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Research Article

New Definition of Fractional Derivative with Some Applications

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Abstract

This research proposes a new definition of fractional calculus based on a generalized fractional derivative. This new definition aims to improve the mathematical structure of fractional calculus and extend its application to fractional-order differential equations. The proposed approach develops modified formulas based on the concepts of Caputo and Riemann-Liouvel, seeking to propose a memory-based structure that provides a flexible framework for analyzing fractional differential equations and modeling systems exhibiting memory effects. The Caputo formula aims to provide a framework that accommodates classical initial conditions, while the Riemann-Liouvel formula maintains the characteristic integrative structure associated with this classical approach. The theoretical properties, applicability, and potential advantages of the proposed operators were examined to establish a coherent extension of generalized fractional calculus. This work contributes to the development of fractional calculus by introducing modified operators that seek to improve conformity with classical definitions while preserving the non-local and memory-based properties that are essential for modeling fractional calculus. The new definitions are more compatible with the previous definitions. Also, they combine the simplicity of the conformal derivative and the non-locality of the Caputo fractional derivative.

Keywords

Generalized Conformable Fractional Derivatives, Caputo and Riemann-Liouville Fractional Derivatives

1. Introduction

The history of the fractional derivative started in the late 17th century when Guillaume de L'Hôpital asked Euler what happens for $\frac{d^n f}{dx^n}$ if $n = \frac{1}{2}$. Many researchers have produced definitions for the derivative of fractional order; most of them are in an integrated form. The most popular among these definitions are Caputo and Riemann Liouville. They have physical meaning and the ability to describe historical memory processes.

1) Riemann-Liouville fractional derivative of order $\alpha \ [n-1,n), n = 1,2,.$ of f is

$$^{RL}D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}dx$$

2) Caputo fractional derivative of order $\alpha \in [n-1,n), n = 1,2,.$ of f is

$$^{C}D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}dx$$

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These definitions satisfy the linearity property, but the commutative property holds only for the R-L fractional integral. The Caputo definition assumes that the function f is differentiable. Like all previous definitions of the conformable fractional derivative, Caputo's and R-L's fractional derivatives do not satisfy important rules such as the derivative of the product of two functions, the quotient of two functions, or the chain rule. They also do not satisfy the Mean Value Theorem or Rolle's Theorem. For further details, see [1-5]. In the period from 2016 to 2020 Fudong Ge and Yangquan Chen [6-11] establish a theoretical investigation to analyze and control the time-fractional diffusion systems, focusing on stability, regional controllability, observability, and feedback control. They explore how to control and monitor time-fractional diffusion processes within specific spatial regions. Regional controllability is verified using time-fractional Riemann-Liouville derivatives.

The classical conformable fractional derivative introduced by Khalil et al. [12] in 2014 provides a simple and localized definition of the fractional operator. However, it's purely local nature limits its ability to describe historical memory processes, and it becomes singular at t = 0. Two years later, Atangana and Paliano [13] proposed a modification of the fractional derivative of Caputo-Fabrizio [14]. The Atangana-Pagliano derivative takes an integral form, is non-local, and has a non-singular kernel. The fractional derivative of Atangana-Pagliano also satisfies the commutative property. This definition has attracted attention in the scientific and engineering fields for reformulating many phenomena and equations using it [15-19]. M. Bouaouid et al [20] studied the nonlocal telegraph equation in frame of the conformable time-fractional derivative. J. Wang and C. Bai [21] investigated the antiperiodic boundary value problems for impulsive fractional functional differential equations in the sense of conformable derivative. S. Li et al [22] studied the existence of solutions of diffusion equation with the general conformable derivative.

Recently, Abu-Shady [23] in 2021 provides a new generalization of the conformable fractional derivative definition that has advantages in comparison with other previous definitions.

3) Conformable Fractional Derivatives

Given a function $f: [a,\infty) \to \mathbb{R}$ and $\alpha \in (0,1]$. Then the conformable fractional derivative of f of order α for all $t > 0$ is defined by:

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon}$$

4) Generalized Conformable Derivative

For a function $f: (0,\infty) \to R$, the generalized conformable derivative of order $0 < \alpha < 1$, for $f(t)$ at $t > o$ is defined by

$$D^{GFD}(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t+\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\varepsilon\ t^{1-\alpha}\right)-f(t)}{\varepsilon}, \quad \beta \in R^+$$

where $D^{GFD}(f)(0) = \lim_{t \to 0^+} D^{GFD}(f)(t)$

These definitions have disadvantages and advantages as we mentioned before. To address these issues, we propose an improvement form, referred to as the improved conformal fractional derivative (ICFD). This operator combines the simplicity of the conformal derivative and the non-locality of the Caputo fractional derivative. Nonlocality was addressed using the limit of Caputo's fractional derivative definition. The first term preserves the local gradient, while the second term introduces historical memory effects through a weak singularity.

The research paper is organized as follows: The next section presents an improvement on the Generalized Conformable Derivative with some definitions and tools. The third section presents some applications, and the final section presents the conclusion.

2. Improvement on the Generalized Conformable Derivative (IGCD)

If we focus on the fractional derivative limit of the Riemann-Liouville and Caputo definitions as $\alpha \to n-1$, we have

$$\lim_{\alpha \to n-1} {}^{RL}D_a^\alpha(f)(t) = \lim_{\alpha \to n-1} \frac{1}{\Gamma(\mathrm{n}-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx = f^{(n-1)}(t)$$

$$\lim_{\alpha \to n-1} {}^{C}D_a^\alpha(f)(t) = \lim_{\alpha \to n-1} \frac{1}{\Gamma(\mathrm{n}-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx = f^{(n-1)}(t) - f^{(n-1)}(a)$$

For $\alpha \in (n-1, n], n = 1, 2, 3, \ldots$, whereas

$$\lim_{\alpha \to n-1} T_\alpha(f)(t) = \lim_{\alpha \to n-1} t^{n-\alpha} f^{(n)} = t f^{(n)}(t)$$

$$\lim_{\alpha \to n-1} D^{GFD}(f)(t) = \lim_{\alpha \to n-1} \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)} t^{n-\alpha} f^{(n)} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)} t f^{(n)}(t)$$

for $\alpha \in (n-1, n], n = 1, 2, 3, \ldots$

This illustrates that the physical meaning of the Riemann-Liouville and Caputo fractional derivatives is entirely different from that of the conformable derivative and the generalized conformable derivative. Therefore, it is recommended to replace these derivatives with the improved generalized conformable fractional derivative, especially when α ∈ (n − 1, n) is close to n − 1, n = 1, 2, ⋯.

Definition 2.1. For a function $f: R \to R$ and a given number $-\infty < a < +\infty$, the improved Caputo-type generalized fractional derivative of order $0 \le \alpha \le 1$ of $f(t)$ is defined as:

$$T_\alpha^C f(t) = \frac{Lim}{\varepsilon\to 0}\left[(1-\alpha)(f(t)-f(a)) + \alpha\frac{f\left(t+\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\varepsilon\ \ t^{1-\alpha}\right)-f(t)}{\varepsilon}\right] \quad (1)$$

If $n<\alpha\le n+1, n=1,2,3,\ldots$

$$T_\alpha^C f(t) = \frac{Lim}{\varepsilon\to 0}\left[\begin{matrix}(n+1-\alpha)\left(f^{(n)}(t)-f^{(n)}(a)\right)+\\ (\alpha-n)\frac{f^{(n)}\left(t+\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\varepsilon\ \ t^{1-\alpha}\right)-f^{(n)}(t)}{\varepsilon}\end{matrix}\right] \quad (2)$$

where $-\infty < a < t < +\infty, \beta > -1, f$ is an $n-$differentiable function.

Definition 2.2. For a function $f: R \to R$ and a given number $-\infty < a < +\infty$, the improved Riemann Liouville-type generalized fractional derivative of order $0\le\alpha\le 1$ of $f(t)$ is defined as:

$$T_\alpha^{RL} f(t) = \frac{Lim}{\varepsilon\to 0}\left[(1-\alpha)f(t) + \alpha\frac{f\left(t+\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\varepsilon\ \ t^{1-\alpha}\right)-f(t)}{\varepsilon}\right] (3)$$

If $n<\alpha\le n+1, n=1,2,3,\ldots$

$$T_\alpha^{RL} f(t) = \frac{Lim}{\varepsilon\to 0}\left[(n+1-\alpha)f^{(n)}(t) + (\alpha-n)\frac{f^{(n)}\left(t+\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\varepsilon\ \ t^{1-\alpha}\right)-f^{(n)}(t)}{\varepsilon}\right] \quad (4)$$

where $-\infty < t < +\infty, \beta > -1, f$ is an $n-$differentiable function.

Remark 2.1. From previous definitions, it is easy to notice that

1) $T_\alpha^C f(t) = T_\alpha^{RL} f(t) + (1-\alpha)f(a)$ when $0<\alpha<1$
2) $T_\alpha^C f(t) = T_\alpha^{RL} f(t) + (n+1-\alpha)f^{(n)}(a)$ when $\alpha\in(n,n+1], n=1,2,\ldots$
3) $\lim_{\alpha\to 0} T_\alpha^C f(t) = \lim_{\alpha\to 0} {}^C D_a^\alpha(f)(t) = \left(f(t)-f(a)\right)$
4) $\lim_{\alpha\to n} T_\alpha^C f(t) = \lim_{\alpha\to n} {}^C D_a^\alpha(f)(t) = \left(f^{(n)}(t)-f^{(n)}(a)\right)$
5) $\lim_{\alpha\to 0} T_\alpha^{RL} f(t) = \lim_{\alpha\to 0} {}^{RL} D_a^\alpha(f)(t) = f(t)$
6) $\lim_{\alpha\to n} T_\alpha^{RL} f(t) = \lim_{\alpha\to n} {}^{RL} D_a^\alpha(f)(t) = f^{(n)}(t)$
7) Furthermore, the improved definitions coincide with the classical derivative when $\alpha=1$ and $\alpha=n+1$. That is, if $\alpha=1$, we have $T_\alpha^C f(t) = T_\alpha^{RL} f(t) = f'(t)$ and if $\alpha=n+1$, we have $T_\alpha^C f(t) = T_\alpha^{RL} f(t) = f^{(n)}(t)$.

Theorem 2.1. If $f(t)$ is a $1-$differentiable function, $0<\alpha\le 1$ then

$T_\alpha^C f(t) = (1-\alpha)(f(t)-f(a)) + \alpha\frac{I(\beta)}{I(\beta-\alpha+1)}t^{1-\alpha}f'(t)$,

and

$$T_\alpha^{RL} f(t) = (1-\alpha)f(t) + \alpha\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}t^{1-\alpha}f'(t)$$

Proof: For the first assertion, let $h = \frac{I(\beta)}{I(\beta-\alpha+1)}\varepsilon\ \ t^{1-\alpha}$, so $\varepsilon = \frac{ht^{1-\alpha}I(\beta-\alpha+1)}{I(\beta)}$. Substituting into Eq. (1), we get

$$T_\alpha^C f(t) = \frac{Lim}{h\to 0}\left[(1-\alpha)(f(t)-f(a)) + \alpha\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}t^{1-\alpha}\frac{f(t+h)-f(t)}{h}\right]$$

$$= (1-\alpha)(f(t)-f(a)) + \alpha\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}t^{1-\alpha}\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$$

$$= (1-\alpha)(f(t)-f(a)) + \alpha\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}t^{1-\alpha}f'(t).$$

The second assertion can be proved similarly.

By using the same technique, one can easily prove the next theorem

Theorem 2.2. If $f(t)$ is an $(n+1)-$differentiable function. For $\alpha\in(n,n+1], n=1,2,.$ then,

$$T_\alpha^C f(t) = (n+1-\alpha)\left(f^{(n)}(t)-f^{(n)}(a)\right) + (\alpha-n)\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}t^{n+1-\alpha}f^{(n+1)}(t)$$

and

$$T_\alpha^{RL} f(t) = (n+1-\alpha)f^{(n)}(t) + (\alpha-n)\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}t^{n+1-\alpha}f^{(n+1)}(t)$$

Based on the results reported in Khalil et al. [12], we can easily prove the following properties

Theorem 2.3. If $f(t)$ is an $(n+1)-$differentiable function. For $\alpha\in(n,n+1], n=0,1,2,.$ then,

1) $T_\alpha^C(rf+sg) = rT_\alpha^C(f) + sT_\alpha^C(g)$
2) $T_\alpha^{RL}(rf+sg) = rT_\alpha^{RL}(f) + sT_\alpha^{RL}(g)$

Theorem 2.4. If $f(t)$ is an $1-$ differentiable function. For $0<\alpha\le 1$ then,

1) $T_\alpha^{RL}(fg) = g(t)T_\alpha^{RL}(f) + f(t)T_\alpha^{RL}(g) - (1-\alpha)f(t)g(t)$
2) $T_\alpha^{RL}\left(\frac{f}{g}\right) = \frac{gT_\alpha^{RL}(f) - fT_\alpha^{RL}(g)}{g^2(t)} + (1-\alpha)\frac{f(t)}{g(t)}$
3) $T_\alpha^{RL}(f(g(t)) = (1-\alpha)f\left(g(t)\right) + \alpha f'\left(g(t)\right)D^{GFD}(g(t))$

Proof. Adding and subtracting the term $(1-\alpha)f(t)g(t)$ to the definition

$T_\alpha^{RL}(fg) = (1-\alpha)fg + \alpha t^{1-\alpha}(fg'+f'g)$will prove the first property. The second property was proved by adding and subtracting the term $(1-\alpha)\frac{f(t)}{g(t)}$ to the result in Theorem 1. The application of the result in Theorem 1 proved property 3.

In general, $T_\alpha^{RL}T_\gamma^{RL}(f) \neq T_{\alpha+\gamma}^{RL}(f)$ also, $T_\alpha^C T_\gamma^C(f) \neq T_{\alpha+\gamma}^C(f)$

Theorem 2.5. We obtain the following improved Caputo-type generalized fractional derivative for $a=0$, of order $0\le\alpha\le 1$ of certain functions $f(t)$

1) Taking $\beta=p$ in definition 1, we get $T_\alpha^C(t^p) = (1-$

$\alpha)t^p + \alpha\frac{\Gamma(p)}{\Gamma(p-\alpha+1)}t^{p-\alpha}$

2) $T_\alpha^C(k) = 0$, for any constant k

3) $T_\alpha^C(e^t) = (1-\alpha)(e^t-1) + \alpha\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}t^{1-\alpha}e^t$

4) $T_\alpha^C(\sin t) = (1-\alpha)\sin t + \alpha\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}t^{1-\alpha}\cos t$

5) $T_\alpha^C(\cos t) = (1-\alpha)(\cos t - 1) - \alpha\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}t^{1-\alpha}\sin t$

Proof: The proof was performed easily using Definition 1.

Definition 2.3. If f is a continuous function, then for $\alpha \in (0,1]$, let

$$I_\alpha = \frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}\int_0^x \frac{f(t)}{t^{1-\alpha}}e^{\frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}(1-\alpha)(t^\alpha-x^\alpha)}dt$$

When $\alpha = 1$ the given integral coincides with the Riemann integral.

Theorem 2.6. If $f(0) = 0$, we get

1) $I_\alpha T_\alpha^{RL}\big(f(t)\big) = T_\alpha^{RL} I_\alpha\big(f(t)\big) = f(t)$

2) $I_\alpha T_\alpha^{C}\big(f(t)\big) = T_\alpha^{C} I_\alpha\big(f(t)\big) = f(t)$

Proof: only the proof of the first assertion will be given, similarly, the second one can be proved.

$$I_\alpha T_\alpha^{RL}\big(f(t)\big) = \frac{(1-\alpha)\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}\int_0^x \frac{f(t)}{t^{1-\alpha}}e^{\frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}(1-\alpha)(t^\alpha-x^\alpha)}dt + \int_0^x f'(t)\,e^{\frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}(1-\alpha)(t^\alpha-x^\alpha)}dt$$

The second term can be performed by integration by parts, where $u = e^{\frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}(1-\alpha)(t^\alpha-x^\alpha)}$ and $dv = f'(t)dt$, we get

$$I_\alpha T_\alpha^{RL}\big(f(t)\big) = \frac{(1-\alpha)\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}\int_0^x \frac{f(t)}{t^{1-\alpha}}e^{\frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}(1-\alpha)(t^\alpha-x^\alpha)}dt + f(t)e^{\frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}(1-\alpha)(t^\alpha-x^\alpha)}\Big|_0^x - \frac{(1-\alpha)\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}\int_0^x \frac{f(t)}{t^{1-\alpha}}e^{\frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}(1-\alpha)(t^\alpha-x^\alpha)}dt = f(x)$$

For the second equality. Let

$(1-\alpha)y(x) + \frac{\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha+1)t^{1-\alpha}}y'(x) = f(x)$; using the integrating factor method to solve this linear first order ode $\mu(x) = e^{\frac{1-\alpha}{\alpha}\frac{}{t^{1-\alpha}}x}$ we get

$$y(x) = \frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}\int_0^x \frac{f(t)}{t^{1-\alpha}}e^{\frac{\Gamma(\beta-\alpha+1)}{\alpha\Gamma(\beta)}(1-\alpha)(t^\alpha-x^\alpha)}dt$$

So, if $I_\alpha(f(x)) = y(x)$ we get

$$T_\alpha^{RL} I_\alpha\big(f(t)\big) = f(x)$$

3. Applications of Improved Generalized Conformable Derivative

We solved some fractional differential equations using the Improved Generalized Conformable Derivative and compared the obtained results with those obtained using the Riemann-Liouville, Caputo, and conformable definitions. The comparison, based on the solution graphs, reveals that the improved generalized Conformable derivative more closely matches the Riemann-Liouville and Caputo definitions than the Conformable definition.

Example 3.1. Consider the following fractional differential equation:

$$y^{(0.5)} = \frac{2}{\Gamma(2.5)}x^{1.5}, y(0) = 0$$

Integrating both sides using the Caputo definition, we get

$$y(x) = \frac{1}{\Gamma(0.5)}\int_0^x (x-t)^{-1/2}\frac{2}{\Gamma(2.5)}t^{3/2}dt = \frac{2}{\Gamma(2.5)\Gamma(0.5)}\int_0^x (x-t)^{-1/2}t^{3/2}dt,$$

by the Beta function formula $y(x) = \frac{2}{\Gamma(2.5)\Gamma(0.5)}x^2\frac{\Gamma(2.5)\Gamma(0.5)}{\Gamma(3)} = x^2$.

The same result was obtained using the Riemann-Liouville definition.

When the conformable fractional derivative was used, the problem transformed to

$y' = \frac{1}{\Gamma(2.5)}x$, by integrating the obtained solution was,

$$y(x) = \frac{1}{\Gamma(2.5)}x^2$$

Now, using our new definition with $\beta = 2.5$, the problem has been reduced to

$$\frac{1}{2}y + \frac{1}{2}\sqrt{x}y' = x^{3/2}$$

$$y' + \frac{1}{\sqrt{x}}y = 2x$$

Using the integrating factor $\mu(x) = e^{2\sqrt{x}}$, we get $y(x) = 2e^{-2\sqrt{x}}\int_0^x e^{2\sqrt{t}\ t}\ dt$, substitute $u = \sqrt{x}$, then integrate by parts three times to get

$$y(x) = 2x^{3/2} - 3x + 3\sqrt{x} - \frac{3}{2} + \frac{3}{2}e^{-2\sqrt{x}}, x > 0$$

Example 3.2. Consider the following fractional differential equation:

$$y^{(1.5)} = \frac{6}{\Gamma(2.5)}x^{1.5}, y(0) = 0, y'(0) = 0$$

Using the Caputo integration definition, we get

$$y(x) = \frac{1}{\Gamma(1.5)}\int_0^x (x-t)^{1/2}\frac{6}{\Gamma(2.5)}t^{3/2}dt = \frac{6}{\Gamma(2.5)\Gamma(1.5)}\int_0^x (x-t)^{1/2}t^{3/2}dt,$$

by the Beta function formula, we get the following solution

$$y(x) = \frac{6}{\Gamma(2.5)\Gamma(1.5)}x^3\frac{\Gamma(2.5)\Gamma(1.5)}{\Gamma(4)} = x^3$$

When the conformable fractional derivative was used, the problem transformed to

$y'' = \frac{6}{\Gamma(2.5)}x$, by integrating the obtained solution was,

$$y(x) = \frac{1}{\Gamma(2.5)}x^3$$

Now, using our new definition with $\beta = 3$, the problem has been reduced to

$$\frac{1}{2}y' + \sqrt{x}y'' = 6x^{3/2}$$

To reduce the order of the equation, let $u = y'$to get the first order linear ordinary differential equation, $u' + \frac{1}{2\sqrt{x}}u = 6x$.

Using the integrating factor $\mu(x) = e^{\sqrt{x}}$, we get

$y'(x) = u(x) = 6e^{-\sqrt{x}}\int_0^x t\ e^{\sqrt{t}}\ dt$, the substitution $v = \sqrt{t}$, yields to,

$$y'(x) = u(x) = 12e^{-\sqrt{x}}\int_0^{\sqrt{x}} v^3\ e^v\ dv$$

Then integrate by parts three times to get

$$y'(x) = u(x) = 12\left(x^{3/2} - 3x + 6\sqrt{x} - 6\right) + 6e^{-\sqrt{x}}, x > 0$$

Integrate and use the initial condition to obtain the following solution:

$$y(x) = 24\left(\frac{1}{5}x^{5/2} - \frac{3}{4}x^2 + 2x^{3/2} - 3x - 6\sqrt{x}\ e^{-\sqrt{x}} - 6e^{-\sqrt{x}}\right) + 144$$

Figure 2 presents the comparison between the three solutions. It reveals clearly that the Improved Generalized Conformable Derivative solution is much better than the solution obtained by using the conformable fractional derivative.

4. Conclusion

This paper proposes an improvement to the Conformable fractional derivative. This improved generalized fractional derivative features a selectable local memory. It exhibits physical behavior closer to that of Riemann-Liouville and Caputo fractional derivatives than to the Conformal fractional derivative. Therefore, this improved generalized fractional derivative has great capabilities for modeling many physical problems.

Abbreviations

ICFD	Improved Conformal Fractional Derivative
${}^{RL}D_a^\alpha$	Riemann Liouville Fractional Derivative
${}^{C}D_a^\alpha$	Caputo Fractional Derivative
T_α	Conformable Fractional Derivative
D^{GFD}	Generalized Conformable Fractional Derivative
T_α^{RL}	Improved Riemann Liouville-type Generalized Fractional Derivative
T_α^{C}	Improved Caputo-type Generalized Fractional Derivative

Author Contributions

Khaled Khalil Jaber: Conceptualization, Formal Analysis, Methodology, writing– original draft, Writing – review & editing

Conflicts of Interest

The author declares no conflicts of interest.

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