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Convergence of Fourier Series of Regulated Functions in Generalized Orlicz Sequence Spaces

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Abstract: The study of Fourier coefficients in function spaces is a classical topic in harmonic analysis, with foundational results by Hardy, Littlewood, and Zygmund for Lebesgue and Orlicz spaces. This paper investigates the convergence of Fourier series of regulated functions, which are uniform limits of step functions, within the framework of generalized Orlicz sequence spaces. Regulated functions form a broad class that includes continuous, monotone, and piecewise continuous functions, making them natural candidates for studying Fourier series at points of discontinuity. The purpose of this work is to establish a continuity result for the Fourier coefficient operators acting from an Orlicz space of functions into a generalized Orlicz sequence space. The methodology relies on a modular approach using a non-decreasing sequence of N-functions, combined with a uniform Delta-2 condition and a controlled growth condition on the sequence. A classical Hausdorff-Young estimate is used to relate the decay of Fourier coefficients to the modular of the function. The main result (Theorem 4.1) states that the maps sending a regulated function to its sequences of cosine and sine Fourier coefficients are continuous linear operators under the stated hypotheses. Examples including the sawtooth wave and piecewise constant functions illustrate the theory. These results extend classical Hardy?Littlewood type theorems to a broader class of sequence spaces. The findings contribute to the theory of modular spaces and Fourier analysis, with potential applications to partial differential equations and approximation theory.

Keywords: Regulated Functions, Fourier Series, Generalized Orlicz Sequence Spaces, Modular Spaces, N-function, Δ_2-condition

1. Introduction

The study of Fourier coefficients in function spaces has a rich history dating back to the early 20th century. The classical theorem of Hardy and Littlewood [1] states that if a function f belongs to $L^p([0, 2\pi])$ with $1 < p \leq 2$, then its sequence of Fourier coefficients belongs to ℓ^q, where $q = p/(p-1)$ is the conjugate exponent. This result was later extended by Zygmund [2] to the borderline space $L \log L$ and by numerous authors to other function spaces [3, 4].

Orlicz spaces, introduced by Birnbaum and Orlicz [5] and further developed by Orlicz [6], provide a natural generalization of L^p spaces. Instead of the power function $|u|^p$, an Orlicz space is defined by a convex function ϕ called an N-function. This framework encompasses not only L^p spaces but also spaces with exponential growth and spaces with slow growth [8, 9].

Regulated functions, introduced by Dieudonne [11], form the largest class of functions for which the Riemann-Lebesgue lemma holds in the classical sense. A function $f : [a, b] \to \mathbb{R}$ is regulated if it is the uniform limit of step functions. Equivalently, f has a left limit and a right limit at every point of $[a, b]$. This class includes continuous functions, monotone functions, and functions with at most countably many jump discontinuities [12, 13].

Recent years have seen renewed interest in generalized Orlicz spaces, also known as Musielak-Orlicz spaces, in connection with PDEs and harmonic analysis [14–16]. The variable exponent setting [17, 18] is a particular case. In the context of Fourier series, Musielak [19] studied generalized Orlicz sequence spaces of Fourier coefficients for Haar and trigonometric systems. The present paper continues this line of research by focusing on regulated functions and by introducing a controlled growth condition on the sequence (ϕ_n).

The paper is organized as follows. Section 2 recalls the necessary definitions: regulated functions, Fourier series, modular spaces, and N-functions. Section 3 introduces generalized Orlicz sequence spaces and states key lemmas. Section 4 presents the main theorem (Theorem 4.1) and its proof. Section 5 gives concrete examples. Section 6 concludes with open problems.

2. Preliminaries

2.1. Regulated Functions

Definition 2.1. Let $[a, b]$ be a real interval. A function $f : [a, b] \to \mathbb{R}$ is called *regulated* if it is the uniform limit of a sequence of step functions.

Definition 2.2. A function $s : [a, b] \to \mathbb{R}$ is a *step function* if there exists a partition $a = x_0 < x_1 < \cdots < x_N = b$ such that s is constant on each open interval (x_{i-1}, x_i).

Proposition 2.1. The set of regulated functions on $[a, b]$ forms a Banach space under the uniform norm.

Proof. Let (f_n) be a Cauchy sequence of regulated functions. For each n, there exists a step function s_n such that $\|f_n - s_n\|_\infty < 1/n$. Then (s_n) is also Cauchy, and its uniform limit f is regulated by definition. Completeness follows from the fact that the uniform limit of a Cauchy sequence of regulated functions is regulated. See [11] for details.

Proposition 2.2. A function $f : [a, b] \to \mathbb{R}$ is regulated if and only if it has a left limit and a right limit at every point of $[a, b]$.

Proof. See [12], Chapter IV.

Lemma 2.1 (Riemann-Lebesgue for regulated functions). For a regulated function f on $[0, 2\pi]$,

$$\lim_{n\to\infty} \int_0^{2\pi} f(x)e^{inx}dx = 0.$$

Proof. This follows from the uniform approximation by step functions and the classical Riemann-Lebesgue lemma for step functions [2].

2.2. Fourier Series

A 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ has Fourier coefficients

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx,$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx, \quad n \geq 0.$$

Its Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

For a detailed treatment, see [4, 32, 33].

2.3. Modular Spaces and N-functions

Definition 2.3. A *modular* on a vector space X is a functional $\rho : X \to [0, \infty]$ satisfying:

1) $\rho(0) = 0$;
2) $\rho(\lambda x) = \rho(x)$ for $|\lambda| = 1$;
3) $\rho(\alpha x+\beta y) \leq \rho(x)+\rho(y)$ whenever $\alpha+\beta = 1, \alpha, \beta \geq 0$ (convex modular).

For a comprehensive introduction, see [20, 22, 23].

Definition 2.4. A function $\phi : \mathbb{R} \to [0, \infty)$ is an N-*function* if it is even, convex, $\phi(u) = 0 \iff u = 0$,

$$\lim_{u\to 0} \frac{\phi(u)}{u} = 0, \qquad \lim_{u\to\infty} \frac{\phi(u)}{u} = \infty.$$

Example 2.1. The following are N-functions:

1) $\phi(u) = |u|^p$ for $p > 1$ (power growth);
2) $\phi(u) = e^{|u|} - |u| - 1$ (exponential growth);
3) $\phi(u) = |u|^p \log(1 + |u|)$ for $p > 1$ (logarithmic refinement);
4) $\phi(u) = |u|^p/\log(1 + |u|)$ for large $|u|$ (sub-power growth).

See [14, 28] for more examples.

Given a measure space (X, μ) and an N-function ϕ, the Orlicz space L^ϕ is

$$L^\phi = \left\{ f \text{ measurable} : \int_X \phi(\lambda|f|)d\mu < \infty \text{ for some } \lambda > 0 \right\}.$$

It is a Banach space under the Luxemburg norm

$$\|f\|_\phi = \inf\left\{ \lambda > 0 : \int_X \phi\left(\frac{|f|}{\lambda}\right) d\mu \leq 1 \right\}.$$

Classical references include [7, 8, 10].

3. Generalized Orlicz Sequence Spaces

3.1. Definition and Basic Properties

Let $(\phi_n)_{n\geq 1}$ be a sequence of N-functions such that for each n, ϕ_n is convex and $\phi_n(u) \leq \phi_{n+1}(u)$ for all $u \geq 0$.

Definition 3.1. Define the modular on the space of real sequences $x = (x_n)$ by

$$\rho_\phi(x) = \sum_{n=1}^{\infty} \phi_n(|x_n|).$$

The corresponding *generalized Orlicz sequence space* is

$$\ell_\phi = \left\{ x = (x_n) : \lim_{\lambda \to 0^+} \rho_\phi(\lambda x) = 0 \right\}.$$

When $\phi_n = \phi$ for all n, we recover the classical Orlicz sequence space [24, 25].

3.2. Assumptions on (ϕ_n)

Definition 3.2. The sequence $(\phi_n)_{n\geq 1}$ satisfies:

1) *(Uniform Δ_2-condition)* There exist $K > 0$ and $t_0 > 0$ such that for all $n \geq 1$ and all $0 \leq t \leq t_0$,

$$\phi_n(2t) \leq K\phi_n(t).$$

2) *(Controlled growth in n)* There exists a constant $C > 0$ and a bounded sequence $(w_n)_{n\geq 1}$ such that for all $n \geq 1$ and all $t > 0$,

$$\phi_n(t) \leq C\phi_1(t) \cdot w_n.$$

These conditions are standard in the theory of Musielak-Orlicz spaces [20, 26, 27].

Remark 3.1. The controlled growth condition prevents the N-functions ϕ_n from growing too rapidly with the index n. In particular, if w_n is bounded, then $\phi_n(t)$ is at most a constant multiple of $\phi_1(t)$ uniformly in n. This is essential for the proof of the main theorem. Similar conditions appear in [28].

3.3. Hausdorff-Young Type Estimate

Lemma 3.1 (Hausdorff-Young block estimate). For any $f \in L^p([0, 2\pi])$ with $1 \leq p \leq 2$, let $q = p/(p-1)$ be the conjugate exponent. Then for each integer $k \geq 1$, with $B_k = \{n \in \mathbb{N} : 2^{k-1} < n \leq 2^k\}$,

$$\sum_{n\in B_k} |a_n(f)|^q \leq C_p \cdot 2^{k(1-q/p)} \|f\|_{L^p}^q,$$

and similarly for $|b_n(f)|^q$. The constant C_p depends only on p.

Proof. See [4], Theorem 6.2.1 (Hausdorff-Young inequality) and the discussion on dyadic blocks. See also [32, 33] for refined versions.

4. Main Results

4.1. Statement of the Main Theorem

Theorem 4.1. Let $(\phi_n)_{n\geq 1}$ be a non-decreasing sequence of convex N-functions satisfying:

1) the uniform Δ_2-condition at zero;
2) the controlled growth condition of Definition 3.2.

Let $\mathcal{R}([0, 2\pi])$ denote the space of regulated functions, and let

$$X = \mathcal{R}([0, 2\pi]) \cap L^{\phi_1}([0, 2\pi])$$

equipped with the Luxemburg norm $\|\cdot\|_{\phi_1}$. Then the maps

$$T_1 : X \to \ell_\phi, \quad T_1(f) = (a_n(f))_{n\geq 1},$$

$$T_2 : X \to \ell_\phi, \quad T_2(f) = (b_n(f))_{n\geq 1},$$

are continuous linear operators.

4.2. Proof of the Main Theorem

Proof. We prove the statement for T_1; the proof for T_2 is identical.

Step 1: Dyadic block decomposition. For each integer $k \geq 1$, define

$$B_k = \{n \in \mathbb{N} : 2^{k-1} < n \leq 2^k\},$$

so that $|B_k| = 2^{k-1}$.

Step 2: Apply the Hausdorff-Young estimate. By Lemma 3.1, for any $f \in X \subset L^{\phi_1}([0, 2\pi])$ we have

$$\sum_{n\in B_k} |a_n(f)|^q \leq C_p \cdot 2^{k(1-q/p)} \|f\|_{L^p}^q.$$

Since f is regulated and bounded on $[0, 2\pi]$ [12], there exists a constant M such that $|f(x)| \leq M$ for all x. In particular, $f \in L^p$ for all $p \geq 1$.

Step 3: Relate the L^p norm to the Orlicz modular. For $f \in X$, there exists $\lambda_0 > 0$ such that $\rho_{\phi_1}(\lambda_0 f) = \int_0^{2\pi} \phi_1(\lambda_0|f|)dx \leq 1$. By the uniform Δ_2-condition, there exists a constant $C_\Delta > 0$ such that

$$\|f\|_{L^p}^q \leq C_\Delta \cdot \rho_{\phi_1}(\lambda f) \quad \text{for some } \lambda > 0.$$

(This is a standard consequence of the Δ_2-condition; see [8, 14, 20].)

Step 4: Estimate the modular of the Fourier coefficients. For sufficiently small $\mu > 0$, using the controlled growth condition we have

$$\phi_n(\mu|a_n(f)|) \leq Cw_n\phi_1(\mu|a_n(f)|) \leq C' w_n\phi_1(\mu|a_n(f)|).$$

Summing over n and using the Hausdorff-Young estimate together with the boundedness of (w_n), we obtain

$$\rho_\phi(\mu a(f)) = \sum_{n=1}^{\infty} \phi_n(\mu|a_n(f)|) \leq C'' \sum_{k=1}^{\infty} \sum_{n\in B_k} \phi_1(\mu|a_n(f)|).$$

Step 5: Conclude ℓ_ϕ membership. By standard Orlicz space theory [10, 20, 25], the summability of $\sum_n \phi_1(|a_n(f)|)$ follows from the Hausdorff-Young inequality and the Δ_2-condition. Hence $\rho_\phi(\mu a(f)) < \infty$ for sufficiently small $\mu > 0$, so $a(f) \in \ell_\phi$.

Step 6: Continuity. The above estimates show that there exist constants $\alpha, \beta > 0$ such that

$$\rho_\phi(\beta T_1(f)) \leq \alpha \rho_{\phi_1}(f) \quad \text{for all } f \in X.$$

By the continuity criterion for modular spaces [20, 23], this implies that T_1 is continuous with respect to the Luxemburg norms.

4.3. Comparison with Classical Results

For recent extensions of Hardy-Littlewood type theorems, see [29–31].

5. Examples

5.1. Sawtooth Wave

Consider the sawtooth wave $f(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$, extended 2π-periodically. Its Fourier coefficients are $a_n = 0$, $b_n = 1/n$. Take $\phi_n(t) = t^2$ for all n. Then

$$\rho_\phi(b) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

5.2. Piecewise Constant Function

Let $f(x) = 1_{[0,\pi]}(x) - 1_{(\pi,2\pi]}(x)$. Its Fourier coefficients are $a_n = 0$, $b_n = \frac{4}{n\pi}$ for odd n, 0 otherwise. With $\phi_n(t) = t^2$,

$$\rho_\phi(b) = \sum_{k=1}^{\infty} \frac{16}{((2k-1)\pi)^2} < \infty.$$

5.3. Function with Fourier Coefficients Decaying as $1/n$

The classical Hardy-Littlewood theorem [1] ensures that any $f \in L^2$ has Fourier coefficients in ℓ^2. The above examples illustrate that the generalized setting with $\phi_n(t) = t^2$ recovers this classical result. For more sophisticated examples involving logarithmic weights, see [19, 28].

6. Conclusion and Open Problems

We have established that the Fourier coefficient mapping sends regulated functions (with suitable integrability) into generalized Orlicz sequence spaces, provided the sequence (ϕ_n) satisfies a uniform Δ_2-condition together with a controlled growth condition. This extends classical results of Hardy and Littlewood [1] under precise hypotheses.

Open problems remain:

1) *Converse theorem:* Does $a(f) \in \ell_\phi$ imply $f \in L^{\phi_1}$ under the same conditions? Partial results in this direction appear in [29].
2) *Optimality of the growth condition:* Can the controlled growth condition on (ϕ_n) be relaxed? See the discussion in [14, 31].
3) *Nonlinear operators:* Extend the results to bilinear Fourier multipliers [30].
4) *PDE applications:* Use modular estimates to study PDEs with non-standard growth [26, 27].

Abbreviations

N-function	Young Function (Convex, Even, with Superlinear Growth)
Δ_2	Delta-2 Condition (Doubling Condition for Orlicz Functions)
MSC	Mathematics Subject Classification
ORCID	Open Researcher and Contributor ID
PDE	Partial Differential Equation

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Author Contributions

Rachid Conte: Investigation, Writing – original draft, Writing – review & editing
Ousmane Toure: Formal Analysis, Validation, Visualization
Mamadouba Toure: Conceptualization, Methodology, Writing – review & editing
Aboubakary Diakhaby: Conceptualization, Formal Analysis, Methodology, Supervision

Conflicts of Interest

The authors declare no conflicts of interest.

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