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Research Article

The *L(j, k)*-Labeling Number and Circular *L(j,k)*-Labeling Number of Distance Graph $D_n(1,5)$

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Abstract

For $j \leq k$, the *L(j, k)*- labeling problem arose form the code assignment problem of the wireless network. That is, let n,j,k be non-negative real numbers with $j \leq k$, an n-*L(j, k)*-labeling of a graph G is mapping f: $V(G) \rightarrow [0, n]$ such that $|f(u)-f(v)| \geq j$ if $d(u, v)=1$, and $|f(u)-f(v)| \geq k$ if t $d(u, v)=2$. The span of f is the difference between the maximum and minimum labeling numbers assigned by *f*. The *L(j, k)*-labeling number of graph G, denoted by $\lambda_{(j,k)}$ (G), is the minimum span of all *L(j, k)*-labeling of G. The infinite distance graph, denoted by D $(d_1,d_2,\ldots,d_k)$, has the set Z of integers as a vertex set and in which two vertices $i,j \in Z$ are adjacent if and only if $|i-j| \in D$. The finite distance graph, denoted by D_n $(d_1,d_2,\ldots,d_k)$, is the subgraph of $D(d_1,d_2,\ldots,d_k)$ induced by vertices $\{0,1,\ldots,n-1\}$. This paper determines the *L(j, k)*-labeling number and the circular *L(j, k)*-labeling number of distance graph D_n (1,5) for $2j \leq k$.

Keywords

The *L*(*j*, *k*)-labeling Number, The Circular *L*(*j*, *k*)-labeling Number, Code Assignment, Distance Graph

1. Introduction

For $j \leq k$, the $L(j,k)$-labeling, first introduced by Yeh [1], arose from the code assignment of wireless network proposed by Bertossi and Bonuccelli [2]. A vertex and its label will represent a station and its corresponding code, respectively. If two stations can transmit to each other directly, their corresponding vertices are connected by an edge, for convenience, these two stations are called *adjacent* stations, and the two stations are at distance two if they are adjacent to a common station. To avoid the interferences in the wireless network, labels of vertices of a graph should satisfy some conditions. For example, a *Packet Radio Network* using CDMA has two types of interference: *Direct* interference and *Secondary* interference. Direct interference is caused by the adjacent stations, and stations at distance two cause the Secondary interference and direct interference simultaneously. In order to avoid these two types of interferences, the adjacent vertices are assigned labels whose difference is at least j and vertices at distance two are assigned labels whose difference is at least k, where $j \leq k$.

Let n, j, k be non-negative real numbers with $j \leq k$, an n-$L(j,k)$-labeling of a graph G is mapping $f: V(G) \rightarrow [0,n]$ such that $|f(u)-f(v)| \geq j$ if $d(u, v)=1$, and $|f(u)-f(v)| \geq k$ if t $d(u, v)=2$. The span of f is the difference between the maximum and minimum labels assigned by f. The $L(j,k)$-labeling number of graph G, denoted by $\lambda_{j,k}(G)$, is the minimum span of all $L(j,k)$-labelings of G.

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$L(j,k)$-labeling numbers of some graphs for $j \le k$ have been introduced in some literatures. For instance, Wu, Shiu and Sun [3] investigated the $L(j,k)$-labeling number of cartesian product graph of a path and a cycle, Wu also studied on $L(j,k)$-labeling number of the strong product graph of path and cycle [4], wheel graph and Cactus graph [5, 6]. Guo and Wu [7] determined the $L(j,k)$-labeling numbers of book graph. Rao et al. studied the $L(j,k)$-labeling numbers of cartesian product graph of arbitrary length of three paths with $k > j$ in reference [8].

Heuvel et al. [9] first introduced the circular $L(j,k)$-labeling of a graph. They used the “circular distance” to instead of the “linear distance”, where the *linear distance* of two point a and b on the line is defined as $|a-b|$, and the *circular distance* of two points a and b on the cycle with circumference σ is defined as $|a-b|_\sigma = min\{|a-b|, \sigma - |a-b|\}$.

Let σ be non-negative real numbers, a *circular* σ-$L(j,k)$-*labeling* of a graph G is mapping $f: V(G) \to [0,\sigma)$ such that $|f(u)-f(v)|_\sigma \ge j$ if $d(u, v)=1$, and $|f(u)-f(v)|_\sigma \ge k$ if $d(u, v)=2$. The *circular* $L(j,k)$-*labeling number* of G, denoted by $\sigma_{j,k}(G)$, is the minimum σ such that there exists a circular σ-$L(j,k)$-labeling of G. The circular $L(j,k)$-labeling number of graphs have been studied in several articles, please refer to [10-16].

Definition 1.1 [17] Let $D=\{d_1,d_2,\dots,d_k\}$, where $d_i(i=1,2,\dots,k)$ are positive integers such that $d_1 < d_2 < \cdots < d_k$. The (infinite) distance graph $D(d_1,d_2,\dots,d_k)$ has the set Z of integers as a vertex set and in which two vertices $i,j \in Z$ are adjacent if and only if $|i-j| \in D$. The finite distance graph $D_n(d_1,d_2,\dots,d_k)$ is the subgraph of $D(d_1,d_2,\dots,d_k)$ induced by vertices $\{0,1,\dots,n-1\}$.

Lemma 1.1 [3] If graph H is an induced subgraph of graph G, then $\lambda_{j,k}(G) \ge \lambda_{j,k}(H)$.

Lemma 1.2 [3] Let j and k be two positive numbers with $j \le k$. Suppose G is a graph and $H$is an induced subgraph of G. Then $\sigma_{j,k}(G) \ge \sigma_{j,k}(H)$.

Lemma 1.3 [3] Suppose f is an $L(j,k)$-labeling of graph G. Let

$$\sigma = max\{max_{d_G(u,v)=2}\{k+|f(u)-f(v)|\}, max_{d_G(u,v)=1}\{j+|f(u)-f(v)|\}\}.$$

If the image of f lies in $[0,\sigma)$, then f is a circular σ-$L(j,k)$-labeling of G.

2. Main Results

In this section, we mainly study on the $L(j,k)$-labeling numbers and the circular $L(j,k)$-labeling numbers of distance graph $D_n(1,5)$ for $n \ge 7$ and $k \ge 2j$. For $1 \le n \le 5$, the distance graph $D_n(1,5)$ is isomorphic to path P_n, and $D_6(1,5)$ is isomorphic to the cycle C_6. The $L(j,k)$-labeling numbers and the circular $L(j,k)$-labeling numbers of path and cycle have been discussed by Niu [18] and Wu, Shiu and Sun [3], respectively, for $k \ge 2j$. For convenience, let $G = D_n(1,5)$.

Theorem 2.1 Let n be an integer and j,k be non-negative real numbers with $k \ge 2j$. For $7 \le n \le 8$, $\lambda_{j,k}(G) = 3k$.

Proof: Define a labeling function f for the graph G as follows.

$$f(i) = \begin{cases} ik, 0 \le i \le 3, \\ [i+1]_4 k, 4 \le i \le n-1. \end{cases}$$

As Figure 1 shows, it is not difficult to verify that f is an $L(j,k)$-labeling of graph G with $span(f) = 3k$, where that is, $\lambda_{j,k}(G) \le 3k$.

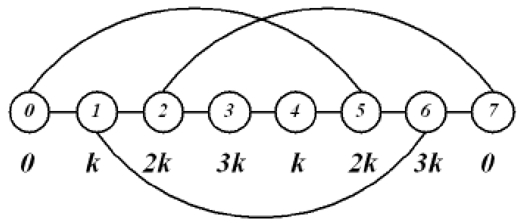

Figure 1. *The labeling f for distance graph $D_8(1,5)$.*

On the other hand, any pair of vertices 0, 2, 4 and 6 are at distance two, according to the definition of $L(j,k)$-labeling, it is at least k for the difference between any two labels of vertices 0, 2, 4 and 6, thus, $\lambda_{j,k}(G) \ge 3k$.

Hence, $\lambda_{j,k}(G) = 3k$.

Theorem 2.2 Let n be an integer and j,k be non-negative real numbers with $k \ge 2j$. For $n \ge 9$, $\lambda_{j,k}(G) = j + 3k$.

Proof: Given a labeling f for graph G as follows.

$$f(i) = [i]_2 j + \left[\left\lfloor \frac{i}{2} \right\rfloor\right]_4 k,\ 0 \le i \le n-1.$$

For example, the labeling f of $D_9(1,5)$ is shown in Figure 2.

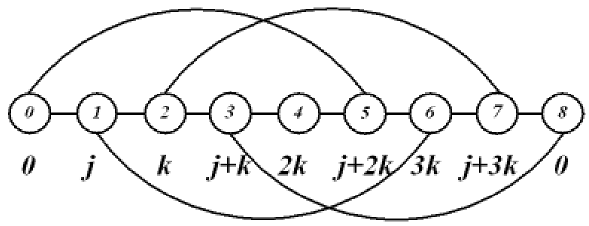

Figure 2. *The labeling f for distance graph $D_9(1,5)$.*

For any vertex i, according to the symmetry of the distance graph, it is needed to verify the labels of vertices $i+1$ and

$i+5$ which are adjacent to vertex i, and the labels of vertices $i+2, i+4, i+6,\ i+10$ which are 2 distance apart from vertex i.

1. The differences between $f(v_{i+1})$ and $f(v_i)$, $f(v_{i+5})$ and $f(v_i)$ are shown as follows.

$$|f(v_{i+1})-f(v_i)| = \left|[i+1]_2 j + \left[\left\lfloor\frac{i+1}{2}\right\rfloor\right]_4 k - [i]_2 j - \left[\left\lfloor\frac{i}{2}\right\rfloor\right]_4 k\right|$$

$$= \left|([i+1]_2-[i]_2)j + \left(\left[\left\lfloor\frac{i+1}{2}\right\rfloor\right]_4 - \left[\left\lfloor\frac{i}{2}\right\rfloor\right]_4\right)k\right|$$

$$= \begin{cases} j, & \text{if } i \text{ is even,} \\ k-j \text{ or } j+3k, & \text{if } i \text{ is odd,} \end{cases} \geq j.$$

$$|f(v_{i+5})-f(v_i)| = \left|[i+5]_2 j + \left[\left\lfloor\frac{i+5}{2}\right\rfloor\right]_4 k - [i]_2 j - \left[\left\lfloor\frac{i}{2}\right\rfloor\right]_4 k\right|$$

$$= \left|([i+5]_2-[i]_2)j + \left(\left[\left\lfloor\frac{i+5}{2}\right\rfloor\right]_4 - \left[\left\lfloor\frac{i}{2}\right\rfloor\right]_4\right)k\right|$$

$$= \begin{cases} j+2k \text{ or } 2k-j, & \text{if } i \text{ is even,} \\ j+k \text{ or } 3k-j, & \text{if } i \text{ is odd,} \end{cases} \geq j.$$

2. The differences between $f(v_{i+2})$ and $f(v_i)$, $f(v_{i+4})$ and $f(v_i)$, $f(v_{i+6})$ and $f(v_i)$, $f(v_{i+10})$ and $f(v_i)$ are shown as follows.

$$|f(v_{i+2})-f(v_i)| = \left|[i+2]_2 j + \left[\left\lfloor\frac{i+2}{2}\right\rfloor\right]_4 k - [i]_2 j - \left[\left\lfloor\frac{i}{2}\right\rfloor\right]_4 k\right|$$

$$= \left|([i+2]_2-[i]_2)j + \left(\left[\left\lfloor\frac{i+2}{2}\right\rfloor\right]_4 - \left[\left\lfloor\frac{i}{2}\right\rfloor\right]_4\right)k\right|$$

$$= k \text{ or } 3k \geq k.$$

$$|f(v_{i+4})-f(v_i)| = \left|[i+4]_2 j + \left[\left\lfloor\frac{i+4}{2}\right\rfloor\right]_4 k - [i]_2 j - \left[\left\lfloor\frac{i}{2}\right\rfloor\right]_4 k\right|$$

$$= \left|([i+4]_2-[i]_2)j + \left(\left[\left\lfloor\frac{i+4}{2}\right\rfloor\right]_4 - \left[\left\lfloor\frac{i}{2}\right\rfloor\right]_4\right)k\right|$$

$$= 2k \geq k.$$

$$|f(v_{i+6})-f(v_i)| = \left|[i+6]_2 j + \left[\left\lfloor\frac{i+6}{2}\right\rfloor\right]_4 k - [i]_2 j - \left[\left\lfloor\frac{i}{2}\right\rfloor\right]_4 k\right|$$

$$= \left|([i+6]_2-[i]_2)j + \left(\left[\left\lfloor\frac{i+6}{2}\right\rfloor\right]_4 - \left[\left\lfloor\frac{i}{2}\right\rfloor\right]_4\right)k\right|$$

$$= k \text{ or } 3k \geq k.$$

$$|f(v_{i+10})-f(v_i)| = \left|[i+10]_2 j + \left[\left\lfloor\frac{i+10}{2}\right\rfloor\right]_4 k - [i]_2 j - \left[\left\lfloor\frac{i}{2}\right\rfloor\right]_4 k\right|$$

$$= \left|([i+10]_2-[i]_2)j + \left(\left[\left\lfloor\frac{i+10}{2}\right\rfloor\right]_4 - \left[\left\lfloor\frac{i}{2}\right\rfloor\right]_4\right)k\right|$$

$$= k \text{ or } 3k \geq k.$$

According to the above verification, f satisfies the definition of $L(j,k)$-labeling, that is, f is an $L(j,k)$-labeling of distance graph G with $span(f) = j+3k$, thus, $\lambda_{j,k}(G) \leq j+3k$.

On the other hand, suppose $\lambda_{j,k}(G) = \lambda < j+3k$. Let f be an $L(j,k)$-labeling of distance graph G, and $I_0 = [0, \lambda-3k]$, $I_1 = [k, \lambda-2k]$, $I_2 = [2k, \lambda-k]$, $I_3 = [3k, \lambda]$. Since vertices 0, 2, 4 and 6 are two distance apart from each other, the differences between any two of $f(0), f(2), f(4), f(6)$ are at least k, that is, $f(0), f(2), f(4), f(6) \in \cup_{i=0}^{3} I_i$. According to the assumption $\lambda < j+3k$, the lengths of interval I_i is less than j, where $i = 0,1,2,3$. For convenience, let $f(0) \in I_a$, $f(2) \in I_b$, $f(4) \in I_c$, $f(6) \in I_d$, where $\{a,b,c,d\} = \{0,1,2,3\}$. Moreover, vertices 2, 4, 6 and 8 are at distances two from each other, then $f(2), f(4), f(6), f(8) \in \cup_{i=0}^{3} I_i$. It implies that $f(8), f(0) \in I_a$. Furthermore, vertices 1, 3, 5 and 7 are also two distances apart mutually, then one of $f(1), f(3), f(5), f(7)$ lies in I_a. However, each of vertices 1, 3, 5 and 7 is adjacent to v_0 or v_8, then $f(1), f(3), f(5), f(7) \notin I_a$ which is a contradiction. Thus $\lambda_{j,k}(G) \geq \mathrm{j}+3k$.

Hence, $\lambda_{j,k}(G) = j+3k$.

Theorem 2.3 Let n be an integer and j, k be non-negative real numbers with $k \geq 2j$. For $n \geq 7$, $\sigma_{j,k}(G) = 4k$.

Proof: Given an labeling f for graph G as follows.

$$f(i) = \left[\frac{i}{2}\right]_4 k,\ 0 \leq i \leq n-1.$$

For any vertex i, according to the symmetry of the distance graph, it is needed to verify the labels of vertices $i+1$ and $i+5$ which are adjacent to vertex i, and the labels of vertices $i+2, i+4, i+6,\ i+10$ which are 2 distance apart from vertex i.

1. The circular distances between $f(v_{i+1})$ and $f(v_i)$, $f(v_{i+5})$ and $f(v_i)$ are shown as follows.

$$|f(v_{i+1})-f(v_i)|_{4k} = \left|\left[\frac{i+1}{2}\right]_4 k - \left[\frac{i}{2}\right]_4 k\right|_{4k}$$

$$= \begin{cases} \left|\frac{k}{2}\right|_{4k} = min\left\{\frac{k}{2}, 4k-\frac{k}{2}\right\} = \frac{k}{2}, \\ \left|\frac{7k}{2}\right|_{4k} = min\left\{\frac{7k}{2}, 4k-\frac{7k}{2}\right\} = \frac{k}{2}, \end{cases} \geq j.$$

$$|f(v_{i+5})-f(v_i)|_{4k}=\left|\left[\frac{i+5}{2}\right]_4 k-\left[\frac{i}{2}\right]_4 k\right|_{4k}$$

$$=\begin{cases}\left|\frac{5k}{2}\right|_{4k}=min\left\{\frac{5k}{2},4k-\frac{5k}{2}\right\}=\frac{3k}{2},\\ \left|\frac{3k}{2}\right|_{4k}=min\left\{\frac{3k}{2},4k-\frac{3k}{2}\right\}=\frac{3k}{2},\end{cases}\geq j.$$

2. The circular distances between $f(v_{i+2})$ and $f(v_i)$, $f(v_{i+4})$ and $f(v_i)$, $f(v_{i+6})$ and $f(v_i)$, $f(v_{i+10})$ and $f(v_i)$ are shown as follows.

$$|f(v_{i+2})-f(v_i)|_{4k}=\left|\left[\frac{i+2}{2}\right]_4 k-\left[\frac{i}{2}\right]_4 k\right|_{4k}$$

$$=\begin{cases}|k|_{4k}=min\{k,4k-k\}=k,\\ |3k|_{4k}=min\{3k,4k-3k\}=k,\end{cases}\geq k.$$

$$|f(v_{i+4})-f(v_i)|_{4k}=\left|\left[\frac{i+4}{2}\right]_4 k-\left[\frac{i}{2}\right]_4 k\right|_{4k}$$

$$=|2k|_{4k}=min\{2k,4k-2k\}=2k\geq k.$$

$$|f(v_{i+6})-f(v_i)|_{4k}=\left|\left[\frac{i+6}{2}\right]_4 k-\left[\frac{i}{2}\right]_4 k\right|_{4k}$$

$$=\begin{cases}|k|_{4k}=min\{k,4k-k\}=k,\\ |3k|_{4k}=min\{3k,4k-3k\}=k,\end{cases}\geq k.$$

$$|f(v_{i+10})-f(v_i)|_{4k}=\left|\left[\frac{i+10}{2}\right]_4 k-\left[\frac{i}{2}\right]_4 k\right|_{4k}$$

$$=\begin{cases}|k|_{4k}=min\{k,4k-k\}=k,\\ |3k|_{4k}=min\{3k,4k-3k\}=k,\end{cases}\geq k.$$

According to the above verification and Lemma 1.3, f is a circular $4k$-$L(j,k)$-labeling of distance graph G, thus, $\sigma_{j,k}(G)\leq 4k$ for $k\geq 2j$.

On the other hand, any pair of vertices 0, 2, 4 and 6 are at distance two, according to the definition of circular $L(j,k)$-labeling, the circular distance between any two labels of vertices 0, 2, 4 and 6 is at least k, thus, $\sigma_{j,k}(G)\geq 4k$.

Hence, $\sigma_{j,k}(G)=4k$ for $k\geq 2j$.

3. Conclusion

In this article, we mainly investigate the $L(j,k)$-labeling numbers and the circular $L(j,k)$-labeling numbers of distance graph $D_n(1,5)$ and obtain the following results.

Let n be an integer and j,k,σ be non-negative real numbers. Then

$$\lambda_{j,k}(G)=\begin{cases}3k, & \text{if } 7\leq n\leq 8.\\ j+3k, & \text{if } n\geq 9.\end{cases}$$

$\sigma_{j,k}(G)=4k$, where $n\geq 7$ and $k\geq 2j$.

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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