International Journal of Theoretical and Applied Mathematics
2026, Vol. 12, No. 5, pp. 90–98
https://doi.org/10.11648/j.ijtam.20261205.11

Research Article

On the Application of an Intertwined Difference Equations and Newton-Lieberstein Algorithmic to Solve Mildly Non-linear Boundary Value Problems (MNBVP)

Dele-Rotimi Adejoke Olumide[1], Akinmuyise Folorunsho Mathew[2], Adebayo Kayode James[3,*]

[1]Department of Mathematics, Bamidele Olumilua University of Education Science and Technology, Ikere-Ekiti, Nigeria
[2]Department of Mathematics, Adeyemi Federal University of Education, Ondo City, Nigeria
[3]Department of Mathematics, Ekiti State University, Ado Ekiti, Nigeria

Abstract

Over the years, numerous researchers have developed numerical or classical methods or the other to obtain solutions for Boundary Value Problems; many of these methods have not been adopted to solve Mildly Non-Linear Boundary Value Problems (MNBVP) basically because of their peculiarities. The MNBVP are problems that are neither entirely linear nor exclusively nonlinear rather for multi-variable functions, not all the constants and the function derivative is greater or equal to zero. This paper discusses the process of intertwining the Approximation Difference Equation with the Newton-Lieberstein method in solving mildly non-linear boundary value problems. The technique requires dual independent processes that involve different Methods. The first technique is aimed at reducing the BVP to a tridiagonal system, while the other is employed to solve the system of equations by applying the Newton-Lieberstein algorithm. The resulting equations that is formed from the nonlinear boundary problems will be nonlinear systems but it will be structurally related to tridiagonal systems. In practical settings, mathematical modelling problems can be expressed in the form of BVPs that arise frequently and majorly in the fields of science and engineering, such as electric circuits, fluid dynamics, the motion of rockets or satellites, and other areas of engineering applications. The intertwined methods will be used to evaluate the approximate solutions without much ado on the number of equations the problem has. The technique was employed to solve some problems with numerical results obtained showed the flexibility, the robustness, and how efficient the coined technique is. The results compare favourably with exact or analytical results.

Keywords

Boundary Value Problem, Mildly Non-Linear, Initial Value Problem, Newton-Lieberstein Method, Tridiagonal, Difference Equation

*Correspondence: Adebayo Kayode James (kayode.adeabyo@eksu.edu.ng)

Received: 12 August 2026; **Accepted:** 21 August 2026; **Published:** 11 September 2026

1. Introduction

From the physical point of view, Boundary Value Problems (BVP) for second-order differential equations are problems where the position at a particular time $x = x_0$ is known. One intends to determine the position at a later time $x = x_1$, such that the motion is described by the given differential equation within the time frame $x_0 < x < x_1$.

With BVP, one has a differential equation with specified functions and/or derivatives at various points, which will be referred to as boundary values. So, our discussion will be narrowed down to a linear differential equation as observed by [1]. An ODE that is a BVP is defined as:

$$y'' = f(x, y, y') \quad x_0 < x < x_1, \tag{1}$$

and for the 2nd-order ODE, the boundary conditions can be any of the following:

Dirichlet or First Kind Boundary Conditions:

$$y(x_0) = y_0, \; y(x_1) = y_1, \tag{2}$$

Regular Boundary Conditions:

$$y(x_0) = y_0, \; y'(x_1) = y_1, \tag{3}$$

Neumann or Second Kind Boundary Conditions:

$$y'(x_0) = y_0, \; y'(x_1) = y_1', \tag{4}$$

Periodic Boundary Conditions:

$$y'(x_0) = y_0' \; y(x_1) = y_1, \tag{5}$$

where x_0 is not the same with x_1 and the values of x_0, x_1, y_0, and y_1 are predetermined constants. As opined by [2], it's sacrosanct to determine the continuous value of $y(x)$ of (1) on the interval $[x_0, x_1]$ and that will satisfy the boundary conditions (2), to which an alternative approach is proposed by [3]. Research has shown that while Initial Value Problems (IVP) have unique solutions, the result is not always the case for BVP as opined by [4]. Some BVPs may have unique, multiple, or no solutions depending on the origin of the problem and nature. BVPs are basic in modeling and have cutting edge practical usage in Engineering, Physics, and other sciences. These involve solving differential equations with specified conditions at the boundaries of the domain, making them necessary for accurately describing real-world phenomena such as heat conduction, fluid dynamics, and structural analysis.

The technique herein introduced three different approximations to the derivatives, y_i', at a grid point x_i, called (i) two-point forward, (ii) two-point backward, and (iii) two-point central, respectively, as:

$$y_i' = \frac{y_{i+1} - y_i}{h} \tag{6}$$

$$y_i' = \frac{y_i - y_{i-1}}{h} \tag{7}$$

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} \tag{8}$$

To determine the second derivatives following the same approximate differential, it will be necessary that an additional relation, $y''(x_k) = y_k''$, be added to the first derivatives (6) through (8). To obtain this, (6) and (7) have the second-order approximation, which is the difference between (6) and (7) as:

$$y_i'' = \frac{y_i'}{h} - \frac{y_i'}{h} = \frac{\frac{y_{i+1} - y_i}{h} - \frac{y_i - y_{i-1}}{h}}{h} = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} \tag{9}$$

usually referred to as three-point central.

Though (6) through (9) have been derived intuitively, each can be derived via Taylor expansion with rigorous and extensive efforts. To approximate the second derivative requires one to use the central difference formula. This is achieved by subdividing $[x_0, x_1]$ into n equal intervals with step lenght $h = \frac{x_1 - x_0}{n}$ and defining the grid points $x_i = x_0 + ih$, where $i = 0, 1, 2, \ldots, n$. The 2nd derivative of $y(x)$ can then be approximated at each point in the grid.

2. Mildly Nonlinear Boundary Value Problems (MNBVP)

The Mildly Nonlinear Boundary Value Problems (MNBVP) appear in different forms depending on the model statement. According to [5], this equation arises mostly in problems in science, engineering, and particularly in the discretization of certain nonlinear differential equations, as it is being mentioned by [6]. However, the most widely studied variant of MNBVP is:

$$y'' = f(x, y, y') \tag{10}$$

$$y(x_0) = y_0, \; y(x_1) = y_1. \tag{11}$$

To ensure the uniqueness of the solution of (10) and (11), one assumes that

$$\frac{\partial f(x,y)}{\partial y} \geq 0, x_0 \leq x \leq x_1, \text{and } -\infty < y < \infty. \tag{12}$$

According to [7] and [8], if (12) holds, then (10) and (11) are referred to as an MNBVP.

Note that for the linear differential equation

$$y'' + P(x)y' + Q(x)y = R(x), x \in I \tag{13}$$

Let $I \in [x_0, x_1]$. Let y_0 and y_1 be specified constants and $P(x), Q(x)$, and $R(x)$ be continuous functions in the interval, $x_0 \leq x \leq x_1$. Considering the linear BVP in (13) subject to the conditions (11), it is pertinent to note that, for numerical reasons, *a priori*, the solution of the problem (8) and (9) thus exists, and the solution is uniquely characterized. With this, in addition to the conditions afore listed herein also in [9] and [10], it is presumed that, if $x_0 \leq x \leq x_1$,

$$Q(x) \leq 0 \tag{14}$$

This is a sufficient condition to ascertain that the solution exists and that it is unique.

If y' is assumed to be an independent variable, then (13) gives rise to

$$\frac{\partial}{\partial y}[-P(x)y' - Q(x)y + R(x)] = -Q(x) \tag{15}$$

Hence, it implies that either the RHS of (13) is

$$-Q(x) \geq 0 \text{ or } Q(x) \leq 0 \tag{16}$$

The solution of (10) and (11) go in the same way as prescribed with the three-point central approximation

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = f(x_i, y_i) \tag{17}$$

Substituting (17) and (8) in (13) for $i = 1, 2, 3, \dots, n-1$ gives rise to

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + P(x_i)\frac{y_{i+1} - y_{i-1}}{2h} + Q(x_i)y_i = R(x_i) \tag{18}$$

The relation so formed in (17) is called the MNBVP. If $f_i(x_i)$ for all $i = 1, 2, \dots, n$, are not all constants, but rather function and the derivative, $f_i'(x_i) \geq 0$, then the algebraic system so formed is referred to as mildly nonlinear. According to [6], unlike linear systems, all mildly nonlinear systems have unique solutions. Solving this class of problems in this paper will require that two different numerical techniques be intertwined.

3. The Intertwined Difference Equations and Newton-Lieberstein Algorithm

While several methods can be employed to find a solution to tri-diagonal systems, not many can be employed to determine solution for mildly non-linear tri-diagonal systems; hence, the intertwined approach is investigated thus:

Step 1a: Determine the nonnegative constant M which satisfies $|P(x)| \leq M$ on $a \leq x \leq b$. The interval is divided into n equal parts using the grid points $a = x_0, x_1, x_2, \dots, x_n = b$, but let the grid step-size, h satisfy the condition

$$Mh < 2. \tag{19}$$

Step 1b: Let $y_i = y(x_i), i = 0, 1, 2, 3, \dots, n$ and recall that $y_0 = \alpha$ and $y_n = \beta$.

Step 1c: At each of the interior grid point, $x_i, i = 1, 2, 3, \dots, n-1$, approximate (13) by (9) and (8) as

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + P(x_i)\frac{y_{i+1} - y_{i-1}}{2h} + Q(x_i)y_i = R(x_i) \tag{20}$$

Or, equivalently, by rearranging (20) as

$$[2 - hP(x_i)]y_{i-1} + [-4 + 2h^2Q(x_i)]y_i + [2 + hP(x_i)]y_{i+1} = 2h^2R(x_i) \tag{21}$$

Step 1d: Write down (21) in order, for $i = 1, 2, 3, \dots, n-1$, and input the determined values for y_0 and y_n. This will definitely yield a tridiagonal linear algebraic system satisfying the conditions afore listed.

Solve the linear algebraic system in Step 1d by the Newton-Lieberstein technique to yield the numerical solution so required.

The Newton-Lieberstein technique for solving mildly nonlinear systems is obtained via the following steps of the algorithm.

Step 2a: Guess an initial variable value for $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \dots, x_n^{(0)}$ and a value for ω in the range $[0, 2]$.

Step 2b: For $k = 0, 1, 2, 3, \dots$, iterate with the relations

$$x_1^{(k+1)} = x_1^{(k)} - \omega\left(\frac{a_{11}x_1^{(k)} + a_{12}x_2^{(k)} - f_1(x_1^{(k)})}{a_{11} - f_1'(x_1^{(k)})}\right) \tag{22}$$

$$x_2^{(k+1)} = x_2^{(k)} - \omega\left(\frac{a_{21}x_1^{(k)} + a_{22}x_2^{(k)} + a_{23}x_3^{(k)} - f_2(x_2^{(k)})}{a_{22} - f_2'(x_2^{(k)})}\right) \tag{23}$$

$$x_3^{(k+1)} = x_3^{(k)} - \omega\left(\frac{a_{32}x_2^{(k)} + a_{33}x_3^{(k)} + a_{34}x_4^{(k)} - f_3(x_3^{(k)})}{a_{33} - f_3'(x_3^{(k)})}\right) \tag{24}$$

$$x_{n-1}^{(k+1)} = x_{n-1}^{(k)} - \omega\left(\frac{a_{n-1,n-2}x_{n-2}^{(k)} + a_{n-1,n-1}x_{n-1}^{(k)} + a_{n-1,n}x_n^{(k)} - f_{n-1}(x_{n-1}^{(k)})}{a_{n-1,n-1} - f_{n-1}'(x_{n-1}^{(k)})}\right) \tag{25}$$

$$x_n^{(k+1)} = x_n^{(k)} - \omega\left(\frac{a_{n,n-1}x_{n-1}^{(k)} + a_{nn}x_n^{(k)} - f_n(x_n^{(k)})}{a_{nn} - f_n'(x_n^{(k)})}\right) \tag{26}$$

Step 2c: In determining the convergence of the method, either of the following conditions could be used as a terminating criterion:

continue the iteration until $\left|x_i^{(k+1)} - x_i^{(k)}\right| < \epsilon, i = 1, 2, 3, \dots, n$ for the prescribed convergence tolerance ϵ.

The method enjoys a quadratic convergence principle that makes the technique said to have converged after n iterations.

where (i) and (ii) are not satisfied, the results could be explicitly compared with results from other methods.

Step 2d: Substitute the values $x_1^{(k+1)}, x_2^{(k+1)}, x_3^{(k+1)}, \dots, x_n^{(k+1)}$ into the original equations to cross check that they are an approximate solution.

Attention will be on theorems to support the linear BVPs of a special type, as in (6) and (7).

4. Transforming BVPs to Tri-diagonal Systems

The general form for the linear algebraic system for $i = 2, 3, 4, \dots\ n$ equations in the n unknowns $x_1, x_2, \dots, x_n$ is given as

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + \ a_{1n}x_1 = f_1(x_1) \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + \ a_{2n}x_2 = f_2(x_2) \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + \ a_{3n}x_3 = f_3(x_3) \\ \quad \vdots \ \vdots \ \vdots \ \ddots \ \vdots \ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + \ a_{nn}x_n = f_n(x_n) \end{array}\right\} \tag{27}$$

In augmented matrix form, (27) is expressed as,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1(x_1) \\ f_2(x_2) \\ f_3(x_3) \\ \vdots \\ f_n(x_n) \end{pmatrix} \tag{28}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}, \ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix},$$

and

$$b = \begin{pmatrix} f_1(x_1) \\ f_2(x_2) \\ f_3(x_3) \\ \vdots \\ f_n(x_n) \end{pmatrix} \tag{29}$$

where A is the coefficient matrix, x is the variable vector, and the solution vector b is as given. Then, (27) can be written in a compact form as

$$Ax = b \tag{30}$$

The system (28) or its equivalent (30) is called a tridiagonal system as opined by [11] and [12], if all the entries are zero except $a_{ii}, a_{j,j+1}$, and $a_{j+1,j}$ for $i = 1, 2, 3, \dots, n$, $j = 1, 2, 3, \dots, n-1$.

A simultaneous algebraic system with nonzero coefficients only on the lead diagonal, on the lower diagonal, and on the upper diagonal is called a tridiagonal system of equations. The term tridiagonal is associated with the coefficient matrix, A, in (28) or (30), provided it's of the form that its lead diagonal a_{ii}, the super-diagonal $a_{j,j+1}$, i.e., the diagonal above the diagonal, and the sub-diagonal $a_{j+1,j}$, i.e., the diagonal below the non-zero entries.

To ensure that the tridiagonal system has a unique solution, in practice, it is important that the following conditions in (31), (32), and (33) must be met.

Theorem 2: If (30) is a tri-diagonal system. Let the main diagonal entries be negative while the sub-diagonal and super-diagonal entries are positive, such that

$$a_{ii} < 0, i = 1, 2, 3, \dots, n \tag{31}$$

$$a_{i+1,i} > 0, i = 1, 2, 3, \dots, n-1 \tag{32}$$

$$a_{i,i+1} > 0, i = 1, 2, 3, \dots, n-1 \tag{33}$$

In addition to the conditions stated above, according to [13] and [15] to ensure (31) through (32) hold, entries on the main diagonal are made to dominate the matrix such that the absolute value of the main diagonal entries is greater than or it is equal to the sum of all other row entries, with strict inequality for at least one row of the coefficient matrix. Precisely, let

$$-a_{11} \geq a_{12} \tag{34}$$

$$-a_{n,n} \geq a_{n,n-1} \tag{35}$$

$$-a_{ii} \geq a_{i,i-1}, i = 2, 3, \dots., n-1, \tag{36}$$

With the strict and absolute inequality condition sustained for at least one of (34) through (36), the resulting linear algebraic system is said to have a unique solution.

The prior knowledge of Theorem 2 as opined by [16] becomes necessary before commencing the computation process because, for systems that do not have unique solutions, computations in such systems will lead to no right or meaningful direction. If a system has more than one solution, i.e., no unique solution, applying specific computational methods could cause a drift from one result to another, thereby confusing.

If $f_i(x_i)$ for all $i = 1, 2, \dots, n$, are not all constants and the derivative, $f_i'(x_i) \geq 0$, then the system is referred to as a mildly nonlinear problem satisfying the conditions in Theorem 2 as in [17]. According to [14], all such mildly nonlinear system has solutions and their solutions are said to be unique.

To illustrate the MNBVP transformation procedure to an argument matrix form that will later be solved using an appropriate numerical method, the following test problems will be considered.

Problem 1: Consider the BVP

$$y'' + \frac{1}{2}(x-3)y' - y = 0 \tag{37}$$

$$y(0) = 11, \text{ and } y(6) = 11 \tag{38}$$

Comparing (37) and (38) with (10), (11), and (13), $0 \leq x \leq 6$ holds for $x_0 \leq x \leq x_1$, $P(x)$ is $\frac{1}{2}(x-3), Q(x)$ is equal to -1, and $R(x)$ corresponds to 0.

It is expedient that one determines if the solution will be unique before rigorous solution exercises are carried out. The solution to the problem is unique. On $0 \leq x \leq 6, \frac{1}{2}(x-3) \leq \frac{3}{2}$, so that one may choose $M = \frac{3}{2}$.

If the interval $0 \leq x \leq 6$ is divided into six equal parts, the grid points are x_0, x_1, x_2, x_3, x_4, x_5, and x_0 as $0, 1, 2, 3, 4, 5,$ and 6 respectively, and the grid size is $h = 1$. Hence, $Mh = \frac{3}{2}$ and (19) is holds. Then, (37) is approximated by (20) for $i = 1, 2, 3, 4,$ and 5 as:

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + \frac{1}{2}(x_i - 3)\frac{y_{i+1} - y_{i-1}}{2h} - y_i = 0. \tag{39}$$

$$(7 - x_i)y_{i-1} - 12y_i + (1 + x_i)y_{i+1} = 0 \tag{40}$$

Expressing each of (40) carefully in order for $i = 1, 2, 3, 4,$ and 5 will degenerate into

$$-12y_1 + 2y_2 = 66$$

$$5y_1 - 12y_2 + 3y_3 = 0$$

$$4y_2 - 12y_3 + 4x_4 = 0$$

$$3y_3 - 12y_4 + 5y_5 = 0$$

$$2y_4 - 12y_5 = 66$$

This actually satisfies the aforementioned conditions, and thereafter the Newton-Lieberstein technique is applied to solve the tridiagonal system, and the result according to [18] and [19] is given as:

$$y_1 = 6, y_2 = y_4 = 3, y_3 = 2, \text{ and } y_5 = 6 \tag{41}$$

The result in (41) is then compared with the analytical result to check for the effectiveness, robustness, and flexibility of the technique.

Problem 2: Solve the Mildly Nonlinear Boundary Value Problem

$$y'' = e^y \tag{42}$$

under the boundary conditions

$$y(0) = 0, y(1) = 0 \tag{43}$$

Imperatively, the Newton-Lieberstein Method will perform better if the solution of the reduced tridiagonal system exists and it is unique.

To investigate if the solution of Problem 2 will exist and determine the uniqueness of the result, the derivative of the function

$$f(x, y) = e^y \tag{44}$$

is sought. Such that

$$\frac{\partial}{\partial y} f(x, y) = e^y > 0. \tag{45}$$

Thus, the existence and the uniqueness of the solution to the problem is substantiated.

Then, to solve the problem numerically, the interval $[0, 1]$ has to be divided into 10 equal parts, and $h = 0.1$ gives rise to

$x_0, x_1, x_2, \ldots, x_{10} =$
$0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$ respectively an in [9].

From the condition in (35), let $y(x_i) = y_i$ such that $y_0 = 0$ and $y_1 = 0$. To approximate y_i where $i = 1, 2, \ldots, n-1$, i.e., $y_1, y_2, \ldots, y_9$, at each interior grid point, the differential equation is approximated by the nonlinear difference equation; introducing (9) into (34) yields

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = e^{y_i} \text{ for } i = 1, 2, \ldots, 9 \tag{46}$$

Simplifying (46) at $h = 0.1$ further gives

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{(0.1)^2} = e^{y_i}$$

$$y_{i-1} - 2y_i + y_{i+1} = 0.01e^y \tag{47}$$

For $i = 1, 2, \ldots, n-1$ and the zero boundary conditions, yields the mildly nonlinear tridiagonal system as

$$\left.\begin{aligned} y_0-2y_1+y_2&=0.01e^{y_1}\\ y_1-2y_2+y_3&=0.01e^{y_2}\\ y_2-2y_3+y_4&=0.01e^{y_3}\\ y_3-2y_4+y_5&=0.01e^{y_4}\\ y_4-2y_5+y_6&=0.01e^{y_5}\\ y_5-2y_6+y_7&=0.01e^{y_6}\\ y_6-2y_7+y_8&=0.01e^{y_7}\\ y_7-2y_8+y_9&=0.01e^{y_8}\\ y_8-2y_9+y_{10}&=0.01e^{y_9}\\ y_9-2y_{10}+y_{11}&=0.01e^{y_{10}} \end{aligned}\right\} \quad (48)$$

Putting the conditions (43) in (48) and simplifying, one obtains (48) as

$$\left.\begin{aligned} -2y_1+y_2&=0.01e^{y_1}\\ y_1-2y_2+y_3&=0.01e^{y_2}\\ y_2-2y_3+y_4&=0.01e^{y_3}\\ y_3-2y_4+y_5&=0.01e^{y_4}\\ y_4-2y_5+y_6&=0.01e^{y_5}\\ y_5-2y_6+y_7&=0.01e^{y_6}\\ y_6-2y_7+y_8&=0.01e^{y_7}\\ y_7-2y_8+y_9&=0.01e^{y_8}\\ y_8-2y_9+y_{10}&=0.01e^{y_9}\\ y_9-2y_{10}&=0.01e^{y_{10}}-1 \end{aligned}\right\} \quad (49)$$

Reducing (49) into an augmented matrix gives:

$$\left.\begin{pmatrix} -2&1&0&0&0&0&0&0&0&0\\ 1&-2&1&0&0&0&0&0&0&0\\ 0&1&-2&1&0&0&0&0&0&0\\ 0&0&1&-2&1&0&0&0&0&0\\ 0&0&0&1&-2&1&0&0&0&0\\ 0&0&0&0&1&-2&1&0&0&0\\ 0&0&0&0&0&1&-2&1&0&0\\ 0&0&0&0&0&0&1&-2&1&0\\ 0&0&0&0&0&0&0&1&-2&1\\ 0&0&0&0&0&0&0&0&1&-2 \end{pmatrix}\begin{pmatrix} y_1\\y_2\\y_3\\y_4\\y_5\\y_6\\y_7\\y_8\\y_9\\y_{10}\end{pmatrix}=\begin{pmatrix} 0.01e^{y_1}\\0.01e^{y_2}\\0.01e^{y_3}\\0.01e^{y_4}\\0.01e^{y_5}\\0.01e^{y_6}\\0.01e^{y_7}\\0.01e^{y_8}\\0.01e^{y_9}\\0.01e^{y_{10}}-1\end{pmatrix}\right\} \quad (50)$$

This point ends the first part of the technique, while the second phase of the intertwining begins by employing the Newton-Lieberstein's Method to solve the argument matrix (50) in which the coefficient matrix of (50), is tridiagonal.

The transformed MNBVP can be solved using [1] however, now being solved following [5] thus:

Step 1: Let $w=1.5$, initial guess values at $k=0$, and

$$y_1^0=0, y_2^0=0, y_3^0=0, y_4^0=0,\ y_5^0=0,$$

$$y_6^0=0, y_7^0=0, y_8^0=0,\ y_9^0=0, \text{ and } y_{10}^0=0.$$

Step 2: Update the variable values using the Newton-Lieberstein algorithm thus:

$$y_1^{(1)}=y_1^{(0)}-w\frac{-2y_1^{(0)}+y_2^{(0)}-f_1\left(y_1^{(0)}\right)}{a_{11}-f_1'\left(y_1^{(k)}\right)}=0-1.5\frac{-2(0)+0-0.01}{-2-0.01}=-0.00746268656716$$

$$y_2^{(1)}=y_2^{(0)}-w\frac{y_1^{(1)}-2y_2^{(0)}+y_3^0-f_2(y_2^{(0)})}{a_{22}-f_2'(x_2^{(k)})}=0-1.5\frac{-0.00746268656716-0+0-0.01}{-2-0.01}=-0.01303185564713$$

$$y_3^{(1)}=y_3^{(0)}-w\frac{y_2^{(1)}-2y_3^{(0)}+y_4^0-f_3(y_3^{(0)})}{a_{33}-f_3'(x_3^{(k)})}=0-1.5\frac{-0.01303185564713-0+0-0.01}{-2-0.01}=-0.01718795197547$$

$$y_4^{(1)}=y_4^{(0)}-w\frac{y_3^{(1)}-2y_4^{(0)}+y_5^0-f_4(y_4^{(0)})}{a_{44}-f_4'(x_4^{(k)})}=0-1.5\frac{-0.01718795197547-0+0-0.01}{-2-0.01}=-0.02028951639960$$

$$y_5^{(1)}=y_5^{(0)}-w\frac{y_4^{(1)}-2y_5^{(0)}+y_6^0-f_5\left(y_5^{(0)}\right)}{a_{55}-f_5'\left(x_5^{(k)}\right)}=0-1.5\frac{-0.02028951639960-0+0-0.01}{-2-0.01}=-0.02260411671612$$

$$y_6^{(1)}=y_6^{(0)}-w\frac{y_5^{(1)}-2x_6^{(0)}+y_7^0-f_6\left(y_6^{(0)}\right)}{a_{66}-f_6'\left(x_6^{(k)}\right)}=0-1.5\frac{-0.02260411671612-0+0-0.01}{-2-0.01}=-0.02433143038516$$

$$y_7^{(1)}=y_7^{(0)}-w\frac{y_6^{(1)}-2y_7^{(0)}+y_8^0-f_7(y_7^{(0)})}{a_{77}-f_7'(x_7^{(k)})}=0-1.5\frac{-0.02433143038516-0+0-0.01}{-2-0.01}=-0.02560242094974$$

$$y_8^{(1)}=y_8^{(0)}-w\frac{y_7^{(1)}-2y_8^{(0)}+y_9^0-f_8(y_8^{(0)})}{a_{22}-f_2'(x_2^{(k)})}=0-1.5\frac{-0.02560242094974-0+0-0.01}{-2-0.01}=-02654997852770$$

$$y_9^{(1)} = y_9^{(0)} \quad - w\frac{y_8^{(1)} - 2y_9^{(0)} + y_{10}^0 \; - \; f_9\left(y_9^{(0)}\right)}{a_{22} \; - \; f_2'\left(x_2^{(k)}\right)} = \quad 0 \quad - \quad 1.5\frac{-\;02654997852770 \; - \; 0 \; + \; 0 \; - \; 0.01}{-2-0.01} = -0.02727610337888$$

$$y_{10}^{(1)} = y_{10}^{(0)} \quad - w\frac{y_9^{(1)} - 2y_{10}^{(0)} - \; f_{10}\left(y_{10}^{(0)}\right)}{a_{22} \; - \; f_2'\left(x_2^{(k)}\right)} = 0 \quad - 1.5\frac{-0.02727610337888 \; - \; 0 \; + \; 0 \; - \; 0.01}{-2-0.01} = - \; 0.02781798759618$$

This process has to be looped 10 times. The expected result is the values of the variables at the 10th iteration. However, this is done using a program, and the summary of the results of the 10 iterations by [17] is as presented below.

$$\text{At } k = 10, \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \end{pmatrix} = \begin{pmatrix} -0.041822363 \\ -0.075229088 \\ -0.100685522 \\ -0.118159167 \\ -0.127522654 \\ -0.128583546 \\ -0.121104998 \\ -0.104818354 \\ -0.079427894 \\ -0.044609196 \end{pmatrix} \tag{51}$$

The solution of the reduced tridiagonal system as, in (51), is the solution of the MNBVP. This technique becomes very relevant, especially when the system of equation has many variables as noted in [12].

5. Observations and Motivations Surrounding the Intertwined Algorithm

At this point, various aspects of the algorithm need to be looked into with a view to substantiate its usage. To carefully x-ray this, Problem 1 will be used as a test case.

1) For the assumption in (19), a clear insight into this is that there is a need to examine (40). Observe that the tridiagonal system as in

$$-4 + 2h^2Q(x_i) \tag{52}$$

which is the primary diagonal entries are the coefficients in (40) for the variable y_i, that is, -12, which is the upper diagonal entries which result are the coefficients in (40) of y_{i+1}, that is $(1 + x_i)$, and the lower diagonal entries which result are the coefficients in (40) of y_{i-1}, that is $(7 - x_i)$.

2) It is observed that, in general, with regards to (21), the lead diagonal entries will be

$$2 + hP(x_i) \tag{53}$$

and the sub-diagonal elements will be

$$2 - hP(x_i). \tag{54}$$

3) Ensure that the existence and the uniqueness conditions of the solution are satisfied as:

$$-4 + 2h^2Q(x_i) < 0 \tag{55}$$

$$2 + 2hP(x_i) > 0 \tag{56}$$

$$2 - 2hP(x_i) > 0. \tag{57}$$

However, since (16) holds, then (55) remains sustained. Hence, by (19),

$$|hP(x)| \leq hM < 2 \tag{58}$$

Such that (56) and (57) are all valid.

4) The assurance of the diagonal dominance has to be established, i.e.,

$$|-4 + 2h^2Q(x_i)| \geq |2 + 2hP(x_i)| \tag{59}$$

$$|-4 + 2h^2Q(x_i)| \geq |2 - 2hP(x_i)| \tag{60}$$

And that the inequality is valid for at least one row. Though this is also valid. It can be noted that

$$|-4 + 2h^2Q(x_i)| = 4 + 2h^2Q(x_i) \geq 4 \tag{61}$$

Moreover, (56) and (57)

$$\begin{aligned} &|2 + 2hP(x_i)| + |2 - 2hP(x_i)| \\ &= 2 + hP(x_i) + 2 - hP(x_i) = 4 \end{aligned} \tag{62}$$

So that for $i = 1, 2, 3, 4,$ and $5,$ but imagine

$$|-4 + 2h^2Q(x_i)| \geq |2 + 2hP(x_i)| + |2 - 2hP(x_i)| \tag{63}$$

But the strict and absolute inequality is holds for the first and last equation and, all the assumptions are satisfied.

6. Conclusion

This research stressed solving MNBVPs that result in tridiagonal systems using the intertwined Difference approximation technique and the Newton-Lieberstein technique. The results demonstrated that the technique provides efficient and accurate solutions with faster convergence.

The key findings include:

1) The Newton-Lieberstein method was able to handle mildly nonlinear systems efficiently.

2) The iterative approach showed improved accuracy with each step.
3) The method proved computationally efficient compared to the classical techniques.

The results from the test problems establish the stability and reliability of the method, evident in the solutions, making it an effective choice for solving MNBVPs in engineering, physics, applied mathematics, and. other sciences.

The findings confirm that the Newton-Lieberstein method is an effective approach for solving tridiagonal linear systems. It enhances accuracy, stability, and efficiency, making it valuable for complex numerical problems. The study contributes to the field of numerical analysis by providing a method that balances computational efficiency with precision, ensuring that BVPs can be solved more effectively.

Abbreviations

IVP	Initial Value Problem
BVP	Boundary Value Problem
MNBVP	Mildly Nonlinear Boundary Value Problem
ODE	Ordinary Differential Equation

Author Contributions

Dele-Rotimi Adejoke Olumide: Formal Analysis, Investigation, Methodology

Akinmuyise Folorunsho Mathew: Data curation, Writing – original draft

Adebayo Kayode James: Conceptualization, Methodology, Resources, Writing – review & editing

Data Availability Statement

The data supporting the outcome of this research work has been reported in this manuscript.

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Mirko Todorovski, Jordan S. A., and Jovica V., (2021), Solving Tridiagonal Symmetric Systems of Equations Using Circuit Theory Approach, IAENG International Journal of Computer Science, 48: 3, IJCS_48_3_24, Volume 48, Issue 3.

[2] Omar, Z. and Suleiman, M. B., (2005), Solving higher order ordinary differential equations using parallel 2-point explicit block method, in Matematika, Jabatan UTM, 21, 15-23.

[3] Nur Zahidah Mukhtar, Zanariah Abdul Majid, Fudziah Ismail, (2011), Numerical Solution for Solving Second Order Ordinary Differential Equations Using Block Method, International Conference Mathematical and Computational Biology, International Journal of Modern Physics: Conference Series, World Scientific Publishing Company, Vol. 9 (2012) 560-565, https://doi.org/10.1142/S2010194512005661

[4] Radzi, H. M., Majid, Z. A., Ismail, F., and Suleiman, M. B., (2011), Four-step implicit block method of Runge-Kutta type for solving first order ordinary differential equations. Proceedings of Fourth International Conference on Modeling, Simulation and Applied Optimization 2011, 139-143.

[5] Rosser, J. B. and Runge-Kutta, (1967), for all seasons, in SIAM Rev. 9, 417-452.

[6] Worland, P. B., (1976), Parallel methods for the numerical solutions of ordinary differential equations, in IEEE Transactions on Computers, 25, 1045-1048.

[7] Majid, Z. A., Suleiman, M. B., Ismail, F., and Othman, M., (2003), 2-point implicit block one-step method half Gauss-Seidel for solving first order ordinary differential equations, in Matematika, Jabatan UTM, 19, 91-100.

[8] Majid, Z. A. Suleiman, M. B., and Omar, Z., (2006), 3-point implicit block method for solving ordinary differential equations, in Bull. Malays. Math. Sci. Soc, 29, 23-31.

[9] Vincenzo Casulli and Paola Zanolli, (2012), Iterative solutions of mildly nonlinear systems, Journal of Computational and Applied Mathematics, 236 (2012) 3937-3947, https://doi.org/10.1016/j.cam.2012.02.042

[10] Prasad Krishna J. S. V. R. and Patil P. V. (2014). Finite Volume Numerical Grid Technique for Solving One and Two Dimensional Heat Flow Problems, Research Journal of Mathematical and Statistical Sciences (RJMSS), Vol. 2, No. 8.

[11] El-Mikkawy M. E. A., (2005), A new Computational Algorithm for Solving Periodic Tri-diagonal Linear Systems, Applied Mathematics and Computation, Vol. 161 No. 2, pp 691-696, https://doi.org/10.1016/j.amc.2003.12.114

[12] El-Mikkawy, M. and Atlan, F. (2014) Algorithms for Solving Linear Systems of Equations of Tridiagonal Type via Transformations. Applied Mathematics, 5, 413-422. https://doi.org/10.4236/am.2014.53042

[13] Krishna Prasad, J. S. V. R. and Parag V. Patil, (2015), Algorithm for Solving Tri-diagonal Finite Volume Discretized Linear Systems, Applications and Applied Mathematics: An International Journal (AAM), Vol. 10, Issue 2 (December 2015), pp. 995-1006, http://pvamu.edu/aamAppl.Appl.Math

[14] Ortega, J. M., Rheinboldt, W. C., (1970), Iterative Solution of Nonlinear Equations in Several Variables. Academic Press, San Diego.

[15] Adebayo K. J., Alabi T. J., Adisa I. O., Ademoroti A. O., and Dele-Rotimi A. O., (2025), Intertwining the Upwind Differencing and the Newton-Lieberstein's Methods to Solve Mildly Non-linear Boundary Value Problems (MNBVP), Quest Journals, Journal of Research in Applied Mathematics Volume 11 ~ Issue 7 (July 2025) pp: 53-56. www.questjournals.org

[16] Adebayo K. J, Adetolaju O. S., Omowaye S. K., (2026), Numerical Solution of Mildly Non-linear Boundary Value Problems (MNBVP) Using Poly-algorithms, IJRAR - International Journal of Research and Analytical Reviews (IJRAR), Volume.13, Issue 3, Page No pp. 864-871, August 2026, Available at: http://www.ijrar.org/IJRAR26C1791

[17] Bai, Z.-Z. (1997), A class of two-stage iterative methods for systems of weakly nonlinear equations. Numer. Algorithms 14(4), 295–319.

[18] Casulli, V. and Zanolli, P., (2010), A nested Newton-type algorithm for finite volume methods solving Richards' equation in mixed form, SIAM J. Sci. Comput. 32 (2010) 2255–2273.

[19] Donald Greenspan, (2008), Numerical Solution of Ordinary Differential Equations for Classical, Relativistic and Nano Systems, Wiley-VCH.

Biography

Dele-Rotimi Adejoke Olumide is a Senior Lecturer in the Department of Mathematical Sciences at Bamidele Olumilua University of Education, Science and Technology (BOUESTI), Ikere-Ekiti, Nigeria. She completed his Ph.D. in Mathematics from Ekiti State University in 2024, and his Master of Mathematics in Control Theory from the same institution in 2016. Her research interests include Optimization, Artificial Intelligence, Operations Research, Computational Mathematics, and Applied Mathematics. She has served in several research groups in her institution. Presently, she is the Head of the Department of Mathematical Sciences in BOUESTI, Ikere Ekiti, Nigeria. She is a member of several Mathematics bodies in and outside Nigeria.

Akinmuyise Folorunsho Mathew is a Senior Lecturer at Adeyemi Federal University of Education, Ondo, Ondo State, in the Department of Mathematics. He completed his Ph.D. in Mathematics from Ekiti State University in 2020, and his Masters of Mathematics in Control Theory from Ekiti State University in the year 2010. He is a member of several Mathematics bodies with research covering Optimization, Control Theory, and applied Mathematics.

Adebayo Kayode James is an Associate Professor at Ekiti State University, Mathematics Department, Faculty of Physical Sciences. He completed his Ph.D. in Mathematics from Ekiti State University in 2016 and his Master of Mathematics in Control Theory from the same institution in 2010. In addition, he is a member of several Mathematics bodies in and outside Nigeria. He has participated in multiple international research collaboration projects in recent years. He currently serves as a reviewer for quite a lot of journals.

Research Field

Dele-Rotimi Adejoke Olumide: Optimization; Artificial intelligence, Operations Research, Computational mathematics, and Applied mathematics

Akinmuyise Folorunsho Mathew: Optimization; Artificial intelligence, Operations Research, Computational mathematics, and Applied mathematics

Adebayo Kayode James: Optimization, Control Theory, Mathematical Modelling, Transportation Theory