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Sufficient Conditions for Wiener Index, Hyper-Wiener Index and Harary Index of the Hamilton Graph

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Abstract: The topological index can be used to depict the structural properties of graphs, and the Hamiltonian problem of graphs has always been a classical problem in graph theory. In this work, we use some known conditions to give some sufficient conditions for Hamilton graphs by the Wiener index, Hyper-Wiener index and Harary index of a graph.

Keywords: Hamilton Graph, Wiener Index, Hyper-Wiener Index, Harary Index

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E, where $V = \{v_1, v_2, ..., v_n\}$. The minimum degree of G is denoted by $\delta(G)$. The distance between vertices v_i and v_j of G is denoted by $d_G(v_i, v_j)$, which is the shortest path length from v_i to v_j in G, where $1 \leq i \leq j \leq n$. For graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ satisfying $V_1 \cap V_2 = \emptyset$, the union of them is denoted by $G_1 \cup G_2$. The join of G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph obtained by adding the edges of each vertex of G_1 to each vertex in G_2. The circle that passes through all vertices in G is called a Hamilton circle. A graph containing a Hamilton cycle is called a Hamilton graph.

The Wiener index[1] of a connected graph G is a topological index that is highly correlated with the physical and chemical properties of molecular compounds. It is defined as

$$W(G) = \sum_{v_i, v_j \in V(G)} d_G(v_i, v_j).$$

Let $D_i = D_G(v_i) = \sum_{v_j \in V(G)} d_G(v_i, v_j)$. Then $W(G) = \frac{1}{2}\sum_{i=1}^{n} D_i$. The hyper-Wiener index of graph G [2, 3] is defined as

$$WW(G) = \frac{1}{2}[\sum_{v_i, v_j \in V(G)} d_G(v_i, v_j) + \sum_{v_i, v_j \in V(G)} d_G^2(v_i, v_j)].$$

Let $DD_i = DD_G(v_i) = \sum_{v_j \in V(G)} d_G^2(v_i, v_j)$. Then $WW(G) = \frac{1}{4}\sum_{i=1}^{n}(D_i + DD_i)$. The Harary index of a connected graph G[4, 5] is defined as

$$H(G) = \sum_{v_i, v_j \in V(G)} \frac{1}{d_G(v_i, v_j)}.$$

Let $\tilde{D}_G(v_i) = \sum_{v_j \in V(G)} \frac{1}{d_G(v_i, v_j)}$.
Then $H(G) = \frac{1}{2}\sum_{v \in V(G)} \tilde{D}_G(v_i)$.

Above topological indices have many applications in graph theory[6, 9]. Some sufficient conditions for Hamiltonian graphs are given by using the Wiener index, hyper-Wiener index and Harary index[7, 8]. There have been many studies on these three kinds of indices in the above literatures, but under many new sufficient conditions for Hamiltonian graphs, we get the sufficient conditions for the updated Hamiltonian Graphs on these three kinds of indices.

Moreover, in this study, we show some different sufficient conditions for a Hamilton graph by using the Wiener index, hyper-Wiener index and Harary index.

2. Main Results

Lemma 2.1. [10] Let graph G be an n order graph with $n > 21$ and $\delta(G) \geq 3$. If $e(G) \geq \binom{n-3}{2}+2$, where $G \notin L_3(n)$ or $G \notin M_3(n)$, then G is a Hamilton graph.

Write K_s and $\bar{K}_s$ for the complete and the edgeless graphs of order s. Given graphs G and H, write $G \vee H$ for their join. Let $L_3(n) := K_1 \vee (K_{n-4} + K_3)$, $M_3(n) := K_3 \vee (K_{n-6} + \bar{K}_3)$.

Lemma 2.2. [11] Let graph G be an n order graph with $n > 6$ and $\delta(G) \geq 6$. If $e(G) \geq \binom{n-3}{2} + 8$, where $G \notin NC \cup NP$, then G is a Hamilton graph.

$NC = \{K_8 \vee 9K_1, K_6 \vee (K_2 + 6K_1), K_7 \vee 8K_1, K_6 \vee (K_{1,7} + K_1), K_{2,8} \vee K_5, G_1,$
$8K_1 \vee K_3 \vee C_4, 8K_1 \vee K_3 \vee (K_1 + K_2), (K_1 \vee 6K_1 + K_2) \vee K_6, K_2 \vee C_4 \vee 7K_1, K_6 \vee 7K_1,$
$K_{2,7} \vee K_4, (2K_2 + K_{1,4}) \vee K_6, G_2, (K_1 + K_2 + K_{1,5}) \vee K_6, K_5 \vee (8K_1 \vee 2K_1 - e), K_{2,2,2} \vee 7K_1\}$.

$NP = \{2K_1 \vee K_4 \vee (K_2 + 6K_1), H_1, K_6 \vee 8K_1, H_2, 2K_1 \vee K_6 \vee 9K_1, K_6 \vee (3K_1 + K_{1,5}), 2K_1 \vee K_4 \vee (2K_1 + K_{1,6}), H_9, H_{10}, K_{1,2} \vee K_{3,7}, H_{23}\}$

Theorem 2.1. Let graph G be an n order graph with $n > 21$ and $\delta(G) \geq 3$. If $W(G) \leq \frac{n^2+5n-16}{2}$, where $G \notin L_3(n)$ or $G \notin M_3(n)$, then G is a Hamilton graph.

Proof. Assuming G is not a Hamilton graph. By Lemma 1.1, we have $e(G) < \binom{n-3}{2} + 2$. unless $G \subset L_3(n)$ or $G \subset M_3(n)$, if $e(G) < \binom{n-3}{2} + 2$,

$$\begin{aligned} W(G) &= \sum_{v_i, v_j \in V(G)} d_G(v_i, v_j) \\ &= \frac{1}{2}\sum_{i=1}^{n} D_i \\ &\geq \frac{1}{2}\sum_{i=1}^{n}(d_G(x_i) + 2(n-1-d_G(x_i))) \\ &= n(n-1) - \frac{1}{2}\sum_{i=1}^{n} d_G(x_i) \\ &= n(n-1) - e(G) \\ &> n(n-1) - [\binom{n-3}{2} + 2] \\ &= \frac{n^2+5n-16}{2}. \end{aligned}$$

This contradicts with theorem condition $W(G) \leq \frac{n^2+5n-16}{2}$. If $G \subset L_3(n)$ or $G \subset M_3(n)$, then graph G is not a Hamilton graph.□

Theorem 2.2. Let graph G be an n order graph, $n > 21$, $\delta(G) \geq 3$. If $WW(G) \leq \frac{n^2+11n-32}{2}$, unless $G \subset L_3(n)$ or $G \subset M_3(n)$, G is a Hamilton graph.

Proof. Reduction to absurdity. Assuming G is not a Hamilton graph, by the lemma 1.1 have $e(G) < \binom{n-3}{2} + 2$, unless $G \subset L_3(n)$ or $G \subset M_3(n)$, if $e(G) < \binom{n-3}{2} + 2$,

$$\begin{aligned} WW(G) &= \frac{1}{2}[\sum_{v_i, v_j \in V(G)} d_G(v_i, v_j) \\ &+ \sum_{v_i, v_j \in V(G)} d_G^2(v_i, v_j)] \\ &= \frac{1}{4}\sum_{i=1}^{n}(D_i + DD_i) \\ &\geq \frac{1}{4}\sum_{i=1}^{n}[d_G(x_i) + 2(n-1-d_G(x_i))] \\ &+ \frac{1}{4}\sum_{i=1}^{n}[d_G(x_i) + 4(n-1-d_G(x_i))] \\ &= \frac{3}{2}n(n-1) - \sum_{i=1}^{n} d_G(x_i) \\ &= \frac{3}{2}n(n-1) - 2e(G) \\ &> \frac{3}{2}n(n-1) - 2[\binom{n-3}{2} + 2] \\ &= \frac{n^2+11n-32}{2}. \end{aligned}$$

This contradicts with theorem condition $WW(G) \leq \frac{n^2+11n-32}{2}$. If $G \subset L_3(n)$ or $G \subset M_3(n)$, then graph G is not a Hamilton graph.□

Theorem 2.3. Let graph G be an n order graph, $n > 21$, $\delta(G) \geq 3$. If $H(G) \geq \frac{n^2-4n+8}{2}$, unless $G \subset L_3(n)$ or $G \subset M_3(n)$, G is a Hamilton graph.

Proof. Reduction to absurdity. Assuming G is not a Hamilton graph, by the lemma 1.1 have $e(G) < \binom{n-3}{2} + 2$, unless $G \subset L_3(n)$ or $G \subset M_3(n)$, if $e(G) < \binom{n-3}{2} + 2$,

$$\begin{aligned} H(G) &= \sum_{v_i, v_j \in V(G)} \frac{1}{d_G(v_i, v_j)} \\ &= \frac{1}{2}\sum_{i=1}^{n} \tilde{D}_G(x_i) \\ &\leq \frac{1}{2}\sum_{i=1}^{n}(d_G(x_i) + \frac{1}{2}(n-1-d_G(x_i))) \\ &= \frac{1}{4}n(n-1) + \frac{1}{4}\sum_{i=1}^{n} d_G(x_i) \\ &= \frac{1}{4}n(n-1) + \frac{1}{2}e(G) \\ &< \frac{1}{4}n(n-1) + \frac{1}{2}[\binom{n-3}{2} + 2] \\ &= \frac{n^2-4n+8}{2}. \end{aligned}$$

This contradicts with theorem condition $H(G) \geq \frac{n^2-4n+8}{2}$. If $G \subset L_3(n)$ or $G \subset M_3(n)$, then graph G is not a Hamilton graph.□

Theorem 2.4. Let graph G be an n order graph, $n > 6$, $\delta(G) \geq 6$. If $W(G) \leq \frac{n^2+5n-28}{2}$, unless $G \in NC \cup NP$, G is a Hamilton graph.

Proof. Reduction to absurdity. Assuming G is not a Hamilton graph, by the lemma 1.2 have $e(G) < \binom{n-3}{2} + 8$, unless $G \in NC \cup NP$, if $e(G) < \binom{n-3}{2} + 8$,

$$\begin{aligned} W(G) &= \sum_{v_i, v_j \in V(G)} d_G(v_i, v_j) \\ &= \frac{1}{2}\sum_{i=1}^{n} D_i \\ &\geq \frac{1}{2}\sum_{i=1}^{n}(d_G(x_i) + 2(n-1-d_G(x_i))) \\ &= n(n-1) - \frac{1}{2}\sum_{i=1}^{n} d_G(x_i) \\ &= n(n-1) - e(G) \\ &> n(n-1) - [\binom{n-3}{2} + 8] \\ &= \frac{n^2+5n-28}{2}. \end{aligned}$$

This contradicts with theorem condition $W(G) \leq \frac{n^2+5n-28}{2}$. If $G \in NC \cup NP$, then graph G is not a Hamilton graph.□

Theorem 2.5. Let graph G be an n order graph, $n > 6$, $\delta(G) \geq 6$. If $WW(G) \leq \frac{n^2+11n-56}{2}$, unless $G \in NC \cup NP$, G is a Hamilton graph.

Proof. Reduction to absurdity. Assuming G is not a Hamilton graph, by the lemma 1.2 have $e(G) < \binom{n-3}{2} + 8$, unless $G \in NC \cup NP$, if $e(G) < \binom{n-3}{2} + 8$,

$$\begin{aligned} WW(G) &= \frac{1}{2}[\sum_{v_i, v_j \in V(G)} d_G(v_i, v_j) \\ &+ \sum_{v_i, v_j \in V(G)} d_G^2(v_i, v_j)] \\ &= \frac{1}{4}\sum_{i=1}^{n}(D_i + DD_i) \\ &\geq \frac{1}{4}\sum_{i=1}^{n}[d_G(x_i) + 2(n-1-d_G(x_i))] \\ &+ \frac{1}{4}\sum_{i=1}^{n}[d_G(x_i) + 4(n-1-d_G(x_i))] \\ &= \frac{3}{2}n(n-1) - \sum_{i=1}^{n} d_G(x_i) \\ &= \frac{3}{2}n(n-1) - 2e(G) \\ &> \frac{3}{2}n(n-1) - 2[\binom{n-3}{2} + 2] \\ &= \frac{n^2+11n-56}{2}. \end{aligned}$$

This contradicts with theorem condition $WW(G) \leq \frac{n^2+11n-56}{2}$. If $G \in NC \cup NP$, then graph G is not a Hamilton graph.□

Theorem 2.6. Let graph G be an n order graph, $n > 6$, $\delta(G) \geq 6$. If $H(G) \geq \frac{n^2-4n+14}{2}$, unless $G \in NC \cup NP$, G is a Hamilton graph.

Proof. Reduction to absurdity. Assuming G is not a Hamilton graph, by the lemma 1.2 have $e(G) < \binom{n-3}{2} + 8$, unless $G \in NC \cup NP$, if $e(G) < \binom{n-3}{2} + 8$,

$$\begin{aligned} H(G) &= \sum_{v_i, v_j \in V(G)} \frac{1}{d_G(v_i, v_j)} \\ &= \frac{1}{2}\sum_{i=1}^{n} \tilde{D}_G(x_i) \\ &\leq \frac{1}{2}\sum_{i=1}^{n}(d_G(x_i) + \frac{1}{2}(n-1-d_G(x_i))) \\ &= \frac{1}{4}n(n-1) + \frac{1}{4}\sum_{i=1}^{n} d_G(x_i) \\ &= \frac{1}{4}n(n-1) + \frac{1}{2}e(G) \\ &< \frac{1}{4}n(n-1) + \frac{1}{2}[\binom{n-3}{2} + 8] \\ &= \frac{n^2-4n+14}{2}. \end{aligned}$$

This contradicts with theorem condition $H(G) \geq \frac{n^2-4n+14}{2}$. If $G \in NC \cup NP$, then graph G is not a Hamilton graph. □

3. Summary

By studying the conclusions in references [10,11] regarding the inclusion of Hamilton cycles in graphs, and combining them with the use of topological indices in reference [9] this paper utilizes the $Wiener$ index, $Harary$ index and $hyper-Wiener$ index provide information on the inclusion of n-order simple graphs under different degrees of minimum degree. The sufficient conditions for the Hamilton cycle in addition, the circle length and circumference are closely related to the topological index, which will be the focus of our future research work. The authors declare no conflicts of interest.

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