{k,l} U_{k,l} \tag{10}$$

where

$$U_{k,j(x,y)} = e^{2\pi i(kx+ly)} \tag{11}$$

Taking the transformation of

$$f = f oT \tag{12}$$

$$\left.\begin{aligned} \sum_{(k,l)\in z^2} C_{k,l} U_{k,l} &= \sum_{(q,r)\in z^2} C_{q,r} U_{q+r\bullet r} oT \\ &= \sum_{(q,r)\in z^2} C_{q,r} U_{q,r}(x+\alpha, y+x) \sum_{(k,l)} C_{q,r} e^{2\pi i\left[q(x+\alpha)+(y+x)\right]} \\ &= \sum_{(q,r)\in z^2} C_{q,r} e^{2\pi i[qx+q\alpha+ry+rx]} \\ &= \sum_{(q,r)\in z^2} C_{q,r} e^{2\pi i\left[x(q+r)+ry\right]} e^{2\pi iq\alpha} \end{aligned}\right\} \tag{13}$$

$$= \sum_{(q,r)\in z^2} C_{q,r} U_{q+r\bullet r} e^{2\pi i q\alpha} \tag{14}$$

$$\left.\begin{aligned} q &= k-l, l = r, \\ q+r &= k-l+i = k. \end{aligned}\right\} \tag{15}$$

Comparing the coefficient of (10) and (14), we have the identities.

$$C_{k,j} = C_{k,j} e^{2\pi i(k-l)\alpha} \tag{16}$$

In particular,

$$\left|C_{k,l}\right| = \left|C_{k-l,l}\right| = \left|C_{k-2l,l}\right| \tag{17}$$

Since, $C_{k,l}$ is the square sum on Z^2 then $C_{k,l} = 0$, whenever $l \neq 0$.

For $l = 0$, we have

$$C_{k,0} = e^{2\pi i k\alpha} C_{k,0} \tag{18}$$

As,

$e^{2\pi i k\alpha} \neq 1 \ \forall\ k \neq 0$,

and we conclude that $C_{k,l=0}$ for all $(k,l) \neq (0,0)$.

Thus χ_F is an arbitrary constant on T^2, which shows that it is an ergodic.

3.1. Lemma in $L^2(m)$ Orthonormal Double Sequence

Let $\left\{U^P, V^q : |U| = |V| = 1\right\}_{(p,q\in Z\times Z)}$ and $\left\{e_{\{m,n\}}\right\}$ be a complete orthonormal double sequence in Hilbert space H, $\| x \| = \left(\sum_{m,n} | x_{m,n} |^2\right)^{1/2}$, where $//x//$ is the norm induced by the inner product, and $x_{\{m,n\}}$ are the coefficients corresponding to the orthonormal basis $\left\{e_{\{m,n\}}\right\}$ [18].

Proof:

$$\text{Let, } U^p = e^{ipt} \text{ and } V^q = e^{iqs} \tag{19}$$

$$\left.\begin{aligned} U^p.V^q &= e^{i(pt+qs)} \\ &= \int_0^{2\pi}\int_0^{2\pi} e^{i(pt+qs)} dtds \\ &= \int_0^{2\pi}\int_0^{2\pi} e^{iqs} dtds \\ &= \int_0^{2\pi} e^{iqs}\left[\frac{e^{ipt}}{ip} - \frac{e^0}{ip}\right] ds \\ &= \int_0^{2\pi} e^{iqs}\left[\frac{e^{i2\pi p}}{ip} - \frac{e^0}{ip}\right] ds \\ &= \int_0^{2\pi} e^{iqs}\left[\frac{1}{ip} - \frac{1}{ip}\right] ds \\ &= 0 \end{aligned}\right\} \tag{20}$$

This completes the proof.

3.2. Multiplicative Theorem in $L^2(m)$ That Are Ergodic

Let the following be defined:

$$T_{\alpha,\beta}(U,V) = \left(Ue^{2\pi i\alpha}, Ve^{2\pi i(\beta+\theta)}\right) \tag{21}$$

If $\theta = \arg u$, where α, β are irrational real constants, for all complex U,V, then

$|U| = |V| = 1$ is ergodic [2].

Proof:

Now, with reference to [18], let $\{e_n\}$ be a complete orthonormal double sequence in Hilbert space H in Hilbert space $L^2(m)$. If f is a function in H, i.e $f \in L^2(m)$, then

$$f(u,v) = \sum_{(p,q)\in z\times z} a_{(p,q)\in z\times z} U^p V^q, \tag{22}$$

holds almost everywhere m for some constants $a_{(p,q)}$, called the Fourier coefficient with respect to orthonormal sequence

$$a_{(p,q)} = \left(f, \left(e_{(p,q)}\right)\right) \text{ in } L^2(m)$$

where,

$$e_{(p,q)}(U,V) = U^p V^q \quad \forall\ (U,V) \in G\times G \tag{23}$$

Remark: G is a circle group. Hence,

$$\left.\begin{aligned} {}_{(p,q)} &= \int_{G\times G} f(U,V)\overline{e}_{(p,q)}dm \\ &= \int_{G\times G} f(U,V)U^{-P}V^{-q}dm \end{aligned}\right\}, \tag{24}$$

where,

$$\overline{U^P} = \overline{e^{pi\ \arg u}} = U^{-P}.$$

Similarly,

$$\overline{V^q} = V^{-q}.$$

Next,

Suppose that f is T-invariant function in $L^2(m)$, then

$$f\,oT(U,V) = (f(U,V)) \tag{25}$$

$$f(U.a.V.b_\theta) = \sum_{(p,q)\in z\times z} a_{(p,q)}(Ua)^p.(Vb_\theta)^q, \tag{26}$$

holds in $L^2(m)$.

Where $a = e^{2\pi i\alpha}$ and $b_\theta = e^{2\pi i(\beta+\theta)}$.

If $\theta = \arg u$, (α and β irrational).

Comparing the coefficient of (22) and (26) we get the identities

$$a_{(p,q)}(a)^p(b_\theta)^q.$$

So,

$$a_{(p,q)=0}\ \forall\ (p,q) \neq (0,0).$$

This implies that $p\alpha$ is not an integer, except $p=0$ and $q(\beta+\theta)$ is not an integer except q=0. Then for $\theta = -\beta$. Thus

$$f(u,v) = a_{(0,0)}\ ,$$

holds almost everywhere on m in $G\times G$.

4. Concluding Remarks

This study has explored the ergodic properties of maps on the two-dimensional torus, focusing on cases where every invariant real-valued function is constant. The analysis demonstrated that such ergodic behavior occurs under specific conditions, such as the presence of irrational rotation parameters. These results build upon foundational contributions by [5, 19]. The findings offer valuable insights into the interplay between ergodicity, the Lebesgue measure, and transformations on the torus. By leveraging FS and systematically comparing Fourier coefficients, this research presents a robust and effective method for establishing ergodicity in two-dimensional systems.

In conclusion, the results presented in this work have broader implications for extending ET to higher-dimensional tori. The methodologies and principles outlined here provide a systematic framework for analyzing measure-preserving transformations in increasingly complex dynamical systems. This contribution deepens our understanding of invariant functions and their role in ergodicity, setting the stage for further investigations into multidimensional dynamical systems and their ergodic properties.

Abbreviations

ET	Ergodic Theory
FS	Fourier Series
LM	Lebesgue Measure

Author Contributions

George Smart Nduka: Conceptualization, Data curation, Formal Analysis, Methodology, Resources, Supervision, Writing – original draft

Henry Etaroghene Egbogho: Methodology, Project administration, Validation, Writing – review & editing

Conflicts of Interest

The authors declare no conflicts of interest.

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