International Journal of Theoretical and Applied Mathematics
2025; 11(1): 1-17
http://www.sciencepublishinggroup.com/j/ijtam
doi: 10.11648/j.ijtam.20251101.11
ISSN: 2575-5072 (Print); ISSN: 2575-5080 (Online)

A Priori and a Posteriori Error Estimates of Finite Element Method for Source Control Problems Governed by a System of Quasi-linear Elliptic Equations

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To cite this article:

Changil Kim, Jayong Ri. (2025). A Priori and a Posteriori Error Estimates of Finite Element Method for Source Control Problems Governed by a System of Quasi-linear Elliptic Equations. *International Journal of Theoretical and Applied Mathematics, 11*(1), 1-17.
https://doi.org/10.11648/j.ijtam.20251101.11

Received: 4 March 2025; **Accepted:** 6 May 2025; **Published:** 21 June 2025

Abstract: In this paper, we consider the optimal source control of a 2-dimensional steady-state thermistor. The problem is described by a system of two nonlinear elliptic partial differential equations with appropriate boundary conditions which model the coupling of the thermistor to its surroundings. The heat source is Joule heat due to variable resistance. The problem is a source optimal control problem that controls the source term necessary to approximate the temperature to a proper target function. First, we derive the optimality condition of the problem. Based on setting the approximation problem of a given control problem in a first order polynomial finite element function space and deriving the optimality condition of the approximation problem, we evaluated a priori error between the optimal control, the optimal state, the conjugate state and its finite element approximation functions. Then, we evaluate the upper bound of a posteriori error estimates that are currently available for error estimation. For a posteriori error estimates, it is necessary to find the convergence of the error indicator. In this paper, we prove the convergence of a posteriori error indicator by obtaining a lower bound estimate of a posteriori error and finding that the total variance error goes to zero. And, we propose a gradient algorithm to find the optimal control and provide a condition for this algorithm to converge. The validity is also demonstrated by adaptive numerical simulations with a detailed problem. The computational results are obtained on three adaptive meshes and the graphs of the finite element solutions are presented.

Keywords: Quasi-linear Elliptic Equations, Source Control, A Posteriori Error Estimates

1. Introduction

Optimal control problems are frequently found in the fields of thermal physics, sociology, economic processes, and etc, and the numerical solutions of optimal control problems are extremely important for better performance of those fields. Therefore, one needs some efficient numerical methods to approximate the solutions of optimal control problems. Finite element method is the most widely used numerical methods for solving optimal control problems. Furthermore, other numerical methods, such as the spectral method, the mixed finite element method and the finite volume method have also been applied to approximate some optimal control problems. For example, there has been done much work on the finite element method for optimal control problems [1-10], the spectral method for optimal control problems [11-13], the mixed finite element method for optimal control problems [14-20], and the finite volume method for optimal control problems [21, 22], a posteriori error control for time-fractional parabolic and diffusion equations derived in [23-27]. A posteriori error estimates of the finite element method for linear parabolic equations with Dirac measure were derived in [1]. In [2, 3], the authors derived a posteriori error estimates of the two-grid finite el-ement method and two-grid finite volume method for quasi-linear elliptic equations. The upper bound of a posteriori error estimates of the Legendre spectral method for source control problems of semi-linear parabolic equations was derived in [4]. In [5, 6], they studied the existence of solution for coefficient control problems of a system of quasi-linear elliptic equations and optimality conditions. Thermistor problems are represented mathematically by quasi-linear elliptic or parabolic equations. In [28], they studied super-convergence analysis of nonconforming FEM(Finite Element Method) for nonlinear time-dependent thermistor problem. In [29], they studied super-convergence analysis of finite element method for time-fractional thermistor problem, and in [30] unconditionally optimal error estimates of Cranc-Nicholson Galerkin method for the nonlinear thermistor equations From the preceding results, we can see that the results related to a posteriori error estimate for semi or quasi-linear differential equations take the majority but not those for optimal control problems of a system described by non-linear differential equations. In [4], for a source optimal control problem of a semi-linear system, a posteriori error estimate was first attempted by Legendre-spectral method under the assumption of strong convexity in the neighborhood of the solution of an objective function and the results were given. However, we have difficulties applying spectral method to the cases in which the solution is piecewise smooth, for this is a numerical method used when the solution is infinitely differentiable. Galerkin finite element method is the one to overcome such difficulties. In [5, 6, 28-30], finite element posteriori error estimates and superconvergence of quasi-linear equations were obtained and the existence of the solution of a control problem of this model, optimality conditions were acquired, but not the error estimates of quasi-linear control problem, as far as I'm concerned. The purpose of this paper is to get a priori and a posteriori error estimates for optimal control, optimal state and conjugate state by an adaptive finite element method for a control problem of a quasi-linear differential equation and to give numerical tests on them. The main results of this paper are priori and posteriori error estimates of approximation solutions by an adaptive finite element method. Numerical testing results on three adaptive triangular meshes have been added. It is fine to compare our results with the ones in [4, 5, 6]. This paper is constructed as follows. In section 1, 2, we give notations and preliminaries and optimality condition, in section 3, upper bound of a posteriori error estimates of Finite element, in section 4, priori error estimates and in section 5, numerical simulations.

2. Notations and Preliminaries

Let Ω be a bounded convex polygonal domain in R^2 with boundary $\Gamma = \Gamma_1 \cup \Gamma_2$. In this paper, we will study the following state model:

$$\begin{aligned} -\Delta y - \sigma(y)|\nabla\varphi|^2 &= g,\ \ \Omega \\ \nabla\cdot(\sigma(y)\nabla\varphi) &= 0,\ \ \Omega \\ \frac{\partial y}{\partial n} + \beta y &= 0,\ \ \Gamma \\ \sigma(y)\frac{\partial\varphi}{\partial n} &= \theta,\ \ \Gamma_1 \\ \varphi &= 0,\ \ \Gamma_2 \end{aligned} \tag{1}$$

, where $g \in L^2(\Omega), g \geq 0$ in $\Omega, \theta \in H^{1/2}(\Gamma_1), \beta \in R_+^1$.

The assumption for function σ, θ is as follows:

[Assumption 1] Let $\sigma \in W^{1,\infty}(R), \sigma(t)$ be a monotonous function.

$$0 < \alpha_1 \leq \sigma(t) \leq \alpha_2,\ \ \alpha_1,\ \alpha_2 \in R, \quad \sigma'(t) \leq 0, \quad t \in R$$

The weak formula of model (1) is

$$\begin{aligned} &\int_\Omega \nabla y\nabla v dx + \int_\Gamma \beta y v dx - \int_\Omega \sigma(y)|\nabla\varphi|^2 v dx = \int_\Omega g v dx, \forall v \in V := H^1(\Omega), \\ &\int_\Omega \sigma(y)\nabla\varphi\nabla\omega dx = \int_{\Gamma_1} \theta\omega d\Gamma, \forall\omega \in V_{\Gamma_2} := \{\omega \in W^{1,3/2}(\Omega)\Big|\, \omega = 0, \Gamma_2\} \end{aligned} \tag{2}$$

We assume that there exist unique weak solutions y, φ of (1) in $H^1(\Omega), W^{1,3}(\Omega)$, respectively. (For this result, we refer to [5] where this assumption may also be compared with $r = 3$ in [5])

Let $\|\cdot\|_k, (\cdot,\cdot)_k$ denote the $H^k(\Omega)$ norm and inner product. We set $H := H^0(\Omega) = L^2(\Omega), U := L^2(\Omega)$, with $k = 0$. We denote by $\|\cdot\|_\Gamma, (\cdot,\cdot)_\Gamma$ or $\|\cdot\|_{0,\Gamma}, (\cdot,\cdot)_{0,\Gamma}$ the $L^2(\Gamma)$ norm and inner product, and by $\|\cdot\|_{0,s}, <\cdot,\cdot>_{0,s}$ $(s > 2)$ the $L^s(\Omega)$-norm and duality product between $L^s(\Omega)$ and $L^{s'}(\Omega)(1/s + 1/s' = 1)$. The cost function is defined as follows:

$$J(y(g), g) := \frac{1}{2}\|y(g) - z_d\|_0^2 + \frac{\nu}{2}\|g\|_0^2, \nu > 0, \nu \in R^1$$

where $\|\cdot\|_0$ denotes the $H = L^2(\Omega)$-norm and z_d is a given element of H, ν is a weight coefficient.

[Problem 1] $\inf\limits_{g\in U} J(y(g), g)$ where $y = y(g)$ is a solution of (1).

For that there exists at least one solution of Problem 1, we refer to [5]. This proof is very similar to [5] and we omit it here.

3. Optimality Conditions

[Theorem 1] Assume that $\bar{g}$ is a solution of Problem 1. There exists $p \in V, q \in V_{\Gamma_2}$ that satisfies the following:

$$\begin{aligned} &(p+\nu\bar{g},\ g)_0 = 0,\ \ \forall g \in U = L^2(\Omega) \\ &(\nabla p, \nabla \upsilon)_0 - (\sigma'(\bar{y})|\nabla\bar{\varphi}|^2 p, \upsilon)_0 + (\sigma'(\bar{y})\nabla\bar{\varphi}\nabla q, \upsilon)_0 + \\ &\qquad + (\beta p, \upsilon)_\Gamma = (\bar{y} - z_d, \upsilon)_0,\ \ \forall \upsilon \in V \\ &(\sigma(\bar{y})\nabla q, \nabla\omega)_0 = 2(p\sigma(\bar{y})\nabla\bar{\varphi}, \nabla\omega)_0,\ \ \forall\omega \in V_{\Gamma_2} \end{aligned} \tag{3}$$

where $\bar{y}, \bar{\varphi}$ are unique solutions of (1), respectively, when $g = \bar{g}$.

(Proof) Solution $\bar{g}$ of problem 1 satisfied

$$(J'(\bar{g}), g)_0 = (y(g) - z_d, \dot{y}_g)_0 + (\nu\bar{g}, g)_0 = 0 \tag{4}$$

Let us Gateaux differentiate in $\bar{g}$ to direction g in (3).

$$\left\{ \begin{aligned} &(\nabla\dot{y}_g, \nabla\upsilon)_0 + (\beta\dot{y}_g, \upsilon)_{0,\Gamma} - (\sigma'(\bar{y})\dot{y}_g|\nabla\varphi|^2, \upsilon)_0 - \\ &\qquad -(2\sigma(\bar{y})\nabla\bar{\varphi}\nabla\dot{\varphi}_g, \upsilon)_0 = (g, \upsilon)_0 \\ &(\sigma'(\bar{y})\dot{y}_g\nabla\varphi, \nabla\omega)_0 + (\sigma(\bar{y})\nabla\dot{\varphi}_g, \nabla\omega)_0 = 0 \end{aligned} \right. \tag{5}$$

where $\dot{y}_g, \dot{\varphi}_g$ are Gateaux differentiations of y, φ in $\bar{g}$ to direction g, respectively.

Now, let $p \in V, q \in V_{\Gamma_2}$ be the solutions of (3), and in (3) substitute $\dot{y}_g, \dot{\varphi}_g$ instead of υ, ω.

$$\left\{ \begin{aligned} &(\nabla p, \nabla\dot{y}_g)_0 - (\sigma'(\bar{y})|\nabla\bar{\varphi}|^2 p, \dot{y}_g)_0 + (\sigma'(\bar{y})\nabla\bar{\varphi}\nabla q, \dot{y}_g)_0 + \\ &\qquad +(\beta p, \dot{y}_g)_\Gamma = (\bar{y} - z_d, \dot{y}_g)_0 \\ &(\sigma(\bar{y})\nabla q, \nabla\dot{\varphi}_g)_0 = 2(p\sigma(\bar{y})\nabla\bar{\varphi}, \nabla\dot{\varphi}_g)_0 \end{aligned} \right. \tag{6}$$

And substitute p, q instead of υ, ω in (5)

$$\left\{ \begin{aligned} &(\nabla\dot{y}_g, \nabla p)_0 + (\beta\dot{y}_g, p)_{0,\Gamma} - (\sigma'(\bar{y})\dot{y}_g|\nabla\varphi|^2, p)_0 - \\ &\qquad -(2\sigma(\bar{y})\nabla\bar{\varphi}\nabla\dot{\varphi}_g, p)_0 = (g, p)_0 \\ &(\sigma'(\bar{y})\dot{y}_g\nabla\varphi, \nabla q)_0 + (\sigma(\bar{y})\nabla\dot{\varphi}_g, \nabla q)_0 = 0 \end{aligned} \right. \tag{7}$$

By comparing the second expression in (6) and (7), we can get

$$(\sigma'(\bar{y})\dot{y}_g\nabla\varphi, \nabla q)_0 = -2(p\sigma(\bar{y})\nabla\bar{\varphi}, \nabla\dot{\varphi}_g)_0$$

Substituting this to the first expression in (6), we can get

$$\begin{aligned} &(\nabla p, \nabla\dot{y}_g)_0 - (\sigma'(\bar{y})|\nabla\bar{\varphi}|^2 p, \dot{y}_g)_0 + (2p\sigma(\bar{y})\nabla\bar{\varphi}, \nabla\dot{\varphi}_g)_0 + \\ &\qquad + (\beta p, \dot{y}_g)_\Gamma = (\bar{y} - z_d, \dot{y}_g)_0 \end{aligned}$$

and comparing this with the first expression in (7), we can get

$$(\bar{y} - z_d, \dot{y}_g)_0 = (g, p)_0$$

Substituting this to (4), we can get optimality condition

$$(p+\nu\bar{g},\ g)_0 = 0,\ \ \forall g \in U = L^2(\Omega)$$

which completes the proof.

4. Upper Bound of a Posteriori Error Estimates of Finite Element Method

Now, we partition Ω into regular triangles $K_j (j = 1, 2, \cdots, M)$ and denote by h_j the diameter of each element K_j. Let $h := \max\limits_{1\le j\le M} h_{K_j}$, and T_h be the family of triangulation, E_h the set consisting of the inter-element edges on the interior of the domain and denote finite element function space of $H^1(\Omega)$ as:

$$\begin{aligned} V_h &:= \{\upsilon_h \in C(\Omega) : \upsilon_h|_K \in P_1(K),\ \forall K \in T_h\} \\ U_h &:= \{\upsilon_h \in L^2(\Omega) : \upsilon_h|_K \in P_s(K),\ \forall K \in T_h\},\ s = 0, 1 \end{aligned}$$

where $P_s(K)$ denotes a polynomial space whose order is less or equal than s on the element K.

The finite element approximation of (1) is as follows:

$$\begin{aligned} &\int_\Omega \nabla y_h \nabla \upsilon_h dx + \int_\Gamma \beta y_h \upsilon_h dx - \int_\Omega \sigma(y_h)|\nabla\varphi_h|^2 \upsilon_h dx = \int_\Omega g_h \upsilon_h dx, \forall \upsilon_h \in V_h, \\ &\int_\Omega \sigma(y_h)\nabla\varphi_h\nabla\omega_h dx = \int_{\Gamma_1} \theta\omega_h d\Gamma, \forall\omega_h \in V_{\Gamma_2,h} := \{\omega_h \in V_h|\,\omega_h = 0, \Gamma_2\} \end{aligned} \tag{8}$$

There exists $y_h \in V_h, \varphi_h \in V_{\Gamma_2,h}$, unique solutions of (8), which are bounded independently of h. Here, $y_h = y_h(g_h) \in V_h, g_h \in U_h$. The finite element approximation problem of Problem 1 is as follows:

[Problem 2]

$$\inf_{g_h \in U_h} J(y_h, g_h)$$

[Lemma 1] $g_h \in U_h$, solution of problem 2, is bounded in $L^2(\Omega)$ independently of h.

(Proof) The boundedness is derived from the fact that J is coercive with respect to g_h.

[Lemma 2] (Optimality system of Problem 2) There exists $p_h \in V_h,\ q_h \in V_{\Gamma_2,h}$, so that g_h, the solution of Problem 2, satisfies:

$$\begin{aligned}
&(p_h+\nu\bar{g}_h,\, g_h)_0=0,\ \forall g_h\in U_h\\
&(\nabla p_h,\nabla\upsilon_h)_0-(\sigma'(\bar{y}_h)|\nabla\bar{\varphi}_h|^2 p_h,\upsilon_h)_0+(\sigma'(\bar{y}_h)\nabla\bar{\varphi}_h\nabla q_h,\upsilon_h)_0+(\,\beta p_h,\upsilon_h)_\Gamma=\,(\bar{y}_h-z_d,\upsilon_h)_0,\ \forall\upsilon_h\in V_h\\
&(\sigma(\bar{y}_h)\nabla q_h,\nabla\omega_h)_0=2(p_h\sigma(\bar{y}_h)\nabla\bar{\varphi}_h\,,\nabla\omega_h)_0,\ \forall\omega_h\in V_{\Gamma_2,h}
\end{aligned}\tag{9}$$

, where $\bar{y_h},\bar{\varphi_h}$ are unique solutions of (8) when $g_h=\bar{g_h}$. The solutions of (9), $p_h\in V_h, q_h\in V_{\Gamma_2,h}$, are bounded independently of h.

[Lemma 3]([1, 4]) Let $I_h : V \to V_h \subset H^1(\Omega)$ denote an orthogonal projection operator. For $\forall\upsilon\in H^1(K),\forall K\in T_h, E\subset\partial K$, we have:

$$\begin{aligned}
&1)\ \|\upsilon-I_h\upsilon\|_{0,K}\le Ch_k\|\upsilon\|_{1,K}, \|\upsilon-I_h\upsilon\|_{0,E}\le Ch_E^{1/2}\|\upsilon\|_{1,K}\\
&2)\ \|\upsilon\|_{0,E}\le C(h_K^{-1/2}\|\upsilon\|_{0,K}+h_K^{1/2}\|\nabla\upsilon\|_{0,K}), \|\nabla\upsilon\|_{0,K}\le Ch_K^{-1}\|\upsilon\|_{0,K}
\end{aligned}\tag{10}$$

[Lemma 4] If $\bar{g}$ is a solution of Problem 1, then

1) $\bar{g}=-\frac{1}{\nu}p\in H^1(\Omega)$

2) The solutions of (8), (9) are bounded independently of h. (Proof) We can get the first claim from the optimality conditions. Then we prove the second result.

We can deduce $\|\varphi_h\|_{W^{1,3}(\Omega)}\le C\|\theta\|_{0,\Gamma}\le\tilde{C}$ for the solution φ_h of (3) (refer to [5]).

By taking $\upsilon_h:=y_h$, we deduce

$$\begin{aligned}
C\,\|y_h\|_1^2 &\le \left|(\sigma(y_h)|\nabla\varphi_h|^2,y_h)_0\right|+\|g_h\|_0\|y_h\|_0\le\\
&\le b_2\left\||\varphi_h|^2\right\|_{1,3/2}\|y_h\|_{0,3}+\ \|g_h\|_0\|y_h\|_1\le C\,\|\varphi_h\|_{1,3}^2\,\|y_h\|_1+\|g_h\|_0\|y_h\|_1,\\
C\,\|y_h\|_1 &\le C\,\|\varphi_h\|_{1,3}^2+\ \|g_h\|_0\le C\,\|\theta\|_{0,\Gamma}^2+\|g_h\|_0, \|y_h\|_1\le C(\|\theta\|_{0,\Gamma}+\|g_h\|_0)\le\tilde{C}
\end{aligned}$$

Taking $\upsilon_h:=p_h, w_h:=q_h$, and considering that $\sigma'(\cdot)\le 0$, the first expression of (4) can be rewritten:

$$\begin{aligned}
&C(\|p_h\|_1^2+\|q_h\|_1^2)\le|(\sigma'(y_h)\nabla\varphi_h\nabla q_h,p_h)_0|+(y_h-z_d,p_h)_0+2(p_h\sigma(y_h)\nabla\varphi_h\,,\nabla q_h)_0\le\\
&\le C\|\varphi_h\|_{1,3}\|q_h\|_1\|p_h\|_{0,6}+\|y_h-z_d\|_0\|p_h\|_0+2b_2\|p_h\|_{0,6}\|\varphi_h\|_{1,3}\|q_h\|_1\le\\
&\le C\|\theta\|_{0,\Gamma}\|q_h\|_1\|p_h\|_1+\|y_h-z_d\|_0\|p_h\|_1\le C_\delta\,\|y_h-z_d\|_0^2+\delta\,\|p_h\|_1^2+C_\delta\,\|\theta\|_{0,\Gamma}^2\,\|q_h\|_1^2\,,\\
&\|p_h\|_1^2+\|q_h\|_1^2\le C\,\|y_h-z_d\|_0^2+\delta\,\|p_h\|_1^2+C\,\|\theta\|_{0,\Gamma}^2\,\|q_h\|_1^2\,,(0<\delta<1)
\end{aligned}$$

We choose θ such that $C\|\theta\|_\Gamma^2<1$. Then the following inequalities hold

$$\|p_h\|_1^2+(1-C\,\|\theta\|_{0,\Gamma}^2)\,\|q_h\|_1^2\le C\,\|y_h-z_d\|_0^2\,, C_1(\|p_h\|_1^2+\|q_h\|_1^2)\le\tilde{C}\,\|y_h-z_d\|_0^2\le C_2$$

where $C_1:=min(2-C\|\theta\|_{0,\Gamma}^2,1)$. This completes the proof.

For simplicity, denote the cost function as

$$\tilde{J}(\,g):=\frac{1}{2}\,\|y(g)-z_d\|_0^2+\frac{\nu}{2}\,\|g\|_0^2\,,$$

then (refer to [4]) we can obtain:

$$\begin{aligned}
&(\tilde{J}'(g),\upsilon)=(\nu g+p,\upsilon)_0,\\
&(\tilde{J}'(g_h),\upsilon_h)=(\nu g_h+p_h,\upsilon_h)_0,\\
&(\tilde{J}'(g_h),\upsilon)=(\nu g_h+p^h,\upsilon)_0,
\end{aligned}$$

with $p^h\in V$ satisfying

$$\begin{aligned}
&(\nabla p^h,\nabla\upsilon)_0+\,(\beta p^h,\upsilon)_\Gamma-(\sigma'(y^h)\left|\nabla\varphi^h\right|^2 p^h,\upsilon)_0+(\sigma'(y^h)\nabla\varphi^h\nabla q^h,\upsilon)_0=(y^h-z_d,\upsilon)_0,\forall\upsilon\in V,\\
&(\sigma(y^h)\nabla q^h,\nabla\omega_h)_0=(p^h\sigma(y^h)\nabla\varphi^h\,,\nabla\omega_h)_0,\forall\omega\in V_{\Gamma_2}
\end{aligned}\tag{11}$$

$$\begin{aligned}
&(\nabla y^h,\nabla\upsilon)_0+(\beta y^h,\upsilon)_\Gamma-(\sigma(y^h)\left|\nabla\varphi^h\right|^2,\upsilon)_0=(g_h,\upsilon)_0,\forall\upsilon\in V,\\
&(\sigma(y^h)\nabla\varphi^h,\nabla\upsilon)_0=(\theta,\upsilon)_{\Gamma_1},\forall\upsilon\in V_{\Gamma_2}
\end{aligned}\tag{12}$$

[Assumption 2] The functional $\tilde{J}$ is strictly convex near g, i.e., there exists a constant C satisfying

$$(\tilde{J}'(v) - \tilde{J}'(g),\, v - g) \geq C \left\| v - g \right\|^2_{L^2(\Omega)}$$

for all v in this neighborhood of g.

Let Assumption 1, 2 be valid in the subsequent theorems.

[Lemma 5] Let $(y,\, \varphi,\, p,\, q;\, g)$, $(y_h,\, \varphi_h,\, p_h, q_h;\, g_h)$ be the solutions of (3) and (9), respectively. Then we have

$$\begin{aligned} &\left\| g - g_h \right\|^2_0 \leq C \left\| p^h - p_h \right\|^2_1 + \eta^2_9, \\ &\eta^2_9 := \left\| g_h + p_h \right\|^2_0, \end{aligned}$$

where p^h is defined by (10) and C denotes a general positive constant independent of h.

(Proof) From the assumption 2, we obtain

$$\begin{aligned} C \left\| g - g_h \right\|^2_0 &\leq (\tilde{J}'(g) - \tilde{J}'(g_h), g - g_h)_0 = \\ &= (g + p, g - g_h)_0 + (g_h + p^h, g_h - g)_0 \leq \\ &\leq (g_h + p^h, g_h - g)_0 \leq \\ &\leq (g_h + p_h, g_h - g)_0 + (p^h - p_h, g_h - g)_0 \end{aligned}$$

Here, the optimality condition(Theorem 1) is taken into account.

It follows from the above inequality that

$$\begin{aligned} C \left\| g - g_h \right\|^2_0 \leq C(\delta) \left\| g_h + p_h \right\|^2_0 + C(\delta) \left\| p^h - p_h \right\|^2_0 + \\ + \delta \left\| g_h - g \right\|^2_0 \end{aligned}$$

Therefore, we can get

$$\left\| g - g_h \right\|^2_0 \leq C \left\| p^h - p_h \right\|^2_0 + C\eta^2_9 \leq C \left\| p^h - p_h \right\|^2_1 + C\eta^2_9$$

which completes the proof.

[Lemma 6] Let $(y_h, \varphi_h, p_h, q_h; g_h), (y^h, \varphi^h, p^h, q^h; g_h)$ be the solutions of (9) and (10), (11), respectively. Then there holds

$$\begin{aligned} \left\| y_h - y^h \right\|^2_1 + \left\| \varphi_h - \varphi^h \right\|^2_1 + \left\| p_h - p^h \right\|^2_1 + \left\| q_h - q^h \right\|^2_1 \leq \\ \leq C(\theta) \sum_{i=1}^{8} \eta^2_i \end{aligned}$$

where

$$\begin{aligned} &\eta^2_1 = \sum_{K \in T_h} h^2_K \left\| \varsigma_K \right\|^2_{0,K}, \\ &\varsigma_K := y_h - z_d + \Delta p_h - \sigma'(y_h)(\left| \nabla\varphi_h \right|^2 p_h - \nabla\varphi_h \nabla q_h) \\ &\eta^2_2 = \sum_{E \in E_h} h_E \left\| \varsigma_E \right\|^2_{0,E}, \\ &\varsigma_E := \begin{cases} \left[\frac{\partial p_h}{\partial n}\right]_E = (\nabla p_h - \nabla \bar{p}_h) \cdot n_K, E \cap \Gamma = \emptyset \\ \frac{\partial p_h}{\partial n} |_E + \beta p_h, E \cap \Gamma \neq \emptyset \end{cases} \end{aligned}$$

$C(\theta)$ is a positive constant only depending on $\|\theta\|_{0,\Gamma}$ and Ω.

The quantity $\left[\frac{\partial p_h}{\partial n}\right]_E$ defined on the edge E measures the jump of $\frac{\partial p_h}{\partial n}$, across the element edge E.

$$\begin{aligned} &\eta^2_3 = \sum_{K \in T_h} h^2_K \left\| \eta_K \right\|^2_{0,K}, \\ &\eta_K := div[\sigma(y_h)\nabla q_h - 2p_h\sigma(y_h)\nabla\varphi_h], \\ &\eta^2_4 = \sum_{E \in E_h} h_E \left\| \eta_E \right\|^2_{0,E}, \\ &\eta_E := \begin{cases} [(\sigma(y_h)\nabla q_h \cdot n - 2p_h\sigma(y_h)\nabla\varphi_h) \cdot n]_E, E \cap \Gamma = \emptyset \\ (\sigma(y_h)\nabla q_h \cdot n - 2p_h\sigma(y_h)\nabla\varphi_h \cdot n) |_E, E \cap \Gamma \neq \emptyset \end{cases} \end{aligned}$$

The quantity $\left[\frac{\partial(\cdot)}{\partial n}\right]_E$ measures the jump of the parameter function $\frac{\partial(\cdot)}{\partial n}$ across the element edge E.

$$\begin{aligned} &\eta^2_5 = \sum_{K \in T_h} h^2_K \left\| \gamma_K \right\|^2_{0,K}, \\ &\gamma_K := div(\sigma(y_h)\nabla\varphi_h), \\ &\eta^2_6 = \sum_{E \in E_h} h_E \left\| \gamma_E \right\|^2_{0,E}, \\ &\gamma_E := \begin{cases} \sigma(u_h) \left[\frac{\partial\varphi_h}{\partial n}\right], \partial K \cap \Gamma = \emptyset \\ \sigma(u_h) \frac{\partial\varphi_h}{\partial n} + \theta, \partial K \cap \Gamma \neq \emptyset, (E \subset \partial K) \end{cases} \end{aligned}$$

The quantity $\left[\frac{\partial(\cdot)}{\partial n}\right]_E$ measures the jump of the parameter function $\frac{\partial(\cdot)}{\partial n}$ across the element edge E.

$$\begin{aligned} &\eta^2_7 = \sum_{K \in T_h} h^2_K \left\| r_K \right\|^2_{0,K}, \\ &r_K := g_h + \Delta y_h + \sigma(y_h)\left| \nabla\varphi_h \right|^2, \\ &\eta^2_8 = \sum_{E \in E_h} h_E \left\| r_E \right\|^2_{0,E}, \\ &r_E := \begin{cases} \left[\frac{\partial y_h}{\partial n}\right]_E = (\nabla y_h - \nabla \bar{y}_h) \cdot n_K, E \cap \Gamma = \emptyset \\ \frac{\partial y_h}{\partial n} |_E + \beta y_h, E \cap \Gamma \neq \emptyset \end{cases} \end{aligned}$$

The quantity $\left[\frac{\partial y_h}{\partial n}\right]_E$ measures the jump of $\frac{\partial y_h}{\partial n}$ across the element edge E.

(Proof) (Part 1) We take

$$\xi := p^h - p_h, \xi_I := I_h\xi \in V_h \subset H^1(\Omega)$$

. By using the assumption of $\sigma(\cdot)$, we have

$$\begin{aligned}
C\|\xi\|_1^2 &\le (\nabla\xi,\nabla\xi)_0+(\beta\xi,\xi)_{0,\Gamma}=(\nabla\xi,\nabla\xi)_0+(\beta\xi,\xi)_{0,\Gamma}-(\sigma'(y^h)\left|\nabla\varphi^h\right|^2\xi,\xi)_0\le\\
&\le(\nabla p^h,\nabla\xi)_0+(\beta p^h,\xi)_{0,\Gamma}-(\sigma'(y^h)p^h\left|\nabla\varphi^h\right|^2,\xi)_0-(\nabla p_h,\nabla\xi)_0-(\beta p_h,\xi)_{0,\Gamma}+(\sigma'(y^h)p_h\left|\nabla\varphi^h\right|^2,\xi)_0=\\
&=(y^h-z_d,\xi)_0-(\sigma'(y^h)\nabla\varphi^h\nabla q^h,\xi)_0-(\nabla p_h,\nabla\xi)_0-(\beta p_h,\xi)_{0,\Gamma}+(\sigma'(y_h)p_h|\nabla\varphi_h|^2,\xi)_0-\\
&\quad-(\sigma'(y_h)\nabla\varphi_h\nabla q_h,\xi)_0]+(\sigma'(y^h)p_h\left|\nabla\varphi^h\right|^2,\xi)_0-(\sigma'(y_h)p_h|\nabla\varphi_h|^2,\xi)_0+(\sigma'(y_h)\nabla\varphi_h\nabla q_h,\xi)_0=\\
&=(y_h-z_d,\xi-\xi_I)_0+(y^h-y_h,\xi)_0-(\sigma'(y^h)\nabla\varphi^h\nabla q^h,\xi)_0+\\
&\quad+[(-(\nabla p_h,\nabla(\xi-\xi_I))_0-(\beta p_h,\xi-\xi_I)_{0,\Gamma}+(\sigma'(y_h)p_h|\nabla\varphi_h|^2,\xi-\xi_I)_0-(\sigma'(y_h)\nabla\varphi_h\nabla q_h,\xi-\xi_I)_0]+\\
&\quad+(\sigma'(y^h)p_h\left|\nabla\varphi^h\right|^2,\xi)_0-(\sigma'(y_h)p_h|\nabla\varphi_h|^2,\xi)_0+(\sigma'(y_h)\nabla\varphi_h\nabla q_h,\xi)_0\\
C\|\xi\|_1^2 &\le \sum_{K\in T_h}(\varsigma_K,\xi-\xi_I)_{0,K}-\sum_{E\in E_h}(\varsigma_E,\xi-\xi_I)_{0,E}+(y^h-y_h,\xi)_0+\\
&\quad+((\sigma'(y^h)\left|\nabla\varphi^h\right|^2-\sigma'(y_h)|\nabla\varphi_h|^2)p_h,\xi)_0+(\sigma'(y_h)\nabla\varphi_h\nabla q_h-\sigma'(y^h)\nabla\varphi^h\nabla q^h,\xi)_0
\end{aligned}$$

It is satisfied following:

$$\begin{aligned}
&((\sigma'(y^h)\left|\nabla\varphi^h\right|^2-\sigma'(y_h)|\nabla\varphi_h|^2)p_h=\\
&=\sigma'(y^h)(\nabla\varphi^h-\nabla\varphi_h)(\nabla\varphi^h+\nabla\varphi_h)p_h+\\
&\quad+(\sigma'(y^h)-\sigma'(y_h))|\nabla\varphi_h|^2p_h\\
&\left|((\sigma'(y^h)\left|\nabla\varphi^h\right|^2-\sigma'(y_h)|\nabla\varphi_h|^2)p_h,\xi)_0\right|\le\\
&\le C\left\|y^h-y_h\right\|_{0,6}\left\||\nabla\varphi_h|^2\right\|_{0,3/2}\|\xi\|_{0,6}+\\
&\quad+C\left\|\nabla(\varphi^h-\varphi_h)\right\|_0(\left\|\nabla\varphi^h\right\|_{0,3}+\|\nabla\varphi_h\|_{0,3})\|\xi\|_{0,6}\le\\
&\le C\left\|y^h-y_h\right\|_1\|\xi\|_1+C\left\|(\varphi^h-\varphi_h)\right\|_1\|\theta\|_{0,\Gamma}\|\xi\|_1\le\\
&\le C\left\|y^h-y_h\right\|_1^2+C\|\theta\|_{0,\Gamma}\left\|\varphi^h-\varphi_h\right\|_1^2\\
&\left|(\sigma'(y_h)\nabla\varphi_h\nabla q_h-\sigma'(y^h)\nabla\varphi^h\nabla q^h,\xi)_0\right|\le\\
&\le\left\|\sigma'(y_h)-\sigma'(y^h)\right\|_{0,\infty}\|\nabla\varphi_h\|_{0,3}\|\nabla q_h\|_{0,3}\|\xi\|_{0,3}+\\
&\quad+C(\left\|\varphi_h-\varphi^h\right\|_1+C\|\theta\|_{0,\Gamma}\left\|q_h-q^h\right\|_1)\|\xi\|_{0,3}\le\\
&\le C(\delta)(\left\|y_h-y^h\right\|_1^2+\left\|\varphi_h-\varphi^h\right\|_1^1+\\
&\quad+C\|\theta\|_{0,\Gamma}\left\|q_h-q^h\right\|_1^2)+\delta\|\xi\|_1^2,
\end{aligned}$$

Note that $\xi-\xi_I\in V=H^1(\Omega)$, and from Lemma 3, we have

$$\begin{aligned}
C\|\xi\|_1^2&\le C(\delta)(\sum_{K\in T_h}\|\varsigma_K\|_{0,K}^2h_K^2+\sum_{E\in E_h}\|\varsigma_E\|_{0,E}^2h_E)+\\
&+C\left\|y_h-y^h\right\|_1^2+C\left\|\varphi_h-\varphi^h\right\|_1^2+\\
&+C\|\theta\|\left\|q_h-q^h\right\|_1^2)+\delta\|\xi\|_1^2.
\end{aligned}$$

$$\begin{aligned}
\left\|p^h-p_h\right\|_1&\le C(\left\|y_h-y^h\right\|_1^2+\left\|\varphi_h-\varphi^h\right\|_1^2+\\
&\quad+\|\theta\|\left\|q_h-q^h\right\|_1^2)+C(\eta_1^2+\eta_2^2)
\end{aligned}\tag{13}$$

(Part 2) We set

$$\xi:=q^h-q_h,\xi_I:=I_h\xi\in V_h\subset H^1(\Omega),\xi\in V$$

$$\begin{aligned}
C\|\xi\|_1^2&\le(\sigma(y^h)\nabla\xi,\nabla\xi)_0=\\
&=(\sigma(y^h)\nabla q^h,\nabla\xi)_0-(\sigma(y^h)\nabla q_h,\nabla\xi)_0=\\
&=2(p^h\sigma(y^h)\nabla\varphi^h,\nabla\xi)_0+((\sigma(y_h)-\sigma(y^h))\nabla q_h,\nabla\xi)_0-\\
&\quad-(\sigma(y_h)\nabla q_h,\nabla\xi)_0+2(p_h\sigma(y_h)\nabla\varphi_h,\nabla\xi)_0-\\
&\quad-2(p_h\sigma(y_h)\nabla\varphi_h,\nabla\xi)_0
\end{aligned}$$

By adding

$$(\sigma(y_h)\nabla q_h,\nabla\xi_I)_0-2(p_h\sigma(y_h)\nabla\varphi_h,\nabla\xi_I)_0=0$$

, we can get

$$\begin{aligned}
&=2(p^h\sigma(y^h)\nabla\varphi^h-p_h\sigma(y_h)\nabla\varphi_h,\nabla\xi)_0+\\
&\quad+((\sigma(y_h)-\sigma(y^h))\nabla q_h,\nabla\xi)_0-\\
&\quad-(\sigma(y_h)\nabla q_h,\nabla(\xi-\xi_I))_0+\\
&\quad+2(p_h\sigma(y_h)\nabla\varphi_h,\nabla(\xi-\xi_I))_0=\\
&=\sum_{K\in T_h}(\eta_K,\xi-\xi_I)_{0,K}-\sum_{E\in E_h}(\eta_E,\xi-\xi_I)_{0,E}+\\
&\quad+2(p^h\sigma(y^h)\nabla\varphi^h-p_h\sigma(y_h)\nabla\varphi_h,\nabla\xi)_0+\\
&\quad+((\sigma(y_h)-\sigma(y^h))\nabla q_h,\nabla\xi)_0\le\\
&\le\sum_{K\in T_h}Ch_K^2\|\eta_K\|_{0,K}^2+\sum_{E\in E_h}Ch_E\|\eta_E\|_{0,E}^2+\delta\|\xi\|_1^2+\\
&\quad+C\left\|y^h-y_h\right\|_1^2+\|\theta\|\left\|p^h-p_h\right\|_1^2+C\left\|\varphi^h-\varphi_h\right\|_1^2\le\\
&\le\eta_3^2+\eta_4^2+\|\theta\|\left\|p^h-p_h\right\|_1^2+C\left\|y^h-y_h\right\|_1^2+\\
&\quad+C\left\|\varphi^h-\varphi_h\right\|_1^2+\delta\|\xi\|_1^2
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\left\|q^h-q_h\right\|_1^2&\le C(\eta_3^2+\eta_4^2)+C\|\theta\|\left\|p^h-p_h\right\|_1^2+\\
&\quad+C\left\|y^h-y_h\right\|_1^2+C\left\|\varphi^h-\varphi_h\right\|_1^2
\end{aligned}\tag{14}$$

(Part 3) We take $\xi:=\varphi^h-\varphi_h,\xi_I:=I_h\xi\in V_h\subset H^1(\Omega)$. Then $\xi\in V$.

$$
\begin{aligned}
&C_1\|\xi\|_1^2 \le b_1\int\limits_\Omega |\nabla\xi|^2 dx \le \int\limits_\Omega \sigma(y^h)|\nabla\xi|^2 dx = \\
&= \int\limits_\Omega \sigma(y^h)\nabla\varphi^h\nabla\xi dx - \int\limits_\Omega \sigma(y^h)\nabla\varphi_h\nabla\xi dx - \int\limits_\Omega \sigma(y_h)\nabla\varphi_h\nabla\xi dx + \int\limits_\Omega \sigma(y_h)\nabla\varphi_h\nabla\xi dx = \\
&= \int\limits_\Omega (\sigma(y^h)\nabla\varphi^h - \sigma(y_h)\nabla\varphi_h)\nabla\xi dx - \int\limits_\Omega (\sigma(y^h)\nabla\varphi_h - \sigma(y_h)\nabla\varphi_h)\nabla\xi dx = \\
&= \int\limits_\Omega \sigma(y^h)\nabla\varphi^h\nabla(\xi-\xi_I)dx - \int\limits_\Omega \sigma(y_h)\nabla\varphi_h\nabla(\xi-\xi_I)dx - \int\limits_\Omega (\sigma(y^h)-\sigma(y_h))\nabla\varphi_h\nabla\xi dx = \\
&= \int\limits_\Gamma \theta(\xi-\xi_I)d\Gamma - \sum_{K\in T_h}\int\limits_K -div(\sigma(y_h)\nabla\varphi_h)(\xi-\xi_I)dx - \\
&\quad - \sum_{K\in T_h}\int\limits_{\partial K}\sigma(y_h)\frac{\partial\varphi_h}{\partial n}(\xi-\xi_I)dl - \int\limits_\Omega (\sigma(y^h)-\sigma(y_h))\nabla\varphi_h\nabla\xi dx \le \\
&\le C(\delta)(\sum_{K\in T_h} h_K^2\|\gamma_K\|_{0,K}^2 + \sum_{E\in E_h} h_E\|\gamma_E\|_{0,E}^2) + C(\delta)\left\|y^h-y_h\right\|_0^2 + \delta\|\xi\|_1^2 \le \\
&\le C(\delta)(\eta_5^2+\eta_6^2) + C(\delta)\left\|y^h-y_h\right\|_0^2 + \delta\|\xi\|_1^2
\end{aligned}
$$

Hence, set $\delta = \frac{C_1}{2}$, by simplifying both sides, we have

$$\left\|\varphi^h-\varphi_h\right\|_1^2 \le C(\eta_5^2+\eta_6^2) + C\left\|y^h-y_h\right\|_1^2 \tag{15}$$

(Part 4) We take

$$\xi := y^h - y_h, \xi_I := I_h\xi \in V_h \subset H^1(\Omega), \xi \in V$$

$$
\begin{aligned}
&C_1\|\xi\|_1^2 \le \int\limits_\Omega |\nabla\xi|^2 dx + \int\limits_\Gamma \beta\xi^2 d\Gamma - \int\limits_\Omega ((\sigma(y^h)-\sigma(y_h))\left|\nabla\varphi^h\right|^2\xi dx = \\
&= \int\limits_\Omega \nabla y^h\nabla\xi dx + \int\limits_\Gamma \beta y^h\xi d\Gamma - \int\limits_\Omega \sigma(y^h)\left|\nabla\varphi^h\right|^2\xi dx - \int\limits_\Omega \nabla y_h\nabla\xi dx - \int\limits_\Gamma \beta y_h\xi d\Gamma + \int\limits_\Omega \sigma(y_h)|\nabla\varphi_h|^2\xi dx - \\
&\quad - \int\limits_\Omega \sigma(y_h)|\nabla\varphi_h|^2\xi dx + \int\limits_\Omega \sigma(y_h)\left|\nabla\varphi^h\right|^2\xi dx = \\
&= \sum_{K\in T_h}\int\limits_K (g_h + \Delta y_h + \sigma(y_h)|\nabla\varphi_h|^2)(\xi-\xi_I)dx - \sum_{K\in T_h}\int\limits_{\partial K}\frac{\partial y_h}{\partial n}(\xi-\xi_I)d\Gamma - \\
&- \int\limits_\Gamma \beta y_h(\xi-\xi_I)d\Gamma + \int\limits_\Omega \sigma(y_h)(\left|\nabla\varphi^h\right|^2 - |\nabla\varphi_h|^2)\xi dx \le \\
&\le \sum_{K\in T_h} C(\delta)\|r_K\|_{0,K}^2\cdot h_K^2 + \sum_{E\in E_h} C(\delta)\|r_E\|_{0,E}^2\cdot h_E + b_2\|\theta\|\left\|\varphi^h-\varphi_h\right\|_1^2 + \delta\|\xi\|_1^2 \le \\
&\le C(\delta)(\eta_7^2+\eta_8^2) + b_2\|\theta\|\left\|\varphi^h-\varphi_h\right\|_1^2 + \delta\|\xi\|_1^2
\end{aligned}
$$

Therefore, set $\delta = \frac{C_1}{2}$, by simplifying both sides, we obtain

$$\left\|y^h-y_h\right\|_1 \le C(\eta_7^2+\eta_8^2) + b_2\|\theta\|_\Gamma\left\|\varphi^h-\varphi_h\right\|_1^2 \le C\sum_{i=5}^8\eta_i^2 \tag{16}$$

Note that

$$\left\|q^h-q_h\right\|_1^2 \le C\sum_{i=3}^8\eta_i^2 + \bar{C}\|\theta\|_\Gamma\left\|p^h-p_h\right\|_1^2$$

Therefore, we can get

$$\left\|p^h-p_h\right\|_1^2 \le C\sum_{i=1}^8\eta_i^2 + \bar{C}\|\theta\|_\Gamma^2(\left\|p_h-p^h\right\|_1^2) \tag{17}$$

where $\bar{C}$ is a positive constant dependent on the maximum value of $|\sigma'(\cdot)|$ and Poincare Constant. Assume that θ satisfies $\bar{C}\,\|\theta\|_\Gamma^2 < 1$. Then we have

$$\left\|p^h - p_h\right\|_1^2 \le \tilde{C}(\theta)\sum_{i=1}^{8}\eta_i^2, \quad \tilde{C}(\theta) := \frac{1}{1-\bar{C}\,\|\theta\|_{0,\Gamma}^2} \tag{18}$$

From (12)-(17), we can derive the desired results. This completes the proof. In the follow theorem give a posteriori error estimates which it is main result of our work.

[Theorem 2] Let $(y,\varphi,p,q;g),(y_h,\varphi_h,p_h,q_h;g_h)$ be the solutions of (3), (9), respectively. Then there holds

$$\|g-g_h\|_0^2 + \|y-y_h\|_1^2 + \|\varphi-\varphi_h\|_1^2 + \|p-p_h\|_1^2 + \\ + \|q-q_h\|_1^2 \le \widehat{C}(\theta)\sum_{i=1}^{9}\eta_i^2$$

where $\eta_i^2 (i=1,2,\cdots,8)$ are the quantities from Lemma 5, 6 and $\widehat{C}(\theta)$ is a positive constant obtained in the process of proof.

(Proof) We have

$$\|g-g_h\|_0^2 \le C\left\|p_h - p^h\right\|_1^2 \le \tilde{C}(\theta)\sum_{i=1}^{9}\eta_i^2 \tag{19}$$

Note that

$$\begin{aligned} \|y-y_h\|_1^2 &\le \left\|y-y^h\right\|_1^2 + \left\|y^h-y_h\right\|_1^2, \\ \|\varphi-\varphi_h\|_1^2 &\le \left\|\varphi-\varphi^h\right\|_1^2 + \left\|\varphi^h-\varphi_h\right\|_1^2, \\ \|p-p_h\|_1^2 &\le \left\|p-p^h\right\|_1^2 + \left\|p^h-p_h\right\|_1^2, \\ \|q-q_h\|_1^2 &\le \left\|q-q^h\right\|_1^2 + \left\|q^h-q_h\right\|_1^2 \end{aligned} \tag{20}$$

By combining (1) and (11) regarding y, y^h, taking $\upsilon = y - y^h$ by the trial function and using the assumption of $\sigma(\cdot)$, we have

$$\int_\Omega \left|\nabla(y-y^h)\right|^2 dx + \beta\int_\Gamma \left|y-y^h\right|^2 d\Gamma - \int_\Omega (\sigma(y)|\nabla\varphi|^2 - \sigma(y^h)\left|\nabla\varphi^h\right|^2)(y-y^h)dx = \int_\Omega (g-g_h)(y-y^h)dx$$

Note that

$$(\sigma(y)|\nabla\varphi|^2 - \sigma(y^h)\left|\nabla\varphi^h\right|^2)(y-y^h) = \sigma(y)(|\nabla\varphi|^2 - \left|\nabla\varphi^h\right|^2)(y-y^h) + \left|\nabla\varphi^h\right|^2(\sigma(y)-\sigma(y^h))(y-y^h)$$

and that $\sigma(\cdot)$ is a decreasing function, we can get

$$\begin{aligned} C_1\left\|y-y^h\right\|_1^2 &\le \int_\Omega \left|\nabla y - \nabla y^h\right|^2 dx + \beta\int_\Gamma \left|y-y^h\right|^2 d\Gamma \le \int_\Omega \sigma(y)(|\nabla\varphi|^2 - \left|\nabla\varphi^h\right|^2)(y-y^h)dx + \\ &+ \int_\Omega (g-g_h)(y-y^h)dx \le C(\|\theta\|_\Gamma\left\|\varphi-\varphi^h\right\|_1 + \|g-g_h\|_0)\left\|y-y^h\right\|_1 \\ \left\|y-y^h\right\|_1 &\le C(\|\theta\|_\Gamma\left\|\varphi-\varphi^h\right\|_1 + \|g-g_h\|_0) \end{aligned} \tag{21}$$

By combining (1) and (11) regarding φ, φ^h, taking $\upsilon = \varphi - \varphi^h$ by the trial function and using the assumption of $\sigma(\cdot)$, we can obtain

$$\begin{aligned} C_1\left\|\varphi-\varphi^h\right\|_1^2 &\le b_1\int_\Omega \left|\nabla\varphi - \nabla\varphi^h\right|^2 dx \le \int_\Omega \sigma(y)\left|\nabla\varphi - \nabla\varphi^h\right|^2 dx = \int_\Omega (\sigma(y^h)-\sigma(y))\nabla\varphi^h\nabla(\varphi-\varphi^h)dx \\ &\le C\left\|y-y^h\right\|_{L^6(\Omega)}\left\|\nabla\varphi^h\right\|_{L^3(\Omega)}\left\|\nabla\varphi-\nabla\varphi^h\right\|_{L^2(\Omega)} \le C\|\theta\|_\Gamma\left\|y-y^h\right\|_1\left\|\varphi-\varphi^h\right\|_1 \end{aligned}$$

Thus, it holds that

$$\begin{aligned} &\left\|\varphi-\varphi^h\right\|_1 \le C\|\theta\|_\Gamma\left\|y-y^h\right\|_1, \left\|y-y^h\right\|_1 \le C\,\|\theta\|_\Gamma^2\left\|y-y^h\right\|_1 + C\|g-g_h\|_0), \\ &\left\|y-y^h\right\|_1 \le \tilde{C}(\theta)\|g-g_h\|_0, \left\|\varphi-\varphi^h\right\|_1 \le C\|\theta\|_\Gamma\tilde{C}(\theta)\|g-g_h\|_0 = \bar{C}(\theta)\|g-g_h\|_0 \end{aligned} \tag{22}$$

where $\tilde{C}(\theta)$ is a positive constant defined in (8) and $\bar{C}(\theta) := C\|\theta\|_\Gamma\tilde{C}(\theta)$.

Similarly, by combining (3) and (11) regarding q, q^h, taking $\upsilon = q - q^h$ and using the assumption of $\sigma(\cdot)$, we can obtain

$$\int_\Omega \sigma(y)\nabla q\nabla\upsilon dx - \int_\Omega \sigma(y^h)\nabla q^h\nabla\upsilon dx = 2\int_\Omega (p\sigma(y)\nabla\varphi - p^h\sigma(y^h)\nabla\varphi^h)\nabla\upsilon dx$$

Note that

$$\sigma(y)\nabla q - \sigma(y^h)\nabla q^h\nabla\upsilon dx = 2\int_\Omega (p\sigma(y)\nabla\varphi - p^h\sigma(y^h)\nabla\varphi^h)\nabla\upsilon dx$$

we have

$$C_1\|q-q^h\|_1 \le b_1\int_\Omega |\nabla q-\nabla q^h|^2 dx \le \int_\Omega \sigma(y)|\nabla q-\nabla q^h|^2 dx \le$$
$$\le \int_\Omega \nabla q^h(\sigma(y^h)-\sigma(y))(\nabla q-\nabla q^h)dx \tag{23}$$

$$\|q-q^h\|_1 \le C(\|y^h-y\|_1+\|\theta\|_\Gamma\|p-p^h\|_1+\|\varphi-\varphi^h\|_1) \le \tilde{C}(\theta)\|g-g\|_0+C\|\theta\|_\Gamma\|p-p^h\|_1$$
$$+2\int_\Omega (p\sigma(y)\nabla\varphi-p^h\sigma(y^h)\nabla\varphi^h) \tag{24}$$
$$(\nabla q-\nabla q^h)dx \le C(\|y^h-y\|_1+\|\theta\|_\Gamma\|p-p^h\|_1+\|\varphi-\varphi^h\|_1)\|q-q^h\|_1$$

Similarly, by combining (3) and (9) regarding p, p^h, setting $\upsilon := p-p^h$ and using the assumption of $\sigma(\cdot)$, we can obtain

$$\int_\Omega |\nabla\xi|^2 dx+\beta\int_\Gamma \xi^2 d\Gamma-\int_\Omega (\sigma'(y)|\nabla\varphi|^2 p-\sigma'(y^h)|\nabla\varphi^h|^2 p^h)\xi dx+$$
$$+\int_\Omega (\sigma'(y)\nabla\varphi\nabla q-\sigma'(y^h)\nabla\varphi^h\nabla q^h)\xi dx=\int_\Omega (y-y^h)\xi dx$$

Then there holds

$$\sigma'(y)|\nabla\varphi|^2 p-\sigma'(y^h)|\nabla\varphi^h|^2 p^h=\sigma'(y)|\nabla\varphi|^2 p-\sigma'(y)|\nabla\varphi|^2 p^h+\sigma'(y)|\nabla\varphi|^2 p^h-\sigma'(y^h)|\nabla\varphi^h|^2 p^h=$$
$$=\sigma'(y)|\nabla\varphi|^2\xi+p^h\sigma'(y)|\nabla\varphi|^2-\sigma'(y^h)|\nabla\varphi^h|^2=$$
$$=\sigma'(y)|\nabla\varphi|^2\xi+\sigma'(y)(|\nabla\varphi|^2-|\nabla\varphi^h|^2)+|\nabla\varphi^h|^2(\sigma'(y)-\sigma'(y^h))$$
$$\sigma'(y)\nabla\varphi\nabla q-\sigma'(y^h)\nabla\varphi^h\nabla q^h=\sigma'(y)\nabla\varphi(\nabla q-\nabla q^h)+\nabla q^h(\sigma'(y)\nabla\varphi-\sigma'(y^h)\nabla\varphi^h)=$$
$$=\sigma'(y)\nabla\varphi(\nabla q-\nabla q^h)+\sigma'(y)(\nabla\varphi-\nabla\varphi^h)\nabla q^h+\nabla\varphi^h(\sigma'(y)-\sigma'(y^h))\nabla q^h$$

Using the above estimations, we can get

$$C_1\|\xi\|_1^2 \le \int_\Omega |\nabla\xi|^2 dx+\int_\Gamma \beta\xi^2 d\Gamma \le C(\|\varphi-\varphi^h\|_1+\|y-y^h\|_1+\|\theta\|_{0,\Gamma}\|q-q^h\|_1)\|\xi\|_1,$$
$$\|p-p^h\|_1 \le C(\|\varphi-\varphi^h\|_1+\|y-y^h\|_1)+C\|\theta\|_\Gamma\|q-q^h\|_1) \le \tilde{C}(\theta)\|g-g_h\|_0+C\|\theta\|_\Gamma\|q-q^h\|_1 \tag{25}$$

From (22) and (23), we have

$$\|p-p^h\|_1 \le C(\|\varphi-\varphi^h\|_1+\|y-y^h\|_1)+C\|\theta\|_\Gamma\|q-q^h\|_1) \le$$
$$\le 2\tilde{C}(\theta)\|g-g_h\|_0+C\|\theta\|_\Gamma^2\|p-p^h\|_1, \|p-p^h\|_1 \le 2\tilde{C}(\theta)^2\|g-g_h\|_0 \tag{26}$$

$$\|q-q^h\|_1 \le C(\|y^h-y\|_1+\|\theta\|_\Gamma\|p-p^h\|_1+\|\varphi-\varphi^h\|_1) \le \tilde{C}(\theta)\|g-g\|_0+C\|\theta\|_\Gamma\|p-p^h\|_1 \le$$
$$\le 2C\|\theta\|_\Gamma\tilde{C}(\theta)^2\|g-g_h\|_0+\tilde{C}(\theta)\|g-g_h\|_0, \|q-q^h\|_1 \le (\bar{C}(\theta)+\tilde{C}(\theta))\|g-g\|_0=\widehat{C}(\theta)\|g-g_h\|_0 \tag{27}$$

From (18)-(25), we get the results. This completes the proof.

5. A Priori Error Estimation

Before getting into a priori error estimation, we need to consider the following result.

[Lemma 7] If $(y,\varphi,p,q;\ g), (y_h,\varphi_h,p_h,q_h;g_h)$ are solutions of (3), (8) respectively, then we have the following inequality.

$$\frac{\nu}{2}\|g-g_h\|_0 \le C\|p-p_h\|_1+Ch$$

, where C is constant independent of h and p_h is the solution of (8).

(Proof) g, g_h satisfy the following estimates.

$$(p+\nu g, w)_0=0,\ \forall w\in U,\ \ g\in U \tag{28}$$

$$(p_h+\nu g_h, w_h)_0=0,\ \forall w_h\in U_h,\ \ g_h\in U_h \tag{29}$$

Choose $\sigma(0\le\sigma\le 1)$, $\tilde{g}\in H^1(\Omega)$.

Let I_h be an orthogonal projection operator as $V\to V_h$.

Define

$$g^\sigma := (1-\sigma)g_h + \sigma\tilde{g}, \quad g_h^\sigma := (1-\sigma)I_h g + \sigma I_h \tilde{g} \tag{30}$$

and with $g \in H^(\Omega)$, we get to know that $g^\sigma \in H^1(\Omega), g_h^\sigma \in U_h$.

By taking $w = g^\sigma,\ w_h = g_h^\sigma$ in (26), (27) and combining them, we get the following:

$$\nu \left\| g - g_h \right\|_0^2 \le (\nu g, g^\sigma - g_h)_0 + (\nu g_h,\, g_h^\sigma - g)_X + (p, g^\sigma - g_h)_0 + (p_h, g_h^\sigma - g)_0 + (p - p_h, g_h - g)_0 \tag{31}$$

Using Cauchy-Schwarz inequality, we obtain

$$(p - p_h, g_h - g)_0 \le \left\| p - p_h \right\|_1^2 + \frac{\nu}{2} \left\| g_h - g \right\|_0^2 \tag{32}$$

Note that the conjugate solutions are bounded independently of h, then $\exists h_0 > 0, \forall h < h_0, 0 < h < 1, \sigma = h$, we have

$$\begin{array}{ll} (\nu g,\, g^\sigma - g_h)_0 = \sigma\nu(g,\, \tilde{g} - g_h)_0 \le \sigma\nu \|g\|_0 (\|\tilde{g}\|_0 + \|g_h\|_0) \le Ch, & \\ (\nu g_h,\, g_h^\sigma - g)_0 = (\nu g_h,\, I_h g - g)_0 + \sigma(\nu g_h,\, I_h(\tilde{g} - g))_0 \le Ch, & \\ (p,\, g^\sigma - g_h)_0 = \sigma(p, \tilde{g} - g_h)_0 \le Ch, (p_h, g_h^\sigma - g)_0 & = (\nu p_h,\, I_h g - g)_0 + \sigma(\nu p_h,\, I_h(\tilde{g} - g))_0 \le Ch \end{array} \tag{33}$$

Thus, the desired result is derived from (29), (30), (31).This can complete the proof. Consider the following:

$$\begin{array}{l} (J'(g), \upsilon) = (\nu g + p, \upsilon)_0, \\ (J'(g_h), \upsilon_h) = (\nu g_h + p_h, \upsilon_h)_0, \\ (J'(g), \upsilon_h) = (\nu g + p^h, \upsilon_h)_0 \end{array}$$

with $p^h := p^h(g) \in V_h$ satisfying

$$\begin{array}{l} (\nabla p^h, \nabla \upsilon_h)_0 + (\beta p^h, \upsilon_h)_\Gamma - (\sigma'(y^h) \left|\nabla\varphi^h\right|^2 p^h, \upsilon_h)_0 + (\sigma'(y^h)\nabla\varphi^h \nabla q^h, \upsilon_h)_0 = (y^h - z_d, \upsilon_h)_0, \forall \upsilon_h \in V_h, \\ (\sigma(y^h)\nabla q^h, \nabla\omega_h)_0 = 2(p^h\sigma(y^h)\nabla\varphi^h\ , \nabla\omega_h)_0, \forall\omega_h \in V_{\Gamma_2,h} \end{array} \tag{34}$$

$$\begin{array}{l} (\nabla y^h, \nabla\upsilon_h)_0 +\ (\beta y^h, \upsilon_h)_\Gamma - (\sigma(y^h)\left|\nabla\varphi^h\right|^2, \upsilon_h)_0 = (g, \upsilon_h)_0, \quad \forall \upsilon_h \in V_h, \\ (\sigma(y^h)\nabla\varphi^h, \nabla\omega_h)_0 = (\theta, \omega_h)_{\Gamma_1}, \ \ \forall\omega_h \in V_{\Gamma_2,h} \end{array} \tag{35}$$

Now we can present the theorem for a priori error estimates of problem 1, which is our main results.

[Theorem 3] Assume that y, φ, p, q, the solutions of (3) belong to $H^2(\Omega)$. If $(y, \varphi, p, q; g), (y_h, \varphi_h, p_h, q_h; g_h)$ are the solutions of (3) and (8), respectively, then we have

$$\|g - g_h\|_0 + \|y - y_h\|_1 + \|\varphi - \varphi_h\|_1 + \|p_h - p\|_1 + \|q_h - q\|_1 \le C(\theta, \nu)\sqrt{h} + Ch$$

where $C(\theta, \nu)$ denotes a general positive constant, independent of h.

(Proof) Denoting $\xi := I_h p - p^h, \varsigma := p - I_h p$, we have $\xi + \varsigma = p - p^h$, where I_h is the operator defined in Lemma 3 and p^h is the unique solution of (10).

1. Estimation of $\|p - p^h\|_1$

$$\begin{array}{l} C \left\| \xi \right\|_1^2 \le (\nabla\xi, \nabla\xi)_0 + (\beta\xi, \xi)_{0,\Gamma} \le (\nabla\xi, \nabla\xi)_0 + (\beta\xi, \xi)_{0,\Gamma} - (\sigma'(y)|\nabla\varphi|^2\xi, \xi)_0 = \\ \quad = (\nabla\pi_h p, \nabla\xi)_0 + (\beta\pi_h p, \xi)_{0,\Gamma} - (\sigma'(y)\pi_h p|\nabla\varphi|^2, \xi)_0 - (\nabla p^h, \nabla\xi)_0 - (\beta p^h, \xi)_{0,\Gamma} + (\sigma'(y)p^h|\nabla\varphi|^2, \xi)_0 = \\ \quad = (\nabla\pi_h p, \nabla\xi)_0 + (\beta\pi_h p, \xi)_{0,\Gamma} - (\sigma'(y)\pi_h p|\nabla\varphi|^2, \xi)_0 - (\nabla p^h, \nabla\xi)_0 - (\beta p^h, \xi)_{0,\Gamma} + (\sigma'(y)p^h|\nabla\varphi|^2, \xi)_0 + \\ \qquad + (\nabla p, \nabla\xi)_0 + (\beta p, \xi)_{0,\Gamma} - (\nabla p, \nabla\xi)_0 - (\beta p, \xi)_{0,\Gamma} + (\sigma'(y)p|\nabla\varphi|^2, \xi)_0 - (\sigma'(y)p|\nabla\varphi|^2, \xi)_0 + \\ \qquad + (\sigma'(y)\nabla\varphi\nabla q, \xi)_0 - (\sigma'(y)\nabla\varphi\nabla q, \xi)_0 = \\ \quad = -(\nabla\varsigma, \nabla\xi)_0 - (\beta\varsigma, \xi)_{0,\Gamma} + (\sigma'(y)|\nabla\varphi|^2\varsigma, \xi)_0 + (y - z_d, \xi)_0 - (y^h - z_d, \xi)_0 - (\sigma'(y^h)\left|\nabla\varphi^h\right|^2 p^h, \xi)_0 + \\ \qquad + (\sigma'(y^h)\left|\nabla\varphi^h\right|^2 p^h, \xi)_0 + (\sigma'(y)|\nabla\varphi|^2 p^h, \xi)_0 - (\sigma'(y)\nabla\varphi\nabla q, \xi)_0 = \\ \quad = -(\nabla\varsigma, \nabla\xi)_0 - (\beta\varsigma, \xi)_{0,\Gamma} + (\sigma'(y)|\nabla\varphi|^2\varsigma, \xi)_0 - ((\sigma'(y)|\nabla\varphi|^2 - \sigma'(y^h)\left|\nabla\varphi^h\right|^2)p^h, \xi)_0 + \\ \qquad + (\sigma'(y^h)\nabla\varphi^h\nabla q^h - \sigma'(y)\nabla\varphi\nabla q, \xi)_0 \end{array}$$

By using Lipschitz continuity of $\sigma'(\cdot) \in L^{\infty}(\Omega), \sigma'(\cdot)$, we have

$$\begin{aligned}
&\left|(\sigma'(y)|\nabla\varphi|^2\varsigma,\xi)_0\right| \le \left\||\nabla\varphi|^2\right\|_{0,3/2}\|\varsigma\|_{0,1/6}\|\xi\|_{0,1/6} \le \\
&\le C\|\varphi\|_{1,3}\|\varsigma\|_1\|\xi\|_1 \le C\|\theta\|_{0,\Gamma}\|\varsigma\|_1\|\xi\|_1, \left|((\sigma'(y)|\nabla\varphi|^2 - \sigma'(y^h)\left|\nabla\varphi^h\right|^2)p^h,\xi)_0\right| \le \\
&\le C\|\theta\|_{0,\Gamma}(\left\|y-y^h\right\|_1 + \left\|\varphi-\varphi^h\right\|_1)\|\xi\|_1, \left|(\sigma'(y^h)\nabla\varphi^h\nabla q^h - \sigma'(y)\nabla\varphi\nabla q,\xi)_0\right| \le \\
&\le C\left\|y-y^h\right\|_{0,4}\left\|\nabla\varphi^h\nabla q^h\right\|_0\|\xi\|_{0,4} + C\left\|\varphi-\varphi^h\right\|_1 + +C\|\theta\|_{0,\Gamma}\left\|q-q^h\right\|_1 \le \\
&\le C(\left\|y-y^h\right\|_1 + \left\|\varphi-\varphi^h\right\|_1 + \|\theta\|_{0,\Gamma}\left\|q-q^h\right\|_1)\|\xi\|_1,
\end{aligned}$$

Considering Lemma 2, we can get the following result.

$$\begin{aligned}
&\|\xi\|_1^2 \le C(\|\varsigma\|_1 + \left\|y-y^h\right\|_1 + \left\|\varphi-\varphi^h\right\|_1) + C\|\theta\|_{0,\Gamma}\left\|q-q^h\right\|_1\|\xi\|_1, \\
&\|\xi\|_1 \le Ch + C(\left\|y-y^h\right\|_1 + \left\|\varphi-\varphi^h\right\|_1) + C\|\theta\|_{0,\Gamma}\left\|q-q^h\right\|_1, \\
&\left\|p-p^h\right\|_1 \le \|\xi\|_1 + \|\varsigma\|_1 \le Ch + C(\left\|y-y^h\right\|_1 + \left\|\varphi-\varphi^h\right\|_1) + C\|\theta\|_{0,\Gamma}\left\|q-q^h\right\|_1
\end{aligned} \tag{36}$$

2. Estimation of $\|q-q_h\|_1^2$
Denoting $\xi := I_h q - q^h,\ \ \varsigma := q - I_h q$, then $\xi + \varsigma = q - q^h$, and we have

$$\begin{aligned}
&\|\xi\|_1^2 \le b_1\|\nabla\xi\|_0^2 \le (\sigma(y)\nabla\xi,\nabla\xi)_0 = (\sigma(y)\nabla\pi_h q,\nabla\xi)_0 - (\sigma(y)\nabla q^h,\nabla\xi)_0 - (\sigma(y)\nabla q,\nabla\xi)_0 + (\sigma(y)\nabla q,\nabla\xi)_0 \\
&= -(\sigma(y)\nabla\varsigma,\nabla\xi)_0 + 2(p\sigma(y)\nabla\varphi,\nabla\xi)_0 - (\sigma(y^h)\nabla q^h,\nabla\xi)_0 - (\sigma(y)\nabla q^h,\nabla\xi)_0 + (\sigma(y^h)\nabla q^h,\nabla\xi)_0 = \\
&= -(\sigma(y)\nabla\varsigma,\nabla\xi)_0 + 2(p\sigma(y)\nabla\varphi,\nabla\xi)_0 - 2(p^h\sigma(y^h)\nabla\varphi^h,\nabla\xi)_0 + ((\sigma(y^h)-\sigma(y))\nabla q^h,\nabla\xi)_0 = \\
&= -(\sigma(y)\nabla\varsigma,\nabla\xi)_0 + 2(p\sigma(y)\nabla\varphi - p^h\sigma(y^h)\nabla\varphi^h,\nabla\xi)_0 + ((\sigma(y^h)-\sigma(y))\nabla q^h,\nabla\xi)_0 \le \\
&\le b_2\|\varsigma\|_1\|\xi\|_1 + C(\|\theta\|_{0,\Gamma}\left\|p-p^h\right\|_1 + \left\|y-y^h\right\|_1 + \left\|\varphi-\varphi^h\right\|_1)\|\xi\|_1,
\end{aligned}$$

$$\begin{aligned}
&\|\xi\|_1 \le Ch + C(\|\theta\|_{0,\Gamma}\left\|p-p^h\right\|_1 + \left\|y-y^h\right\|_1 + \left\|\varphi-\varphi^h\right\|_1), \\
&\left\|q-q^h\right\|_1 \le \|\xi\|_1 + \|\varsigma\|_1 \le Ch + C(\|\theta\|_{0,\Gamma}\left\|p-p^h\right\|_1 + \left\|y-y^h\right\|_1 + \left\|\varphi-\varphi^h\right\|_1)
\end{aligned} \tag{37}$$

Consequently, we obtain the following inequality.

$$\left\|p-p^h\right\|_1 \le Ch + C(\left\|y-y^h\right\|_1 + \left\|\varphi-\varphi^h\right\|_1) + C\|\theta\|_{0,\Gamma}^2\left\|p-p^h\right\|_1$$

Now choose θ such that $1 - C\|\theta\|_{0,\Gamma}^2$. Then there holds

$$\begin{gathered}
\left\|p-p^h\right\|_1 \le \tilde{C}(\theta)h + \tilde{C}(\theta)(\left\|y-y^h\right\|_1 + \left\|\varphi-\varphi^h\right\|_1), \\
(\tilde{C}(\theta) := C/(1 - C\|\theta\|_{0,\Gamma}^2))
\end{gathered} \tag{38}$$

3. Estimation of $\|y-y^h\|_1$
Let y^h be the solution of (11). Denoting $\xi := I_h y - y^h,\ \ \varsigma := y - I_h y$, we have $\xi + \varsigma = y - y^h$.
By using Lipschitz continuity of $\sigma(\cdot)$ and Lemma 2, we have

$$\begin{aligned}
&C_1\|\xi\|_1^2 \le (\nabla\xi,\nabla\xi)_0 + (\beta\xi,\xi)_{0,\Gamma} - (\sigma(\pi_h y) - \sigma(y^h)|\nabla\varphi|^2,\xi)_0 = \\
&\quad = (\nabla\pi_h y,\nabla\xi)_0 - (\nabla y^h,\nabla\xi)_0 + (\beta\pi_h y,\xi)_{0,\Gamma} - (\beta y^h,\xi)_{0,\Gamma} - (\sigma(\pi_h y)|\nabla\varphi|^2,\xi)_0 + (\sigma(y^h)|\nabla\varphi|^2,\xi)_0 + \\
&\qquad + (\nabla y,\nabla\xi)_0 - (\nabla y,\nabla\xi)_0 + (\beta y,\xi)_{0,\Gamma} - (\beta y,\xi)_{0,\Gamma} - (\sigma(y)|\nabla\varphi|^2,\xi)_0 + (\sigma(y)|\nabla\varphi|^2,\xi)_0 = \\
&\quad = -(\nabla\varsigma,\nabla\xi)_0 - (\beta\varsigma,\xi)_{0,\Gamma} - ((\sigma(\pi_h y) - \sigma(y))|\nabla\varphi|^2,\xi)_0 - (g,\xi)_0 + (g,\xi)_0 + (\sigma(y^h)(|\nabla\varphi|^2 - \left|\nabla\varphi^h\right|^2),\xi)_0 \\
&\quad \le \|\varsigma\|_1\|\xi\|_1 + \beta\|\varsigma\|_{0,\Gamma}\|\xi\|_{0,\Gamma} + C\|\varsigma\|_{0,6}\left\||\nabla\varphi|^2\right\|_{0,3/2}\|\xi\|_{0,6} + C\left\|\varphi-\varphi^h\right\|_{0,3}\left\|\nabla(\varphi+\varphi^h)\right\|_{0,3}\|\xi\|_{0,3} \le \\
&\quad \le C(\|\varsigma\|_1 + \|\theta\|_{0,\Gamma}\|\varsigma\|_1 + \|\theta\|_{0,\Gamma}\left\|\varphi-\varphi^h\right\|_1)\|\xi\|_1
\end{aligned}$$

$$\begin{aligned}
&\|\xi\|_1 \le C(\|\varsigma\|_1 + \|\theta\|_{0,\Gamma}\|\varsigma\|_1 + \|\theta\|_{0,\Gamma}\left\|\varphi-\varphi^h\right\|_1) \le Ch + C\|\theta\|_{0,\Gamma}\left\|\varphi-\varphi^h\right\|_1, \\
&\left\|y-y^h\right\|_1 \le \|\xi\|_1 + \|\varsigma\|_1 \le Ch + C\|\theta\|_{0,\Gamma}\left\|\varphi-\varphi^h\right\|_1
\end{aligned}$$

4. Estimation of $\left\|\varphi-\varphi^h\right\|_1^2$

φ^h is the solution of (9). Denoting $\xi := I_h\varphi-\varphi^h, \varsigma := \varphi - I_h\varphi$, we have $\xi+\varsigma=\varphi-\varphi^h$. Then, we have

$$
\begin{aligned}
&C_1\|\xi\|_1^2 \le b_1(\nabla\xi,\nabla\xi)_0 \le (\sigma(y)\nabla\xi,\nabla\xi)_0 = \\
&= (\sigma(y)\nabla\pi_h\varphi,\nabla\xi)_0 - (\sigma(y)\nabla\varphi^h,\nabla\xi)_0 + (\sigma(y)\nabla\varphi,\nabla\xi)_0 - (\sigma(y)\nabla\varphi,\nabla\xi)_0 = \\
&= -(\sigma(y)\nabla\varsigma,\nabla\xi)_0 + (\theta,\xi)_{0,\Gamma} - (\sigma(y^h)\nabla\varphi^h,\nabla\xi)_0 + ((\sigma(y^h)-\sigma(y))\nabla\varphi^h,\nabla\xi)_0 = \\
&= -(\sigma(y)\nabla\varsigma,\nabla\xi)_0 + (\theta,\xi)_{0,\Gamma} - (\theta,\xi)_{0,\Gamma} + ((\sigma(y^h)-\sigma(y))\nabla\varphi^h,\nabla\xi)_0 \le \\
&\le b\|\varsigma\|_1\|\xi\|_1 + C\|\theta\|_{0,\Gamma}\left\|y^h-y\right\|_1\|\xi\|_1\|\xi\|_1 \le C\|\varsigma\|_1 + C\|\theta\|_{0,\Gamma}\left\|y^h-y\right\|_1 \le \\
&\quad \le Ch^\varepsilon + C\|\theta\|_{0,\Gamma}\left\|y^h-y\right\|_1\left\|\varphi-\varphi^h\right\|_1 \le \|\xi\|_1+\|\varsigma\|_1 \le Ch + C\|\theta\|_{0,\Gamma}\left\|y^h-y\right\|_1 \\
&\left\|y-y^h\right\|_1 \le Ch + C\|\theta\|_{0,\Gamma}^2\left\|y^h-y\right\|_1
\end{aligned}
$$

From the assumptions on θ, there holds the following inequality

$$\left\|y-y^h\right\|_1 \le \tilde{C}(\theta)h^\varepsilon \tag{39}$$

and thus we deduce the following result.

$$
\begin{aligned}
\left\|\varphi-\varphi^h\right\|_1 &\le Ch + C\tilde{C}(\theta)\|\theta\|_{0,\Gamma}h \le \bar{C}(\theta)h \\
\bar{C}(\theta) &:= C + C\tilde{C}(\theta)\|\theta\|_{0,\Gamma}
\end{aligned} \tag{40}
$$

Now we can estimate $\left\|y^h-y_h\right\|_1, \left\|\varphi^h-\varphi_h\right\|_1, \left\|p^h-p_h\right\|_1, \left\|q^h-q_h\right\|_1$. From (4)-(6), we can get the following results.

$$
\begin{aligned}
&\left\|y^h-y_h\right\|_1 \le C_1\widehat{C}(\theta)\|g-g_h\|_0, \\
&\left\|\varphi^h-\varphi_h\right\|_1 \le C_2\widehat{C}(\theta)\|g-g_h\|_0, \\
&\left\|p^h-p_h\right\|_1 \le C_3\widehat{C}(\theta)\|g-g_h\|_0, \\
&\left\|q^h-q_h\right\|_1 \le C_4\widehat{C}(\theta)\|g-g_h\|_0
\end{aligned} \tag{41}
$$

where $C_i (i=1,2,3,4)$ are constants independent of θ, h.

On the other hand, we also have

$$
\begin{aligned}
&\|y-y_h\|_1 \le \left\|y-y^h\right\|_1 + \left\|y^h-y_h\right\|_1, \\
&\|\varphi-\varphi_h\|_1 \le \left\|\varphi-\varphi^h\right\|_1 + \left\|\varphi^h-\varphi_h\right\|_1, \\
&\|p-p_h\|_1 \le \left\|p-p^h\right\|_1 + \left\|p^h-p_h\right\|_1, \\
&\|q-q_h\|_1 \le \left\|q-q^h\right\|_1 + \left\|q^h-q_h\right\|_1
\end{aligned} \tag{42}
$$

According to Lemma 4 and (36)-(40), we have

$$
\begin{aligned}
\frac{\nu}{2}\|g-g_h\|_0^2 &\le C\left\|p^h-p\right\|_1^2 + Ch \le \\
&\le C(\theta)\|g-g_h\|_0^2 + C(\theta)h^2 + Ch \le \\
&\le C(\theta)\|g-g_h\|_0^2 + Ch
\end{aligned}
$$

Choose ν such that $\nu > 2C(\theta)$. Then we have

$$
\begin{aligned}
(\tfrac{\nu}{2} - C(\theta))\|g-g_h\|_0^2 &\le Ch, \\
\|g-g_h\|_0 &\le C(\theta,\nu)\sqrt{h}
\end{aligned} \tag{43}
$$

By summing up (34)-(41), we can get the desired result. This completes the proof.

6. Numerical Simulations

Numerical Experiments use gradient algorithm based on optimality condition. State equation is solved by Newton-Laphson method. We use only coarse mesh for convenience in the numerical process.

6.1. Initial Data

The computational domain is taken to be $\Omega = (-1,1)\times(-1,1)$. The differential model, objective function and measure data are given as follows:

$$
\begin{aligned}
-\Delta y - \sigma(y)|\nabla\varphi|^2 = g, &\quad \Omega \\
\nabla\cdot(\sigma(y)\nabla\varphi) = 0, &\quad \Omega \\
\frac{\partial y}{\partial n} + \beta y = 0, &\quad \Gamma \\
\sigma(y)\frac{\partial\varphi}{\partial n} = \theta, &\quad \Gamma_1 \\
\varphi = 0, &\quad \Gamma_2
\end{aligned}
$$

$$
\begin{aligned}
J(y(g),g) &:= \frac{1}{2}\|y(g)-z_d\|_0^2 + \frac{\nu}{2}\|g\|_0^2 \\
r &= \sqrt{x^2+y^2}, \nu = 5e-7 \sim 1e-5
\end{aligned}
$$

The target value function $z_d(x,y)$ within the objective functional is defined as follows:

$$
\begin{aligned}
&z_d(r) = (r^2\log(r+\varepsilon)-2)/4 + r/4 + r^3/6 + 0.25)/ \\
&\quad /(6.28*0.2) + (4-9r)/6.28, \\
&\theta = 0.0061, \quad \beta = 0.01
\end{aligned}
$$

Now we can propose the following algorithm.

1. Compute the approximation of the optimal control on the primitive mesh by the finite element method.
2. Get the average of a posteriori errors $\|r_{h,k}\|_{0,k}, \|\gamma_{h,k}\|_{0,k}$ from Lemma 6 in each element, which is used as a posteriori error estimator pos.
3. Mark the elements with $\|r_{h,k}\|_{0,k},\ \|\gamma_{h,k}\|_{0,k}$ larger than pos.
4. Solve the optimal control problem on a mesh generated by refining the marked elements.

The iterations will stop when the relative error of the objective function $bb := {|J(n+1)-J(n)|}/{J(n+1)}$ or $L^2(\Omega)$-norm of difference between the target value function and the solution of state equation $\|y - z_d\|^2_{L^2(\Omega)}$ is less than the given tolerance $1e-10 \sim 1e-6$.

6.2. Numerical Analysis

6.2.1. Numerical Test(I)

For convenience, we use the same parameters as in the code. $nn, ns, ne, ssj, bb, \varepsilon, pos, eff = pos/\varepsilon, t$ stand for iteration number, node number of the mesh, triangle number, value of the objective function, relative error of the objective function, $\|y(u) - z_d\|_0/ne$, a posteriori error indicator, the effectiveness index, computing time, respectively.

The first mesh refinement is carried out by dividing the domain Ω into 10×10 squares uniformly and further splitting every square into two right triangles.

We choose $ns = 121, ne = 200, nband = 12, \nu = 1e-8$.

In the gradient algorithm for finding the optimal control g, the control is initialized as

$$g^{(0)}(r) = \begin{cases} 0.101, & r > 1 \\ 0.102, & r < 1 \end{cases}$$

and the iterations will stop when the relative error of the objective function is less than $1e-5$.

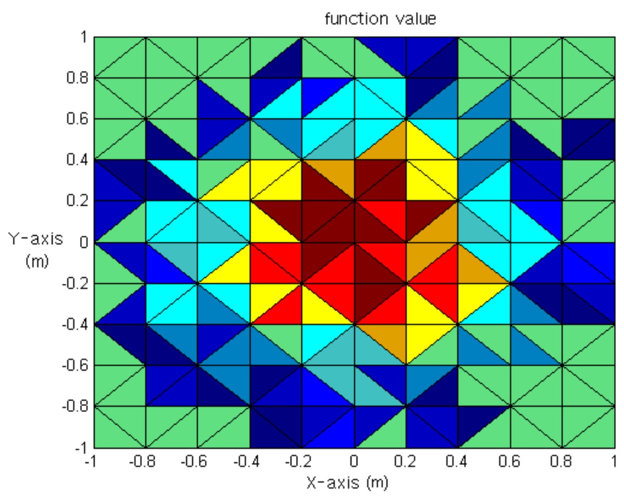

Figure 1. Mesh 1.

Then we can get the following results:

We have $ssj = 1.682e-5, pos = 3.15e-4$. The relative error of objective function becomes $bb = 1.94e-4$ and the computing time is around two minutes.

The numerical result is shown in Figure 1 $\sim$ Figure 7 about the approximation optimal state, the corresponding co-state and the approximation optimal control.

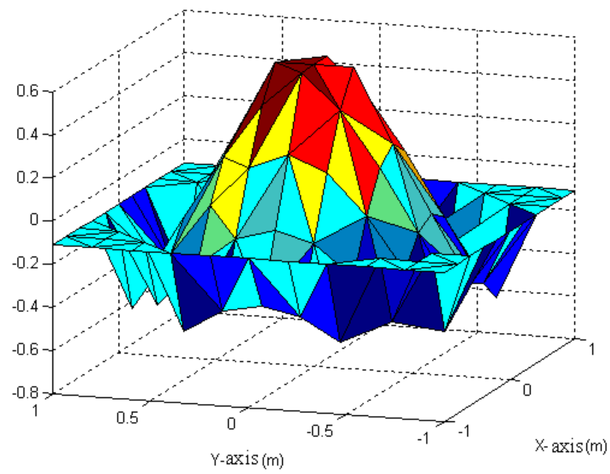

Figure 2. Target value function z_d.

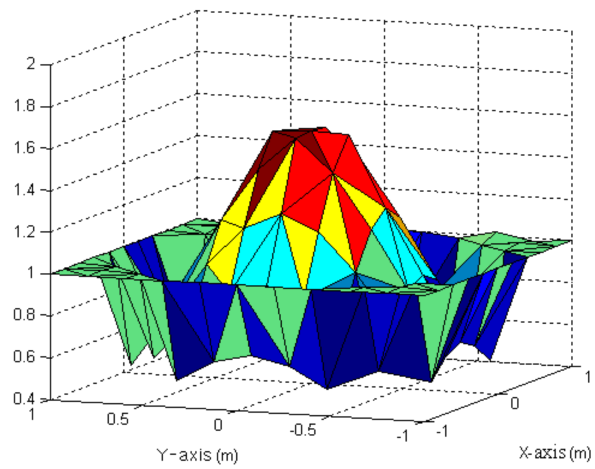

Figure 3. Optimal state y_h.

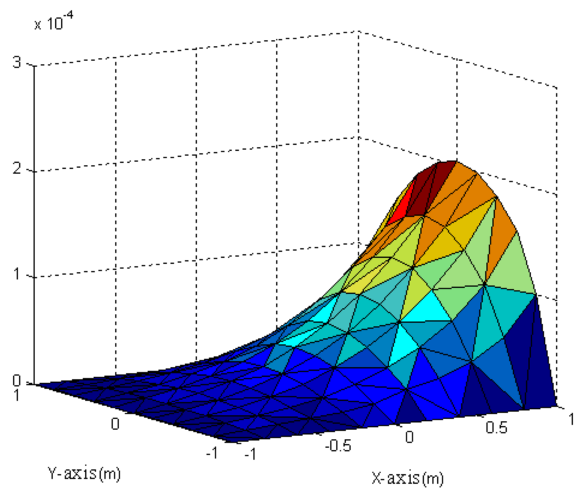

Figure 4. Optimal state φ_h.

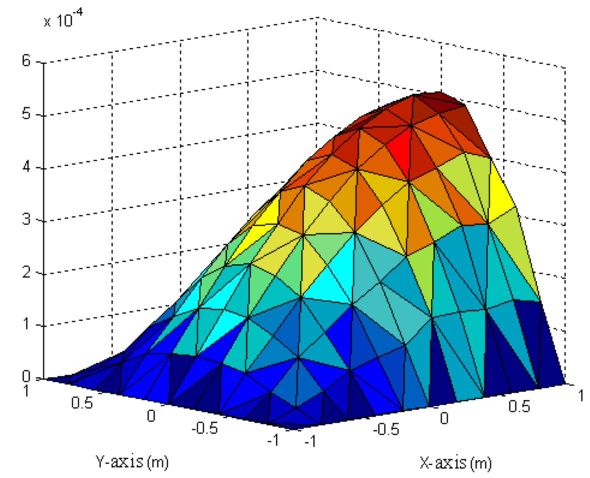

Figure 5. *Co-state p_h.*

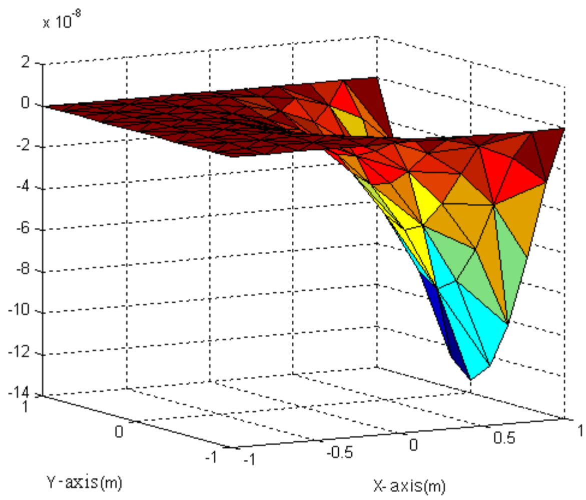

Figure 6. *Co-state q_h.*

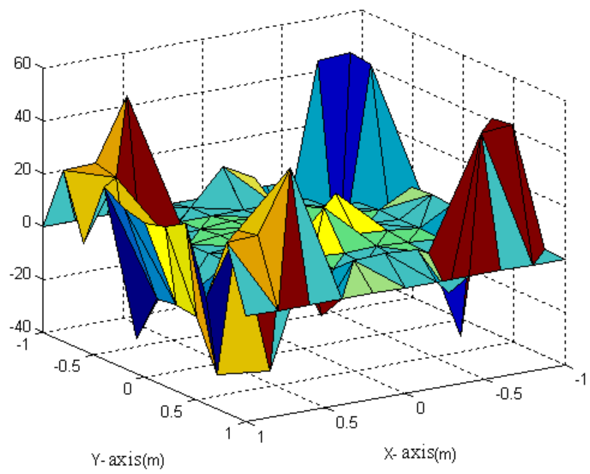

Figure 7. *Optimal control g_h.*

6.2.2. Numerical Test(II)

We mark the elements in which a posteriori errors are larger than the $pos = 3.15e{-}4$(when $ns = 121, ne = 200$) to update the mesh.

We perform the computation on the three refined meshes

$$\begin{aligned} &ns = 161, \quad ne = 280, \quad nband = 40, \quad \nu = 1e-8, \\ &ns = 185, \quad ne = 328, \quad nband = 40, \quad \nu = 1e-8, \\ &ns = 233, \quad ne = 424, \quad nband = 40, \quad \nu = 1e-9 \end{aligned}$$

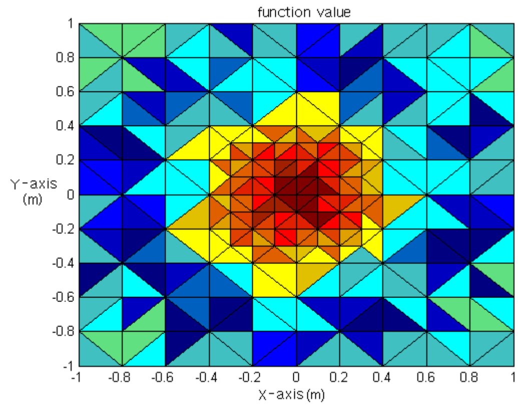

Figure 8. *Mesh 2.*

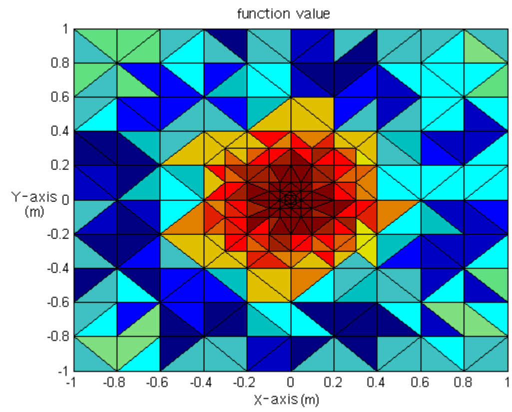

Figure 9. *Mesh 3.*

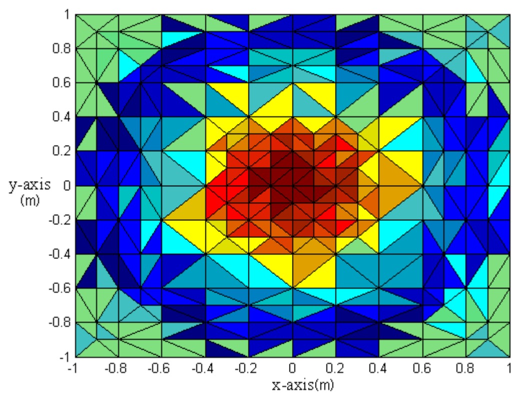

Figure 10. *Mesh 4.*

In Mesh 4, we refine the initial mesh uniformly around the origin and the circle with $r = 1$. The iterations will stop when $\varepsilon \leq 1e-6 \sim 1e-4$and the other initial data are the same as in Numerical test (I).

The numerical result for three cases are obtained as follows:

$$ssj = 1.542e-5, \quad \varepsilon = 1.50e-5, \quad bb = 1.175e-5,$$
$$pos = 2.3622e-4,$$
$$ssj = 1.542e-5, \quad \varepsilon = 1.50e-5, \quad bb = 1.666e-5,$$
$$pos = 2.004e-4,$$
$$ssj = 1.542e-5, \quad \varepsilon = 1.50e-5, \quad bb = 2.726e-5,$$
$$pos = 1.921e-4.$$

For the case of mesh $3(ns = 233, ne = 424)$, the numerical result is shown in Figure 10 $\sim$ Figure 14 about approximation optimal state, the corresponding co-state and the approximation optimal control.

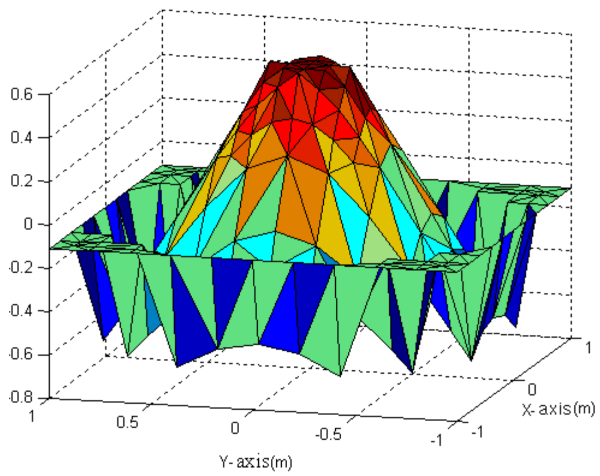

Figure 11. *Target value function z_d.*

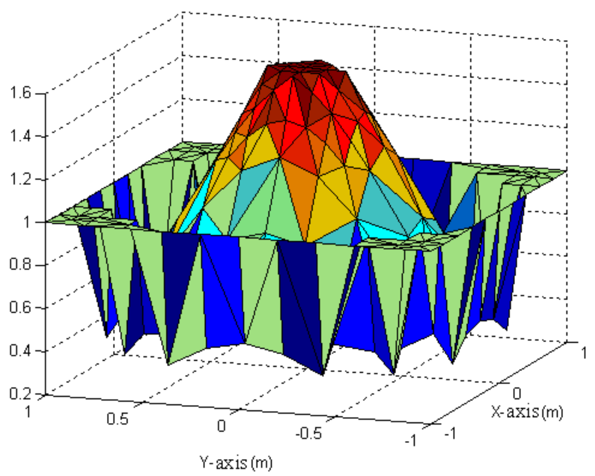

Figure 12. *Optimal state y_h.*

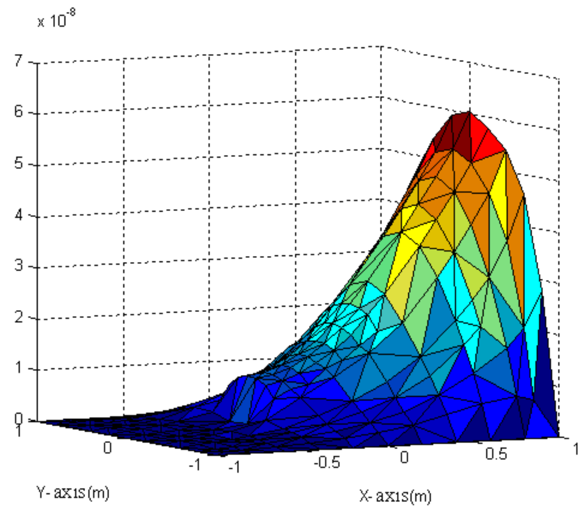

Figure 13. *Optimal state φ_h.*

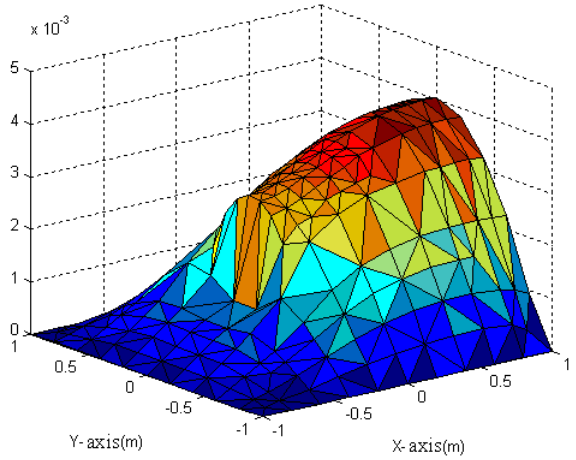

Figure 14. *Co-state p_h.*

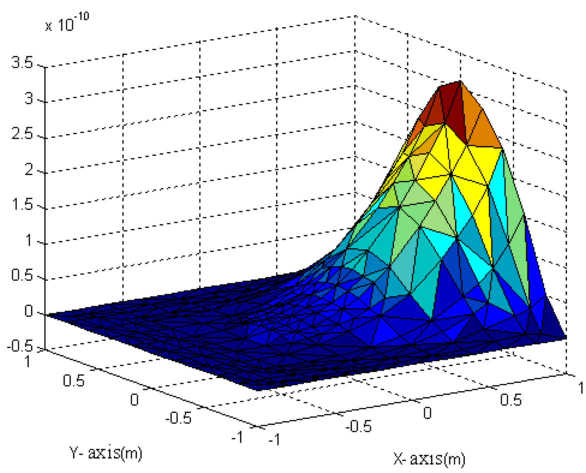

Figure 15. *Co-state q_h.*

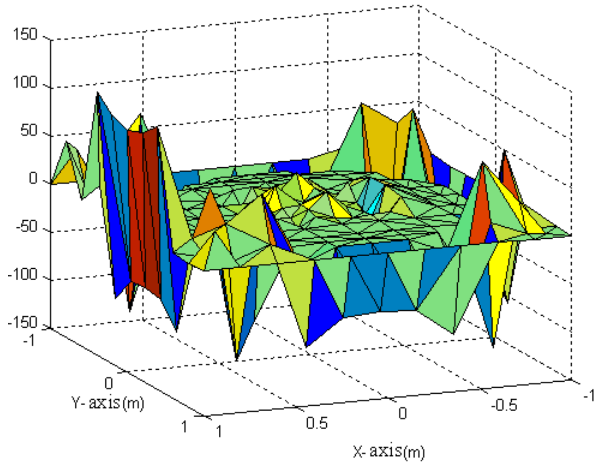

Figure 16. *Optimal control* g_h.

We can see $\varepsilon = \|y(u) - z_d\|_0 / ne$, a posteriori error indicator and the effectiveness index for different nodes and elements in the following table.

Table 1. *Result for numerical test.*

No	ns	ne	ε	pos	eff
1	121	200	$1.5e-5$	$3.15e-4$	21.0
2	161	280	$1.5e-5$	$2.362e-4$	15.7
3	185	328	$1.5e-5$	$2.004e-4$	13.4
4	233	424	$1.5e-5$	$1.921e-4$	12.8

In the table above eff represents pos/ε and the stopping condition is $\varepsilon = 1.5e-5$.

7. Conclusion

In this paper, the priori and the upper bound of a posteriori error estimates of finite element approximation solution of source control problem governed by a system of quasi-linear elliptic equation with homogeneous Dirichlet boundary conditions was derived with an accuracy of about $O(h^{1/2})$ in H^1-norm on triangular mesh.

Acknowledgments

The authors would like to thank the University of Science, Pyongyang, DPR Korea for their help during preparation of this book.

Funding

The authors have no relevant financial or non-financial interests to disclose.

Conflicts of Interest

The authors have no competing interests to declare that are relevant to the content of this article.

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