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Research Article

# Probability of Ruin in Finite Time Given by a Variable Memory Hawkes Process in the Univariate Case with Brownian Perturbation

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## Abstract

Our article relates to the field of actuarial science, where the analysis of the probability of ruin is a fundamental issue for insurance companies. The stability of reserves is a key factor in ensuring the sustainability of insurance companies, and understanding the mechanisms that influence this risk allows for the optimization of management strategies. The main objective of this study is to establish an expression to calculate the probability of ruin over a finite time horizon. We use the Hawkes process to model the dynamics of claims arrivals, and we introduce Brownian motion at the level of reserve R(t) to incorporate unexpected variations in compensations. By adopting the assumption that the arrival of claims and their amounts, which follow an exponential distribution, are independent. Then, considering the claims modeled by $\alpha$-stable distribution. The key ideas developed in this article are based on several aspects: The Hawkes process is used to describe the frequency of claims, taking into account the impact of past events on the future dynamics of losses. A stochastic oscillation (Brownian motion) is integrated into the model to reflect variations in the financial reserve. With the previous elements, a mathematical expression for the probability of ruin in a finite time is formulated to assess the level of risk that a reserve faces over a given period.

## Keywords

Hawkes Process, α-stable Distribution, Finite Time Ruin Probability, Brownian Motion, Claim

## 1. Introduction

In the field of insurance, ruin theory aims to mathematically analyze the stochastic fluctuations inherent in insurers' financial computations. Risk is defined as the probability that a company's reserve becomes negative at some point. While multiple risk measures exist, the probability of ruin remains one of the most compelling metrics to examine. The finite-horizon ruin probability constitutes a fundamental concept in risk theory, particularly pertinent to insurance firms. It quantifies the likelihood that an insurer will face insolvency within a predefined time frame, due to claims exceeding its financial reserves. Unlike the ruin probability over an infinite horizon, which considers an indefinite period, the finite-horizon perspective focuses on a specific duration, such as one year or more.

The studies referenced in [5-8] concentrated on the approximation of risk probability over a finite time horizon. Their approach is founded on modeling the reserve available at time t through a process incorporating Brownian perturba-

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tions. Additionally, they assume that claim costs are independent and identically distributed.

In this article, we focus on a risk model that specifically examines the probability of ruin over a finite time horizon under Brownian perturbations [3]. Our objective is to derive an expression for the ruin probability within this finite horizon, wherein claim inter-arrivals follow a variable memory Hawkes process [1], coupled with an exponential distribution of claim amounts and Brownian oscillations affecting the reserve level R(t). This perspective proves particularly relevant, as claim settlements are often not entirely resolved, and the risk of unforeseen claims exerting a substantial impact on a company's financial stability remains a significant concern.

The reserve model [9, 10] that we use is:

$$R(t) = u + ct - \sum_{i=1}^{N(t)} X_i + \sigma w(t) \tag{1}$$

Where $u \geq 0$ is the initial reserve, $r$ is the interest rate, $c \geq 0$ the premium rate, $\sigma$ the volatility factor and $\{w_t; t \geq 0\}$ is the diffusion disturbance which is a standard Brownian motion.

We also draw inspiration from the studies conducted in [11-17] to effectively advance our research. Initially, we shall revisit the fundamental elements essential for the progression of our work. Subsequently, we will present the findings obtained within the scope of this study. Naturally, we shall conclude with a comprehensive summary of our results and their implications.

## 2. Preliminaries

Definition 2.1 (Moment of Ruin):

The time of risk noted $\tau$ associated with an initial reserve u is defined by:

$$\tau = inf\{t \geq 0; R(t) < 0\} \tag{2}$$

This is the first moment when the reserve process becomes negative. The univariate reserve model with Brownian disturbance that we use is defined by:

$$R(t) = u + ct - \sum_{i=1}^{N(t)} X_i + \sigma \int_0^t e^{r(t-s)} ds \tag{3}$$

The ruin hypotheses of the risk model are as follows:

1) $u \geq 0$ is the initial reserve of the insurance company.
2) $c > 0$ is the contribution rate or the premium.
3) The $\{X_i, i \in \mathbb{N}^*\}$represent the amount of the company's claims (compensation) or the amount spent on claims and independent of the process $(N(t))$. $F_X$ designates their common distribution function, $f_X$ their density. $(X_i)_{\, i\geq 1}$represents the amount of the $i^{eme}$ claim and has an exponential distribution of parameter $\delta$.
4) The $\{t_i, i \in \mathbb{N}^*\}$The arrival times of the claims play an important role in calculating the probability of ruin.
5) $\{B_t; t \geq 0\}$ is the diffusion disturbance which is a standard Brownian motion.

Definition 2.2 (Finite time probability):

The probability of ruin at a finite horizon is therefore defined by:

$$\psi(u, T) = \mathbb{P}(\inf_{1\leq t\leq T}\{R(t) < 0; R(0) = u\}) \tag{4}$$

Definition 2.3 (Hawkes process):

The arrival time laws of a Hawkes process can be established using the intensity function $\lambda(t)$ of the Hawkes process. The intensity function $\lambda(t)$ is a function that describes the infinitesimal probability of an arrival given the history of previous events.

$$\lambda(t) = \lambda + \int_0^t \mu(t - t_i) dN(s) \tag{5}$$

When the event of interest occurs, the intensity of the process is modified by the function μ. In a way, this function can be interpreted as a response to the jump in the process. Its introduction in the expression of intensity allows extending the possibility of modeling by point processes to a large number of random phenomena. The function μ can be increasing or decreasing. In this article, we consider the decreasing exponential function μ defined by $\forall t > 0$:

$$\mu(t) = \alpha e^{-\beta t}$$

For more details (see in [1]).

Definition 2.4 (standard Brownian motion):

The process $\{B_t; t \geq 0\}$ taking values in $\mathbb{R}^d$ is called standard Brownian motion if:

1) for all $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables $B_{t_i} - B_{t_{i-1}}$ are independent (independent increments).
2) for all $i \geq 1$, the increment $B_{t_i} - B_{t_{i-1}}$ has a Gaussian distribution in $\mathbb{R}^d$ with mean zero and covariance matrix $cov(t_i - t_{i-1})$.

Definition 2.5 ($\alpha$-stable law):

A random variable X is said to be distributed according to a stable law if and only if there exist four unique parameters: $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, $\gamma > 0$ and $\mu \in \mathbb{R}$ such that the characteristic function of X, denoted by $\phi(t) = exp(-\gamma|t|^\alpha)$. For more details (see in [4]). We can cite as an example the Hawkes process which encompasses four unique parameters: α; λ; β and μ.

Lemma 2.1:

Let $l \in R_0$ bounded in $[x_0, \infty]$ for $x_0 \geq 0$. Then

$$\int_{x_0}^x t^{-\beta} l(t) dt \sim (1-\beta)^{-1} x^{-\beta+1} l(x)$$

when $x \to \infty$ with $0 < \beta < 1$

$$\int_{x_0}^x t^{-\beta} l(t) dt \sim (\beta - 1)^{-1} x^{-\beta+1} l(x)$$

when $x \to \infty$ with $1 < \beta$

Proposition 2.6 (Wiener Integral):

Let them be $T \in \mathbb{R}_+$ and $(B_t)_{t\in\mathbb{R}_+}$ a standard one-dimensional Brownian motion.

1) For everything $(a,b) \in {\mathbb{R}_+}^2$ as such $a \le b$, so, $\int_a^b dB_s = B_b - B_a$.
2) If $f: [0,T] \to \mathbb{R}$ is a Borel function such that $\int_0^T |f(s)|^2 ds < +\infty$, then $\int_0^T f(s) dB_s$ is well defined and is a centered Gaussian random variable with variance $\int_0^T |f(s)|^2 ds$.
3) If $f, g: [0,T] \to \mathbb{R}$ are two Borel functions of square integrable over [0, T], then $\mathbb{E}\left(\int_0^T f(s)dB_s \int_0^T g(s)dB_s\right) = \int_0^T f(s)g(s)dB_s$

Theorem 2.7

Consider a non-standard renewal Risk model with an inter-arrival time $\{\theta_k; k \ge 0\}$ with a common Erlang distribution H and let $(X_i)_{i=1;\dots;n}$ and $(Y_i)_{i=1;\dots;n}$ be iid random variables with respective distributions F and G. Suppose F is an α-stable distribution and G is a β-stable distribution with $0 < \alpha$ ; $0 < \beta$. Let $T \in \Lambda$, then the probability of ruin in finite time is:

$$\psi(u,T) \sim a(T)\bar{F}(u) + b(T)\bar{G}(u) \tag{6}$$

We draw inspiration from these results to assess the probability of univariate ruin in finite time, assuming independence between the inter-arrival of claims and the amount of claims. Then considering the law of claims modeled by an α-stable distribution.

## 3. Results

In this section, we present the results obtained in the context of this article, which among other things expresses the ruin probability at a finite horizon, where the inter-arrivals of claims follow a Hawkes process with variable memory and an exponential distribution of the claim amounts accompanied by a Brownian oscillation at the level of the reserve.

*Theorem 3.1*

Consider the perturbed risk model with standard Brownian motion defined by the equation (3) in which time the interarrival $\{T_k; k \ge 0\}$ follows a common Hawkes distribution and let $(X_i)_{i=1;\dots;n}$ iid r. v with a distribution F, then the probability of failure in finite time is defined by:

$$\psi(u,T) \simeq e^{-\delta u} \int_0^T e^{-r\alpha s}\left[\frac{\lambda}{r\alpha}\left(1 - e^{-r\alpha(s-T)}\right) - \frac{\gamma e^{\beta s}}{r\alpha+\beta}\left(e^{(s-T)(r\alpha+\beta)} - 1\right)\right] d\lambda_s \tag{7}$$

The proof of this theorem will be given later, as the elements for its demonstration are not yet gathered.

Lemma 3.1

The probability of ruin at the finite horizon $\psi(u,T)$ satisfies for all $u \ge 0$ and $T > 0$:

$$\psi(u,T) \le \mathbb{P}\left(\sum_{i=1}^{N(t)} X_i > u + ct + \sigma \int_0^t e^{r(t-s)} ds\right) \tag{8}$$

Proof

By using the equations (3) and (4), we have:

$$\psi(u,T) = \mathbb{P}\left(\inf_{1\le t\le T}\left\{u + ct - \sum_{i=1}^{N(t)} X_i + \sigma \int_0^t e^{r(t-s)} ds\right\} < 0\right)$$

$$= \mathbb{P}\left(sup_{1\le t\le T}\left\{-u - ct + \sum_{i=1}^{N(t)} X_i - \sigma \int_0^t e^{r(t-s)} ds\right\} < 0\right)$$

which means that

$$\psi(u,T) \le \mathbb{P}\left(\sum_{i=1}^{N(t)} X_i > u + ct + \sigma \int_0^t e^{r(t-s)} ds\right)$$

The following lemma gives us an approximation of the formula. (8).

Lemma 3.2

For everything $u \ge 0$ and $T > 0$:

$$\psi(u,T) \le \mathbb{P}\left(\sum_{i=1}^{N(t)} X_i > u + ct + \sigma \int_0^t e^{r(t-s)} ds\right)$$

$$\sim \mathbb{P}\left(\sum_{i=1}^{N(t)} X_i > u\right) \tag{9}$$

Proof

According to the equation (8), we have:

$$\psi(u,T) \le \mathbb{P}\left(\sum_{i=1}^{N(t)} X_i > u + ct + \sigma \int_0^t e^{r(t-s)} ds\right)$$

According to lemma (A.5) in [12], we obtain:

$$\psi(u,T) \ge \sum_{i=1}^{N(t)} \left(\int_0^T \mathbb{P}\{X_i > u\} ds\right)$$

like

$$\psi(u,T) \le \mathbb{P}\left(\sum_{i=1}^{N(t)} X_i > u + ct + \sigma \int_0^t e^{r(t-s)} ds\right)$$

and $(X_i)_{i\geq 1}$ is an $\alpha$-stable random variable, then according to lemma 2.1, we deduce the equation (9).

Lemma 3.3

for everything $u \geq 0$ and $T > 0$:

$$\mathbb{P}\left(\sum_{i=1}^{N(t)} X_i > u + ct + \sigma \int_0^t e^{r(t-s)} ds\right)$$

$$\sim \int_0^T e^{-r\alpha s} \int_0^{T-s} e^{-r\alpha t} \bar{F}(u) h(t) d\lambda_s dt \tag{10}$$

Proof

If F is an α-stable distribution, according to [18], We have:

$$\mathbb{P}\left(\sum_{i=1}^{N(t)} X_i > u\right) \to \sum_{i=1}^{\infty} \mathbb{P}\{X_i > u\}$$

and

$$\sum_{i=1}^{\infty} \mathbb{P}\{X_i > u\} \sim \int_0^{\infty} \bar{F}(u + ct) dH(t) d\lambda_s$$

Suppose that if $t \to \infty$, then $N(t) \to \infty$. By a change of variable, we have:

$\sum_{i=1}^{\infty} \mathbb{P}\{X_i > u\} \sim \int_0^T \bar{F}(u + ct) dH(t) d\lambda_s$

Let's ask

$$\varphi(u) = \sum_{i=1}^{\infty} \mathbb{P}\{X_i > u\}$$

so

$$\varphi(u) \geq \int_0^{\infty} \int_0^{\infty} \bar{F}[(1+\varepsilon)(u + ct)] dH(t) d\lambda_s$$

when $\varepsilon \to 0$, we get (10)

Lemma 3.4

The finite-time ruin probability $\psi(u, T)$ is defined for all $u \geq 0$ and $T > 0$:

$$\psi(u, T) \simeq a(T) \bar{F}(u) \tag{11}$$

with

$$a(T) = \int_0^T e^{-r\alpha s} \int_0^{T-s} e^{-r\alpha t} h(t) d\lambda_s dt$$

Proof

We know that

$$\psi(u, T) \sim \int_0^T e^{-r\alpha s} \int_0^{T-s} e^{-r\alpha t} \bar{F}(u) h(t) d\lambda_s dt$$

So

$$\psi(u, T) \simeq \bar{F}(u) \int_0^T e^{-r\alpha s} \int_0^{T-s} e^{-r\alpha t} h(t) d\lambda_s dt$$

and posing

$$a(T) = \int_0^T e^{-r\alpha s} \int_0^{T-s} e^{-r\alpha t} h(t) d\lambda_s dt$$

we get the equation (11).

Proof of theorem 3.1

Since the inter-arrival time $\{T_k; k \geq 0\}$ is a sequence of random variables following a Hawkes distribution, we have:

$$h(t) = \lambda + \gamma \int_0^T e^{-\beta(t-s)} d\lambda_s$$

and posing

$$\mathrm{M(s)} = \int_0^{T-s} e^{-r\alpha t} h(t) dt$$

$$= \int_0^{T-s} e^{-r\alpha t} \left(\lambda + \gamma \int_0^T e^{-\beta(t-s)} d\lambda_s\right) dt$$

$$= \int_0^{T-s} \lambda e^{-r\alpha t} dt + \int_0^{T-s} \gamma e^{\beta s} \times e^{-(r\alpha+\beta)t} dt d\lambda_s$$

$$M(s) = \frac{\lambda}{r\alpha}\left(1 - e^{-r\alpha(s-T)}\right) - \frac{\gamma e^{\beta s}}{r\alpha+\beta}\left(e^{(s-T)(r\alpha+\beta)} - 1\right)$$

So

$$a(T) = \int_0^T e^{-r\alpha s} M(s) d\lambda_s$$

$$a(T) = \int_0^T e^{-r\alpha s}\left[\frac{\lambda}{r\alpha}\left(1 - e^{-r\alpha(s-T)}\right) - \frac{\gamma e^{\beta s}}{r\alpha+\beta}\left(e^{(s-T)(r\alpha+\beta)} - 1\right)\right] d\lambda_s \tag{12}$$

F being the distribution of the amounts of claims following an exponential law, we obtain:

$$\bar{F}(u) = 1 - \left(1 - e^{-\delta u}\right) = e^{-\delta u} \tag{13}$$

The equations (11), (12) and (13), gives (7).

## 4. Conclusions

In this paper, we introduce a risk model incorporating Brownian perturbations, where claim arrivals are governed by a Hawkes process. This framework enables us to derive an expression for the finite-horizon ruin probability, considering claim inter-arrivals characterized by a Hawkes variable memory process [1], alongside an exponential distribution of claim amounts and a Brownian oscillation at the reserve level R(t). We rigorously examine its theoretical properties and establish a pure diffusion approximation, facilitating the computation of ruin probabilities. Furthermore, we apply asset-liability management techniques to analyze the impact of a Hawkes claim arrival process on optimal claims for an insurer within an incomplete market. Naturally, the assumption of a one-dimensional Hawkes process with exponential intensity represents merely an initial step. Future research could therefore explore the application of multidimensional Hawkes processes. In this study, we confined our analysis to portfolio modeling with claims belonging to a single subclass. A promising avenue for generalization would involve employing a multidimensional marked Hawkes process to examine a portfolio of claims across different subclasses, their mutual interactions, and their temporal evolution. This holds particular significance for an insurance company in the context of reserve estimation, as it could enable the classification of claims based on their initial attributes, thereby refining the assessment of the required capital reserves. Similar examples have been processed by [19, 20].

## Author Contributions

**Souleymane Badini:** Conceptualization, Data curation, Formal Analysis, Investigation, Methodology, Project administration, Resources, Software, Supervision, Visualization, Writing – original draft

**Frederic Bere:** Validation, Writing – review & editing

## Conflicts of Interest

The authors declare no conflicts of interest.

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