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Multi-objective Linear Programming: A Survey

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Abstract: This paper presents a state-of-the-art survey of major algorithms suggested to solve Multiple Objective Linear Programming (MOLP). We have comprehensively reviewed MOLP papers that have appeared since 1964. The algorithms are considered in two broad categories: Non-Interactive algorithms and interactive ones. Interactivity in our view is an essential feature of usable tools. It enhances applicability and robustness of methods on one hand, but hinders them on the other by the mere fact that intervention is required. Note that the Non-Interactive algorithms include the Simplex, Interior Point, Objective Space based algorithms and relevant Nature-inspired population-based stochastic algorithms which are becoming more and more prominent. Note also that in the objective space methods, the simplex algorithm or the dual simplex algorithm are being invoked during the search process. This suggest that they should be put in the simplex based class. However, for more clarity and given that there is a strong trend to refer to them as objective space methods, we prefer to put them on their own since their underlying philosophy is different from that of the simplex based methods. While the Interactive ones only consist of the Simplex and Interior Point algorithms. An illustration of representative algorithms of each category and a tabulated summary of all algorithms are included.

Keywords: Multiple Objective Linear Programming, Simplex Based Methods, Interior Point Based Methods, Objective Space Based Methods, Heuristics, Meta-heuristics

1. Introduction

As stated in [114] that MOLP is a branch of Multiple Criteria Decision Making (MCDM) that seeks to optimize two or more linear objective functions subject to a set of linear constraints. It has been an active area of research since the 1960s because of its relevance in practice, [55]. Indeed, many decision making problems that arise in the real world have more than one objective function. Consequently, it has been widely applied in many fields of human endeavour and has been a useful tool in decision making, [98]. We noted in [100] that it can be formally written as

$$\begin{aligned} \min \quad & c_1 x = f_1 \\ & \vdots \\ & c_q x = f_q \\ \text{subject to} \quad & x \in X = \{x \in \mathbb{R}^n : Ax = b, b \in \mathbb{R}^m, x \geq 0\} \end{aligned} \tag{1}$$

or alternately as a linear vector optimization problem

$$\begin{aligned} \min \quad & Cx \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{2}$$

with C a $q \times n$ criterion matrix consisting of the rows c_k, $k = 1, 2, ..., q$, A an $m \times n$ constraint matrix and the right hand side vector $b \in \mathbb{R}^m$. The feasible set in the decision space is $X = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ and in the objective space it is $Y = \{Cx : x \in X\}$. The set Y is also referred to as the image of X, [98]. We also noted in [99] in practice, this problem is typically solved by the Decision Maker (DM) in conjunction with the analyst who look for a most preferred solution in the feasible region X.

In MOLP, optimizing all the objectives at the same time is not possible because of the conflicting nature of the objectives [102], and the concept of optimality in MOLP is replaced with that of efficiency [38]. Therefore, the purpose of MOLP is to obtain either all efficient solutions, or a subset of efficient solutions, or a most preferred efficient solution depending on the purpose for which it is needed, [114].

A nondominated point in the objective space is the image of an efficient solution in the decision space; and the set of all nondominated points forms the nondominated set, [98].

An efficient solution of an MOLP problem is a solution that cannot improve any of the objective functions without deteriorating at least one of the other objectives. A weakly efficient solution is one that cannot improve all the objective functions simultaneously, [98].

Mathematically, let $\hat{x} \in X$ be a feasible solution of (2) and let $\hat{y} = C\hat{x}$:

1) $\hat{x}$ is called efficient if there is no $x \in X$ such that $Cx \leq C\hat{x}$ and $Cx \neq C\hat{x}$; correspondingly, $\hat{y} = C\hat{x}$ is called nondominated.
2) $\hat{x}$ is called weakly efficient if there is no $x \in X$ such that $Cx < C\hat{x}$; and $\hat{y} = C\hat{x}$ is called weakly nondominated, [54].

The set of all efficient solutions and the set of all weakly efficient solutions of (2) are denoted by X_E and X_{WE} respectively. They are called the efficient decision set and the weakly efficient decision set of (2) respectively [28]. The sets $Y_N = \{Cx : x \in X_E\}$ and $Y_{WN} = \{Cx : x \in X_{WE}\}$ are the nondominated and weakly nondominated sets in the objective space of (2), respectively.

The nondominated faces in the objective space of a given problem constitutes the nondominated frontier and the efficient faces in the decision space of the problem constitutes the efficient frontier, [114].

In the last few decades a number of algorithms have been suggested for MOLP. Such as the Multicriteria Simplex Method and it variants (MSM), the Interior Point Methods (IPM) and the Objective Space Methods (OSM) have been developed to solve the problem. MSM and IPM work in the decision space which are in turn used to obtain the nondominated points, while the objective space methods as the name implies works in the objective space of the problem and returns the set of all nondominated points and extreme directions.

The aim of this paper is to present a state-of-the-art survey of major algorithms suggested to solve MOLP problems. The rest of the paper is organized as follows. Section 2 is the motivation. Section 3 is a review of MOLP literature which covers the non-interactive and 2 interactive methods, the non-interactive algorithms include: the simplex based methods, the interior-point based methods, the objective space based methods and relevant heuristic approaches while the interactive approaches also include the simplex and interior point based approaches with illustrations of representative algorithms in each category. In section 4, we present a tabulated summary of all algorithms. Section 5 discusses future research work. Conclusion is presented in section 6. Finally, appendix is presented in section 7.

2. Motivation

As stated earlier, that this paper is aimed at presenting a state-of-the-art review of major algorithms proposed to solve Multiple Objective Linear Programming (MOLP) problems; and having extensively and comprehensively reviewed MOLP papers on non-interactive and interactive algorithms that have appeared since 1964; it was conspicuously observed that there is a complete absence of survey papers or articles that are dedicated to the field of MOLP. This is the gap, we intend to fill here. We have carried out a thorough look at the existing literature and found out that, there are a lot of survey papers on Non-linear and discrete multi-objective optimisation problems and not a single one suggested for the linear multi-objective optimisation case. See [109, 110, 125] and the references therein. This paper has been necessitated by the lack of survey articles that are devoted or propose for the linear multi-objective optimisation problem (aka MOLP). The introduction of this survey paper on MOLP would serve as a reference material as well as assist the potential researchers and practitioners in the field to choose which of the MOLP solution approaches is best suitable or appropriate to apply in their situation when solving MOLP problems or engaging in future innovative research.

3. MOLP: A Review

Before going any further, let us reiterate that the algorithms considered are in two broad categories, namely:

1) Non-Interactive Algorithms
   a) Simplex based methods and it variants, (MSM)

b) Objective space based methods, (OSM)
c) Interior point based approaches, (IPM)
d) Heuristics and Meta-heuristics approaches (HMA)

2) Interactive Algorithms
a) Simplex based methods and it variants, (MSM)
b) Interior point based approaches, (IPM).

Note that in the objective space methods, the simplex algorithm or the dual simplex algorithm are being invoked during the search process. This suggest that they should be put in the simplex based class. However, for more clarity and given that there is a strong trend to refer to them as objective space methods, we prefer to put them under the simplex since most of them use the simplex algorithm, though their underlying philosophy is different from that of the simplex based methods.

3.1. Simplex-based Algorithms

The first to publish research activity in MOLP appears to be Schonfeld [118]. The author proposed an algorithm for the determination of efficient solutions of the problem using parametric programming. He also slightly supplemented Gale *et al.* [62] duality concept in MOLP and later, extended this concept to multi-objective nonlinear programming problems in [119]. Linear programming models with two objectives (Bicriterion) were studied in Geoffrion [65]. A parametric linear programming algorithm was used to solve them. It was noted that solutions are not extreme points of the feasible region, pointing to the fact that one should not rely only on algorithms that consider extreme points of the feasible region as in ordinary LP, [98].

We stated in [99] that Eiselt and Sandblom [57] noted that, Evans and Steuer [58], Philip [104], and Zeleny [146] all derived generalized version of the simplex method known as the MSM for generating the entire efficient solutions of the problem. That of Philip [104] first determines if an extreme point is efficient and subsequently checked if the efficient extreme point found is the only one that exists. If not, the algorithm finds them all. This MSM approach however, may fail at a degenerate vertex. Philip [105] modified it to overcome this difficulty.

Evans and Steuer [58] presented a MSM for generating all the efficient extreme points and unbounded efficient edges for MOLPs. The algorithm first establishes that the problem is feasible and has efficient solutions. Thereafter, generates these solutions by moving from one efficient extreme point to adjacent efficient extreme points until all the efficient extreme points have been found. The algorithm utilizes a test problem to determine pivots that lead to efficient extreme points. Though this algorithm is prone to generating a huge number of efficient solutions and the computational efforts required to generate them grows exponentially with the problem size. However, it was noted in [117], that the algorithm has been the most dominant software for computing all efficient extreme points of the problem. It was implemented as a software called ADBASE in [131].

In [146], was presented two approaches to the problem. The parametric linear programming approach and a generalization of the single-objective simplex method to MSM. In the parametric approach, the author extended the method in [65] to solve more than two objectives. He decomposes the parametric space into finite subspaces with optimal weights corresponding to efficient extreme points. It was however noted that, the approach might be inefficient for degenerate problems as two or more bases may be associated with the same extreme point, [98]. This method also make use of a test problem to determine the efficiency of extreme points like the approach in [58]. The LP test problem utilized here determines efficient vertices as soon as they are generated, while in [58], the test problem determines the pivots that leads to efficient extreme points.

Yu and Zeleny [142] applied Zeleny's approach in [146] to generate the set of all efficient extreme for the problem and presented a formal procedure for testing the efficiency of extreme points. They also reported that a relationship exists between MSM and parametric programming, that the decomposition of the parametric space into subspaces associated with efficient extreme points can be obtain as a byproduct of applying the MSM to identify these efficient extreme points. In [143], Yu and Zeleny formally presented a generalization of the ordinary simplex method to MSM which was briefly discussed in [146]. They adopted a top-bottom search strategy where the efficient solutions of the problem are generated based on the information provided by the efficient faces. Numerical illustrations with three objectives were used to demonstrate the effectiveness of the method. it was noted however that the time required to compute the entire efficient set depends on the problem's dimension as was also reported in [58]. In a similar paper, Yu and Zeleny [144] applied their approach in [143] to parametric linear programming. We noted in [98] that two basic forms of parametric linear programs and their computational procedures for computing the efficient set were discussed. The direct decomposition of the parametric space and the indirect algebraic method. From a numerical experience the indirect algebraic method outperforms the direct decomposition approach, making it suitable for solving parametric LPs.

Belenson and Kapur [23] developed a technique which applies LP approach to solve two person zero-sum games with mixed strategies. The problem was formulated as a weighted sum MOLP problem and solved with the simplex method. Relatively small numerical illustrations with two objectives were used to demonstrate the applicability of the method. However, the number of individual LPs to be solved during the process may be too large which makes the approach difficult to solve larger instances.

Isermann [72] presented a variant of MSM that solves fewer LPs when determining the entering variables. The algorithm first establishes whether an efficient solution for the problem exists, and subsequently determines the set of all efficient solutions. This algorithm also uses a test problem that determines pivots leading to efficient vertices. The algorithm was implemented as a software called EFFACET in [73] and was reported to be efficient, but only available for a few years (in the 80s) unlike the ADBASE which is still available.

Ecker and Kouada [53] proposed a procedure for obtaining an initial efficient solution or it does not exist for MOLP problems. It was noted in [72] that the efficient solution obtained with this procedure is not necessarily an efficient basic solution of the problem.

[61] presented a parametric linear programming algorithm for generating the set of all efficient vertices and higher-dimensional faces (hyperplanes) of MOLP problems. The problem of determining efficient faces and higher dimensional faces were not resolve in [58] and [104]. In the procedure, efficient extreme points are generated through the use of a subprogram (auxiliary problem) which itself is an ordinary LP. The algorithm also determines higher-dimensional efficient faces for degenerate problems which were only discussed in [72] and [146] but not solved. The efficient faces are generated based on the information provided by the efficient extreme points (i.e. following a bottom-up) search strategy and the test problem employed, allows simultaneous determination of efficient faces and vertices making it different from other approaches.

In [120], Seiford and Yu extended MSM to solve a multiple criteria and multiple constraint levels (right hand side) problem (i.e. multiple discrete right hand sides). The authors regarded the multiple criteria and multiple constraint simplex method as a symmetric extension of MSM and used it to effectively generate all the potential solutions of the problem. This algorithm is only suitable for problems with multiple discrete right hand sides.

A detailed discussion on the field of MCDM in general and its models was presented in [147]. The author also gave a complete account of MSM as well as different real MOLP problem formulations. These problems were also solved effectively using his MSM approach in [146].

In [130], Steuer introduced a Vector Maximum Approach (VMA) based on the algorithm in [58] to generate the set of all efficient extreme points and unbounded efficient edges for parametric and non-parametric MOLPs. Different methods for obtaining an initial efficient extreme point as well as different LP test problems were also presented with numerical illustrations.

Ehrgott [54] also used the MSM in [58] to solve parametric MOLP problems with two and three objective functions. The set of all efficient solutions was determined effectively.

Philip [105], noted that his approach in [104] may fail at a degenerate vertex. He modified this approach in order to overcome this difficulty by solving a subprogram which determines the efficiency of such a vertex.

In [52], was proposed a method for finding all efficient extreme points for the problem based on the simplex method. Earlier before now, algorithms usually start from an initial efficient extreme point to adjacent efficient extreme point and this require solving an LP subproblem, see [58], [104], [146], [53], [72] and [61]. The proposed method does not require the solution of any LP subproblem to test for the efficiency of extreme points and the feasible region does not need to be bounded. The algorithm enumerates all efficient extreme points and appears to have advantage in speed over other well-known methods,[98].

In a different paper, Ecker *et al.* [51] presented a method for finding all the maximal efficient faces of the problem similar to the approach discussed in [72]. The method first determines the set of maximal efficient faces incident to a given efficient vertex (i.e. containing the efficient vertex), and ensures that previously generated efficient faces are not regenerated which dramatically improves computation time. Efficient faces are also generated following a bottom-up search strategy as in [61]. The proposed approach was illustrated with a degenerate example posed in [143] to demonstrate its effectiveness, and was computationally more efficient than the method in [143].

In [26], Benson validated Isermann's method in [72] for finding an initial efficient extreme point for the problem and also proposed a method to do the same.

Strijbosch *et al.* [133] proposed a simplified MOLP algorithm (MOLP-S) based on the simplex method to trace the efficient solutions of the problem. The proposed algorithm was computationally more efficient than the ADBASE software of Steuer [130] upon comparison.

Armand and Malivert [19] proposed two algorithms for determining the set of all efficient extreme points for degenerate MOLPs. The first algorithm enumerates the set of efficient vertices while the second generates the set of all efficient faces following a bottom-up search strategy. The method was tested on a number of degenerate problems for which efficient extreme points were generated. It was noted that, there was no computational experience in the literature for a comparison to be made. However, a numerical example with five objectives and eight constraint that was solved in [143] was used to show the efficacy of their method. In a similar paper, Armand [18], proposed another algorithm for finding same. Here, he used the lexicographic pivoting rule to determine all maximal efficient edges incident to degenerate and non-degenerate vertices. This algorithm was computationally superior to that in [19] when compared.

A method called moving optimal method for finding the efficient solutions of MOLP problems based on the simplex method was proposed in [102]. The algorithm proceeds by solving q single objective LPs of a given problem to obtain the optimal solution of each LP. Using the optimal solution of the first LP as an initial solution of the next and repeating the process, efficient solution line segments are generated. An instance posed in [47] was used to demonstrate its effectiveness. The proposed method was found to be better than that in [47].

In [134], Suprajitno presented an interval arithmetic simplex-based algorithm for MOLPs. According to the author, the information needed to solve real world problems is uncertain, hence the need for using intervals to define the data. The problem is formulated as an interval MOLP problem to start with and solved using a modified simplex algorithm. The procedure was found to be effective when tested on interval MOLPs posed in [101] for which efficient solutions were obtained.

Sayin [116] presented a method for finding the efficient set of the problem based on the results in [143]. The approach incorporates a simple LP test that identifies efficient faces and also employs a top-bottom search strategy to generate maximal efficient faces. Computational experiments show that the computational effort increases with the problem dimension and the number of variables appeared to have a significant effect on computation time.

In [140], Yan *et al.* investigated the structure of efficient and weakly efficient solutions as well as proposed a method for finding them. The algorithm determines finite number of weights corresponding to weighted-sum problems and solves many LPs in the process of obtaining weakly efficient and efficient solutions.

Pourkarimi *et al.* [106] improves on the algorithm of Yan *et al.* [140] by presenting a method with less computations in the determination of maximal efficient faces using a bottom-up search strategy. The proposed algorithm avoids degenerate problems and gives a representation of the maximal efficient faces. A numerical example was used to demonstrate its computational efficiency over the method in [140].

Kim and Thien [81] presented an algorithm for generating the set of all efficient extreme points and rays for MOLP. The authors used a bottom-top search strategy as well as a linear system to determine the efficiency of extreme points. An example introduced by Yu and Zeleny [143] and also solve in [19] was used to test the approach. The results agrees with those in [19]. The algorithm was also tested on 20 randomly generated problems for which the computational requirements increases with the problem size. It was noted however, that the number of objectives have a significant effect on the computational time and the efficient extreme points which contradict what was reported in [116]. In a similar paper, Kim *et al.* [82] extended the approach in [81] to MOLPs associated with linear multiplicative (nonconvex global optimization) programming problems for generating same.

Cardoso and Climaco [36] introduced a method where the two dimensional simplex tableau is replaced with a three dimensional tableau. Contrary to MSM which allows displacement between zero dimensional faces (extreme points), this approach enables displacement between higher dimensional faces and leads to the direct determination of efficient faces of higher dimensions without necessarily determining its vertices.

We also noted in [98] a modification of the parametric simplex algorithm for single objective LP to solve (bounded) bicriterion linear programming problems was presented by Ruszczynski and Vanderbei in [112]. The approach was applied to a large mean-risk portfolio optimization problem for which the nondominated portfolios were generated.

Ehrgott *et al.* [56], introduced a primal-dual simplex algorithm for bounded MOLPs. This algorithm finds a subset of the efficient solutions used to approximate the whole efficient frontier. The algorithm starts with a coarse partitioning of the weight space which continues in each iteration as well as solve a costly LP in each iteration. A vertex enumeration is then perform in the last step to obtain efficient solutions. Numerical illustrations were used to show the applicability of the algorithm, [98].

We stated in [98] that Rudloff *et al.* [111] presented a parametric simplex algorithm for MOLP. The algorithm is a generalization of the algorithm in [112] and is similar to that in [56]. It works for any dimension, solve bounded and unbounded problems (unlike the algorithm in [56] and [112]), and does not find all the efficient solutions just like that in [56]. Instead, it finds a subset of them that allows to generate the whole efficient frontier. This algorithm was compared with an objective space approach based on Benson's [29] algorithm and Evans-Steuer's MSM, [58]. Numerical experiments show that the proposed algorithm outperforms Benson's algorithm for non-degenerate problems. However, Benson's algorithm is better for highly degenerate problems. The parametric simplex algorithm was also found to be computationally more efficient than Evans-Steuer's MSM, [58].

Luc [90] presented his book on MOLP which introduces readers to the fundamental concepts of MOLP and constructed different dual problems for the problem. He also discussed the MSM, OSM and normal cone approaches to MOLP. Though, we are only concerned in this section with the MSM. The author noted that, apart from the work of Zeleny [146] and Steuer [130] nearly all other books on the subject are devoted to nonconvex problems, which motivated his work. The MSM was thoroughly discussed with illustrative examples presented.

Recently, we presented an extension of the MSM of Evans and Steuer [58] to generate the set of all nondominated points of the problem devoid of redundant ones in [99]. This extended version was then compared to Benson's outer approximation algorithm [29] which also computes the set of all nondominated points of the problem. Numerical results show that the total number of nondominated points returned by the extended version is the same with that returned by Benson's algorithm for most of the problems considered. In [100] we presented the results of a computational investigation of the extended version multi-objective simplex algorithm (EMSA) [99], Arbel's affine scalling interior MOLP algorithm (ASIMOLP) [4] and Benson's outer approximation algorithm (BOA) [29]. Numerical results on a collection of 60 MOLP instances, ranging from small to medium and realistic problems suggests that EMSA and BOA are superior to ASIMOLP in terms of the quality of the most preferred solution they returned; however, ASIMOLP outperforms them in terms of computing efficiency.

Jayanti and Bhatacharya [97] reformulated a portfolio optimization problem into an MOLP problem by maximizing the short and long term returns as well as annual dividend and minizing the associated risk in the form of semi-absolute deviation. The solution of the resulting MOLP was obtained using the mini-max goal programming approach.

3.2. The Multicriteria Simplex Algorithm

A typical simplex-based algorithm is the Multicriteria Simplex Algorithm (MSA) of Evans and Steuer [58]. The version described here can be found in Ehrgott [54]. This algorithm works in the decision space and finds all efficient extreme points of the MOLP problem. Its implementation stores a list of efficient bases L_1 to be processed, a list L_2 of efficient bases for output, and a list of efficient nonbasic variables N_E. The algorithm is initialized by solving two LPs to determine whether the MOLP is feasible as well as to verify if it has efficient solutions. If the feasible region X is not empty and the set of efficient solutions X_E exists, the algorithm pivots among efficient bases to determine all efficient extreme points of the problem.

Algorithm 1 Multicriteria Simplex Algorithm [54]

Input: A, b, C **: Problem data**
Initialize: $Set\ L_1 \leftarrow \emptyset,\ L_2 \leftarrow \emptyset;$
  $Phase\ I:\ Solve\ the\ LP\ min\{e^T z : Ax + Iz = b, x, z \geq 0\}. If\ the\ optimal\ value$
    $of\ this\ LP\ is\ nonzero,\ STOP,\ X = \emptyset;$
    $Otherwise\ let\ x^0\ be\ a\ basic\ feasible\ solution\ of\ MOLP$
  $Phase\ II:\ Solve\ the\ LP\ min\{u^T b + w^T Cx^0 : u^T A + w^T C \geq 0, w \geq e\}. If\ this$
    $LP\ is\ infeasible,\ STOP,\ X_E = \emptyset;$
    $Otherwise\ let\ (\hat{u}, \hat{w})\ be\ an\ optimal\ solution;$
    $Find\ an\ optimal\ basis\ B\ of\ the\ LP\ min\{\hat{w}^T Cx : Ax = b, x \geq 0\};$
    $Set\ L_1 \leftarrow \{B\},\ L_2 \leftarrow \emptyset.;$
while $L_1 \neq \emptyset$ **do : The list of efficient bases to be processed**
  $Choose\ B \in L_1,\ L_1 \leftarrow L_1 \backslash \{B\}, L_2 \leftarrow L_2 \bigcup \{B\}$**;**
  $Compute\ \tilde{A},\ \tilde{b},\ and\ R\ according\ to\ B$**;**
  $N_E \leftarrow N$**;**
  for $all\ j \in N$
    $Solve\ the\ LP\ max\{e^T v : Ry - r^j \sigma + Iv = 0; y, \sigma, v \geq 0\}$**.**
    $If\ this\ LP\ is\ unbounded\ N_E \leftarrow N_E \backslash \{j\}$**;**
    for $all\ j \in N_E$
      for $all\ i \in B$
        if $B' \leftarrow (B \backslash \{i\}) \bigcup \{j\}\ is\ feasible\ , B' \notin L_1 \bigcup L_2$ **then;**
          $L_1 \leftarrow L_1 \bigcup B'$**;**
        endif;
      endfor;
    endfor;
  endfor;
endwhile.
Output: L_2 **: The list of efficient bases.**

3.3. Illustration of the Multicriteria Simplex Algorithm

Consider the following MOLP adapted from Junior and Lins [74], which we solve using a Matlab implementation of MSA provided in [111].

$$
\begin{aligned}
&\min f_1 = -x_1 \\
&\min f_2 = \qquad -x_2 \\
&\text{Subject to} \\
&\qquad 6x_1 + 10x_2 \leq 60 \\
&\qquad x_1 \qquad\quad \leq 7 \\
&\qquad\qquad\quad x_2 \leq 5 \\
&\qquad x_1, \quad x_2 \geq 0
\end{aligned}
\tag{3}
$$

The efficient extreme points were found to be $x^1 = (7.0,\ 1.8)^T$, $x^2 = (1.6,\ 5.0)^T$, $x^3 = (1.6,\ 5.0)^T$, $x^4 = (7.0,\ 1.8)^T$ and the corresponding objective function values are $f^1 = (-7.0, -1.8)^T$ and $f^2 = (-1.6, -5.0)^T$ respectively, where $x^1 = (x_1^1, x_2^1)^T, ..., x^4 = (x_1^4, x_2^4)^T \in X_E$ and $f^1 = (f_1^1, f_2^1)^T$, $f^2 = (f_1^2, f_2^2)^T \in Y_N$. The algorithm is prone to generating more solutions than necessary due to the way it operates and due to the fact that MSA may find the same efficient solutions in more than one iteration, as in this case, $x^1 = x^4$ and $x^2 = x^3$ are repetitive of what has already been found; $x^1 = x^4$ and $x^2 = x^3$. Solutions x^3 and x^4 are redundant and would be little or no use to the DM, [99]. The feasible region in the decision space is shown in Figure 1.

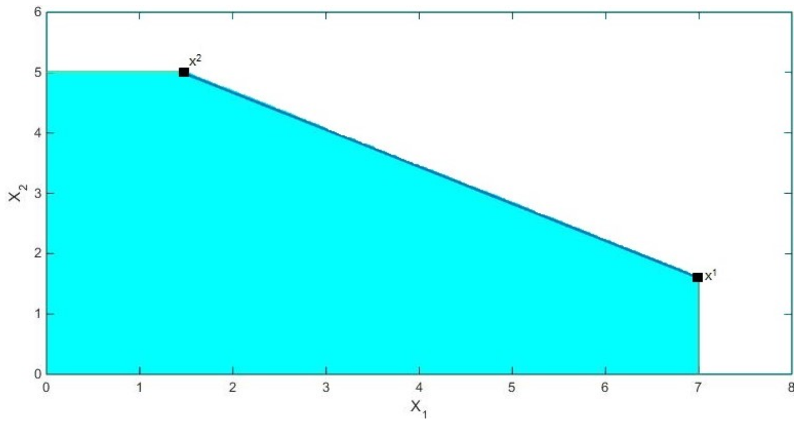

Figure 1. *Edge connecting the two efficient points in the decision variable space.*

3.4. The Parametric Simplex Algorithm

Another simplex-based algorithm for MOLP is that of Rudloff, Ulus and Vanderbei [111]. This algorithm also works in the decision space and does not intend to find all efficient extreme points of an MOLP problem unlike Algorithm 1. Instead it finds a solution in the sense of Löhne [87], that is, it finds a subset of efficient extreme points and directions that allows to generate the whole efficient frontier. In addition, this algorithm performs pivoting for only one leaving variable among the set of all possible leaving variables thereby making it computationally more efficient than Algorithm 1, [111].

Algorithm 2 Parametric Simplex Algorithm [111]

Input: A, b, P
Initialize: $Find\, D^0\, and\, the\, index\, set\, of\, entering\, variables\, J^{D^0}$;
$BD \leftarrow \{D^0\},\ \bar{X} \leftarrow \{x^0\}, R \leftarrow \emptyset, VS \leftarrow \emptyset,\ \bar{X}^h \leftarrow \emptyset,\ E^{D^0} \leftarrow \emptyset.$
while $BD \neq \emptyset$ **do**
  $Let\, D \in BD\, with\, nonbasic\, variables\, N\, and\, index\, set\, of\, entering\, variables\, J^D$;
  for $j \in J^D$ **do**
    $Let\, x_j\, be\, the\, entering\, variable$;
    if $B^{-1}Ne^j \leq 0$ **then**
      $Let\, x^h\, be\, such\, that\, x^h_B = -B^{-1}Ne^j\, and\, x^h_N = e^j$;
      $\bar{X}^h \leftarrow \bar{X}^h \bigcup \{x^h\}$
      $P^T[\bar{X}^h] \leftarrow P^T[\bar{X}^h] \bigcup \{-Z_N^T e^j\}$
    else
      $Pick\, i \in argmin_{i \in B, (B^{-1}N)_{ij} > 0} \frac{(B^{-1}b)_i}{(B^{-1}N)_{ij}}$;
      if $(j, i) \notin E^D$ **then**
        $Perform\, the\, pivot\, with\, entering\, variable\, x_j\, and\, leaving\, variable x_i$;
        $Call\, the\, new\, dictionary\, \bar{D}\, with\, nonbasic\, variables\, \bar{N} = N \bigcup \{i\} \backslash \{j\}$;
        if $\bar{D} \notin VS$ **then**
          if $\bar{D} \in BD$ **then**
          $E^{\bar{D}} \leftarrow E^{\bar{D}} \bigcup \{(i, j)\}$;
          else
            $Let\, \bar{x}\, be\, the\, basic\, solution\, for\, \bar{D}$;
            $\bar{X} \leftarrow \bar{X} \bigcup \{\bar{x}\}$;
            $P^T[\bar{X}] \leftarrow P^T[\bar{X}] \bigcup \{P^T \bar{x}\}$;
            $Compute\, the\, index\, set\, of\, entering\, variables\, J^{\bar{D}}\, of\, \bar{D}$;
            $Let\, E^{\bar{D}} = \{(i, j)\}$;
            $BD \leftarrow BD \bigcup \{\bar{D}\}$;
          endif
        endif
      endif
    endif
  endfor
  $VS \leftarrow VS \bigcup \{D\}, \quad BD \leftarrow BD \backslash \{D\}$
endwhile
Output: $(\bar{X}, \bar{X^h}) : A\, finite\, solution\, of\, P$;
$(P^T[\bar{X}], P^T[\bar{X}^h] \bigcup \{y^1, ..., y^t\}) : V\, representation\, of\, \boldsymbol{P}$.

3.5. Illustration of the Parametric Simplex Algorithm

Using Problem 3 of section 2.1.3, the efficient extreme points found are $x^1 = (7.0, 1.8)^T$, $x^2 = (1.6, 5.0)^T$ and the nondominated points are $f^1 = (-7.0, -1.8)^T$ and $f^2 = (-1.6, -5.0)^T$ respectively. The solution was obtained using a Matlab implementation of Algorithm 2 in [111]. Here, two efficient extreme points were found unlike Algorithm 1 that returned efficient extreme points found twice. Because the algorithm finds a subset of efficient extreme points that allows to generate the whole efficient frontier. The feasible region is the same as in Figure 1.

3.6. Interior Point Based Algorithms

Simplex-based approaches make explicit use of the vertices of the feasible region. Interior point approaches, however, generate iterates in the interior of the feasible region. Various interior point approaches have been suggested to solve the problem. The difference between these approaches depends on the methodology employed to assess the relative preference of points used to derive a combined search direction along which one heads towards the next iterate.

The first to modify a variant of Karmarkar's [77] interior point algorithm to solve an MOLP problem appears to be Abhyankar *et al.* [1]. The authors presented an interior point method using the method of centers. The proposed algorithm used parameterization of ellipsoid in n-dimensional space to approximate the efficient frontier of the problem in polynomial time.

Arbel [4] also modified and adapted a variant of Karmarkar's [77] algorithm resulting in he so called Affine Scaling Interior MOLP (ASIMOLP) algorithm. He used the convex combination of individual directions to derive a combined direction along which to step toward the next iterate. Specifically, the algorithm generates step direction vectors based on the objectives of the problem. The relative preference of these directions is then assessed using a utility (or preference) function to obtain the points used in combining them into a single direction vector that moves the current iterate to a new one. The process is repeated until the algorithm converges to an efficient solution after meeting some termination conditions. Arbel [5] proposed another ASIMOLP algorithm. This approach offers another means of assessing preference information to establish a combined search direction rather than using the DM's utility function. The Analytic Hierarchy Process (AHP) developed in [113] was applied to obtain the relative preference of the individual directions used to derive a combined direction along which the next step is taken. It is based on weighing the step direction vectors for each of the objectives in order to find a combined direction. This process continues to generate search directions and new feasible points at each iteration, until the algorithm converges to a point on the efficient frontier. In [6], Arbel proposed yet another ASIMOLP algorithm based on the AHP. The derived preference information is applied to the projected gradients in order to obtain anchoring points and cones used in searching for a most preferred solution. The boundaries of the constraints polytope are constantly probed to mae more directions available, which enables one to arrive at a most preferred solution.

Arbel and Oren [15] modified Affine Scaling Primal Algorithm (ASPA) and presented a procedure for generating the combined search directions in interior point MOLP problems. The approach assumes that the DM has an implicitly known utility function. He then generates search directions that approximate the gradient of this function, which are later combine to arrive at the next iterate. This continues in a sequence of steps until the algorithm converges to an efficient solution.

Similarly, Arbel's [8] presented a modification of ASPA to generate search directions in form of projected gradients which improves each objective separately. These gradients are then used to derive a combined direction that leads to a new iterate. By taking a full step along the combine direction enables one to reach a boundary (anchor) point. The algorithm continue to generate anchor points in each iteration. The anchor points together with the current iterate defines a cone of opportunities that provides the boundary points at which to terminate the process. The formal principles behind interior point LP approaches was introduced in [7], the author provided the basic details of ASPA which are used in the implementation of the ASIMOLP algorithm.

Wen and Weng [137] modified the approach in [4] to resolve zigzagging when solving MOLPs. Though the modified algorithm may only yield an inefficient solution instead of an efficient one.

Arbel and Korhonen [12] introduced a new starting mechanism which makes it possible to start an algorithm from a feasible or infeasible solution. This starting mechanism can be applied to the simplex method as well. The problem is first augmented so as to obtain an initial feasible solution. This augmentation is controlled by a nonnegative parameter which verifies the efficiency of the final solution. The parameter is then forced to zero at the end of the iterative process thereby achieving an efficient solution.

Zhong and Shi [149] applied ASIMOLP algorithm to find the solution of multiple criteria and multiple constraint levels problem. The approach involves partitioning the columns of the right hand side matrix (with each column representing the right hand side vector) into different MOLP problems. The MOLP problems are then solved by ASIMOLP and their efficient solutions combined through a convex combination to obtain a most preferred efficient solution of the problem.

Lin *et al.* [86] also presented a modification of the algorithm in [4]. They adopted the utility function trade-off method to weigh the objectives involved and compared the modified algorithm with that of Wen and Weng in [137] and the simplex method. Numerical experiments show that their algorithm is superior to that in [137] and on computing efficiency, the interior point based algorithms outperform the simplex-based ones for large problems.

In [59], Fliege presented a method for approximating the solution of the problem based on a warm-start strategy using the path-following primal interior point algorithm. With the warm-start strategy, points from the iteration history of scalar problems already solved are used as starting points until an approximate solution to the problem is obtained.

Arbel and Sadka [17] presented a derivation of the Euclidean center (and its weighted version) which is defined as the point in the decision space which is the centre of the largest sphere that can be inscribed in the constraints polytope. By assigning weights to the different decision variables, one traverses the entire efficient frontier, thereby solving the problem.

Weng and Wen [138] proposed an interior point MOLP algorithm to solve the problem. The algorithm is based on ASPA and computes a weighted sum of the different search directions involved. These search directions are then combined into a single direction which is projected to obtain the anchoring points. Computational experiments show that the proposed algorithm is suitable for solving large instances.

A detailed account of the algorithm in [4] was provided in Bufardi [34]. Also discussed were the modifications needed for adopting single objective interior point algorithm to MOLPs. Numerical illustrations with two and three objectives were used to demonstrate the effectiveness of the algorithm.

3.7. The Affine Scaling Interior MOLP Algorithm

A typical interior-point MOLP algorithm is the Affine Scaling Interior MOLP Algorithm whose general form is found in [5]. It works in the decision space and returns an efficient face of the feasible region with infinitely many efficient points. The point of convergence to the efficient frontier is taken as the most preferred efficient solution of the problem. Although this algorithm is very efficient as noted in [74], it may not be the best in this class. It was chosen because of its simplicity and most of the studies conducted especially by Arbel and his team, are based on this algorithm.

Algorithm 3 Affine Scaling Interior MOLP Algorithm

0: **Input:** A, b, C
1: **Initialize:** $Choose\ x_0 > 0$, $Stopping\ tolerance\ \sigma\,(0 < \sigma < 1)$, $Step\ size\ factor\ \rho\,(0 < \rho < 1)$,
$Converged = 0$, $k \leftarrow 0$;
2: **while** $Converged \neq 1$ **do**
3: $k \leftarrow k + 1$
4: $D \leftarrow diag(x_0)$
5: $y_i(k) \leftarrow (AD^2A^T)^{-1}AD^2c_i$
6: $dx_i(k) \leftarrow D^2(c_i^T - A^T y_i(k))$
7: **if** $dx_i(k) \geq 0, 1 \leq i \leq q$, **stop**
8: **else**
9: $\alpha_i = Min[\frac{-x_i}{dx_i(k)}, \forall dx_i(k) < 0]$
10: $x_i(k+1) \leftarrow x(k) + \rho\alpha_i dx_i(k)$
11: $dx = \sum_{i=1}^{q} p_i\alpha_i dx_i$
12: $x_{new} \leftarrow x(k) + \rho dx(k)$
13: $v_{new} \leftarrow C^T x_{new}$
14: $x_{end} \leftarrow x(k) + dx(k)$
15: $dx_{end} \leftarrow x_{end} - x_{new}$
16: **if** $k > 1$, **do**
17: $dx = \sum_{i=1}^{q} p_i\alpha_i dx_i + p_{end}dx_{end}$
18: $x_{new} \leftarrow x(k) + \rho dx(k)$
19: $v_{new} \leftarrow C^T x_{new}$
20: $x_{end} \leftarrow x(k) + dx(k)$
21: $dx_{end} \leftarrow x_{end} - x_{new}$
22: **else**
23: **if** $\|dx_{end}\| \leq \sigma$, **stop**
24: **else**
25: $x_0 \leftarrow x_{new}$
26: **Go to step 4**
27: **endif**
28: **endif**
29: **endif**
30: **endwhile**
31 **Output:** x_{new}
v_{new}

3.8. *Illustration of Affine Scaling Interior MOLP Algorithm*

Consider again Problem (3) of section 2.1.3, for which the efficient solutions found using our Matlab implementation of Algorithm 4 are $x^* = (x_1^*, x_2^*)^T = (3.6418, 3.7913)^T$ and the values of the objective functions are $v^* = (v_1^*, v_2^*)^T = (-3.6418, -3.7913)^T$ respectively. The search path as generated by Algorithm 4 is shown in Figure 3.

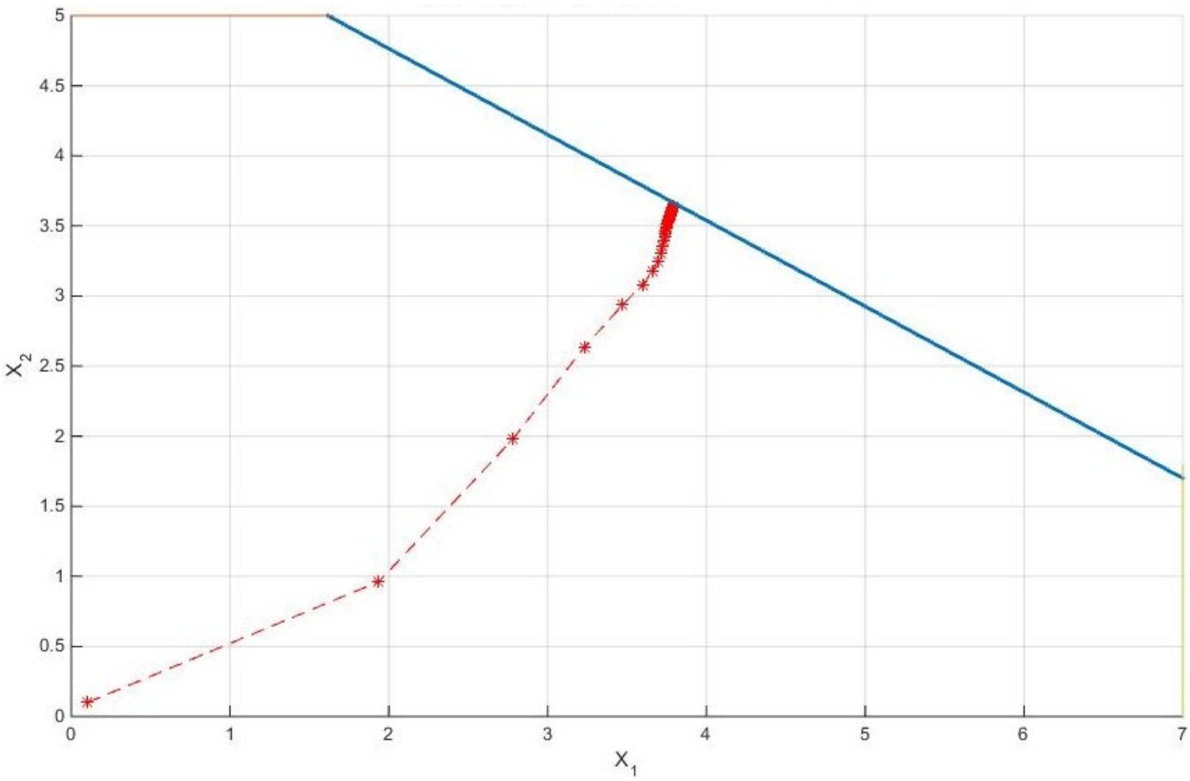

Figure 2. *The edge joining the two nondominated points in the objective space.*

4. Objective Space Based Algorithms

We have so far discussed the simplex based methods for MOLP. We now look at objective space-based approaches.

We noted in [100] that following the various difficulties arising from solving MOLP problems with the decision-set based vector optimization approaches, researchers in the field began to look into the possibility of solving the problem in the objective space. The first attempt at exploring the possibility of solving MOLP problems in the objective space was due to Dauer [43]. He presented an analysis of the objective space of the problem and obtained a characterization between a face of the objective space and the corresponding faces of the decision space. A necessary and sufficient condition for a face to be efficient was developed. Numerical illustrations show the collapsing that occurs when mapping the decision space onto the objective space, thereby forming the basis for a new procedure for MOLP.

In a similar paper, Dauer and Liu [45] presented another analysis of the objective space approach and developed a procedure for determining the nondominated extreme points and edges in this space. They observed that not all extreme points of the constraint space necessarily map to extreme points in the objective space, and the technique analyzes a simpler structure than those analyzed by algorithms based on the decision space. In [46], Dauer and Saleh determined the nondominated extreme points for MOLP by solving a resulting single-objective nonparametric LP. The optimal basic solution of this LP was used to obtain the corresponding efficient extreme point in the objective space.

An algorithm for the construction of an Efficiency Equivalent Polyhedron (EEP) having the same efficient structure with the feasible set X was presented in [63]. The proposed algorithm generates a nonredundant linear inequality representation of the EEP associated with the problem and use it to obtain the Maximal Efficient Faces (MEF) of the problem. Similarly, Dauer and Gallagher [44] presented an algorithm known as the MEF algorithm for determining higher dimensional MEF of the problem. The algorithm utilizes a nonredundant inequality representation of the EEP generated by the algorithm in [63] as input, and work in conjunction with it to obtain the MEF of the problem. The proposed algorithm is limited to only problems with two and three objectives.

In [29], Benson presented a detailed account of decision set-based approaches and proposed a finite algorithm for generating the set of all nondominated points in the objective space. According to the author, the algorithm was the first to generate all the nondominated points in the objective space of the problem. Computational results suggests that the objective space based approach is better than the decision set-based approach. A further analysis of an outcome set-based algorithm for MOLP was presented by Benson [27]. He show that the outer approximation algorithm also generates the set of all weakly nondominated points, thereby enhancing the usefulness of the algorithm as a decision aid in MOLP. Another of Benson's [28] suggestions is a hybrid approach for solving the problem in the objective space. The hybrid approach incorporates a simplicial partitioning technique (i.e. partitioning the outcome space into simplices that lie in each face) of Ban [22] into an outer approximation procedure, so as to generate the set of all nondominated extreme points in the objective space. This algorithm is quite similar to that in [29]. The difference between the two is in the manner in which the nondominated vertices are found. While a vertex enumeration procedure is employed in [29], a simplicial partitioning technique is used in the latter to serve the purpose.

Benson and Sun [32] proposed a weight set decomposition algorithm for generating the set of all nondominated points using the idea presented in Zeleny [146]. The authors validated Zeleny's weight set decomposition approach to resolve the issue of one-to-one correspondence with the efficient decision set X_E. This approach proves to be computationally less demanding than the decision set-based methods.

Kim [80] noted that not all the data in the feasible set X plays a role in determining the efficient decision set X_E of the problem. He therefore, suggested working with an EEP smaller in size than the feasible set X. Based on this observation, he proposed an outer approximation algorithm for constructing an EEP which is simpler to work with and used it to enumerate all the nondoninated extreme points of the problem. This algorithm is similar to that in [63] in the sense that, both work with nonredundant inequalities and differ in the definition of an EEP. Whereas in the former, it was described as being associated with the objective space Y, here it is equal to the efficient decision set X_E.

Tantawy [135] proposed a method based on the simplex algorithm for detecting nondominated points of the problem in the objective space and presented a condition for an efficient extreme point to have a corresponding nondominated point in the objective space. The author suggested that the DM should rely on the extreme points generated in the objective space because they are fewer in number compared to those generated in the decision space.

Foroughi and Jafari [60] presented a modification of the algorithm in [140] for constructing an efficient structure of the problem. The authors showed that if in one stage of this method, the weighted sum problem has no finite optimal solution, the algorithm would be ineffective. The modified method overcomes these difficulties without changing the complexity of the initial method.

Shao and Ehrgott [121] proposed a modification of the approach in [29] for solving the problem in the objective space. Benson suggested a bisection method that requires the solution of many LPs in a single step. It was shown that solving one LP could achieve the desired effect. This modification often improves computation time and provides an approximation to the whole nondominated solutions set.

Heyde and Lohne [70] presented a geometric approach to duality in MOLP similar to the theory of convex polytopes [67], which states that two polytopes A and A^* in $\mathbb{R}^q$ are dual to each other if there exists a one-to-one mapping Φ between the set of all faces of A and all faces of A^* such that Φ is inclusion-reversing, that is, faces F_1 and F_2 of A satisfy $F_1 \subseteq$

F_2 iff the faces $\Phi(F_1)$ and $\Phi(F_2)$ satisfy $\Phi(F_1) \supseteq \Phi(F_2)$, [67]. They authors show that there exists a similar one-to-one mapping Φ between the set of all minimal faces of the primal problem and maximal faces of the dual problem, such that Φ is inclusion-reversing. With this mapping, they computed the minimum number of faces in the primal and the corresponding maximum number of vertices in the dual problem and vice versa. Heyde *et al.* [71] presented a duality theory for MOLP using a set-valued dual objective function map. Numerical application using a bicriterion portfolio optimization problem reveals the practical relevance of the duality theory.

Ehrgott *et al.* [55] modified and presented a dual variant of the algorithm of Benson [29] using results from the geometric duality theory of Heyde and Lohne [70]. Numerical evidence suggests that the dual variant of the algorithm may have computational advantage. In a similar paper, Shao and Ehrgott [122], proposed an approximate dual variant of Benson's algorithm [29] for obtaining an approximate solution to the problem in the objective space. The proposed algorithm was applied to the beam intensity optimization problem of radio therapy treatment planning for which approximate nondominated points were obtained. This numerical illustration also show that the approach is faster than directly solving the primal problem.

A framework for MOLP based on oriented projective geometry which allows unbounded polyhedra to be treated as bounded is presented by Burton and Ozlen [35]. The framework when applied to Benson's outer approximation algorithm resulted in a new algorithm that works in an oriented projective space, and yields a reduction in the number of non-efficient vertices as well as running time.

In [69], Hamel *et al.* introduced new versions and extensions of Benson's algorithm [29] for solving MOLP in the objective space. The primal and dual algorithms of Benson's type were presented in which only one LP problem has to be solved in each iteration rather than two or three as was necessary in previous versions. Numerical applications revealed considerable reduction in computational time.

Csirmaz [42] also presented an improved version of BOA that solve one LP and a vertex enumeration problem in each iteration. This version was applied to four random variables with a joint distribution so as to map their entropy region. Numerical illustrations show the applicability of the approach in obtaining the nondominated set for medium and large problems.

Lohne [87] presented a detailed and comprehensive account of set-valued and vector-valued approaches to MOLP in the objective space. The exposition covers the general theory of linear vector optimization, scalarization techniques, duality and extended variants of Benson's algorithm. The author also introduced a solution concept that takes into account the polyhedral structure of the problem. He suggested that a solution of the problem is a finite subset of the feasible set which should consists not only a set of efficient extreme points but also directions.

Shao and Ehrgott [124] extended the primal and dual variants of Benson's algorithm [29] to solve linear multiplication (nonconvex global optimization) programming problems. The authors first improved their primal objective space algorithm in [123] which was based on Benson's outer approximation algorithm, by requiring only one LP problem to be solved in each iteration. Thereafter, sandwiched the dual variant of Benson's algorithm in [55] to obtain a modified version. Based on this modified version, a dual objective space algorithm that also solve one LP was proposed for the problem. Numerical results suggest that the proposed algorithm is superior to the global optimisation algorithm for linear multiplicative programming problem presented in [64].

Similarly, Lohne *et al.* [88] extended the primal and dual variants of Benson's algorithm [29] to solve convex vector optimization problems approximately. Numerical illustrations yielded approximate nondominated solutions to the problem, [98].

Recently Dörfler *et al.* presented an algorithm for approximately solving bounded vector optimization problems in the objective space. The proposed algorithm is an improvement and a modification of the algorithm in [69]. The authors introduced a new selection procedure for the vertex enumeration in order to improve the overall efficiency of the algorithm. Numerical illustrations suggests that the new method may be faster than that in [69].

4.1. Benson's Outer Approximation Algorithm

Benson's outer approximation algorithm for MOLP as contained in [87], is a typical objective space based algorithm. It is the most popular and commonly used objective space method for MOLP. Most objective space methods are usually based on this algorithm. It can be regarded as a primal-dual method because it also solves the dual problem simultaneously. Here, we are only concern with the solution of the primal problem. The algorithm finds the set of all nondominated extreme points as well as extreme directions of a given problem in the objective space.

Algorithm 4 Benson's Outer Approximation Algorithm [87]

Input: A, b, P
$a\ finitely\ generated\, solution\ (\{0\}, \bar{X}^h)\ to\ P^h$;
$a\ finitely\ generated\, solution\ \bar{T}^h\ to\ D^{*h}$;
Initialize: $\hat{p} \leftarrow P(solve(P_1(0))) + e$;
$\bar{T} \leftarrow \{(solve(D_1(w)), w) | (u, w) \in \bar{T}^h\}$;
while $z \neq 0$ **do**
$X^d \leftarrow \{D^*(u, w) | (u, w) \in \bar{T}\}$;
$X^p \leftarrow vert(X^d)$;
$\bar{X} \leftarrow \emptyset$;
for $i = 1\, to\, |X^p|$ **do**
$v \leftarrow X^p[i]$;
$(x, z) \leftarrow solve(P_2(v))$;
$\bar{X} \leftarrow \bar{X} \cup \{x\}$;
if $z > 0$ **then**
$(x, \delta) \leftarrow solve(R(v))$;
$y \leftarrow \delta v + (1 - \delta)\hat{p}$;
$(u, w) \leftarrow solve(D_2(y))$;
$\bar{T} \leftarrow \bar{T} \cup \{(u, w)\}$;
break;
endif;
endfor;
endwhile
Output: $(\bar{X}, \bar{X^h})\, a\ finitely\ generated\, solution\ \ to\ P$;
$\bar{T}\, a\ finitely\ generated\, solution\ \ to\ D^*$.

4.2. Illustration of Benson's Outer Approximation Algorithm

For continuity of exposition, consider again Problem (3). The nondominated points found using an existing MATLAB implementation of Algorithm 3 are $f^1 = (-7.0, -1.8)^T$ and $f^2 = (-1.6, -5.0)^T$ where f^1 and $f^2 \in Y_N$, [99]. These Points are shown in Figure 3.

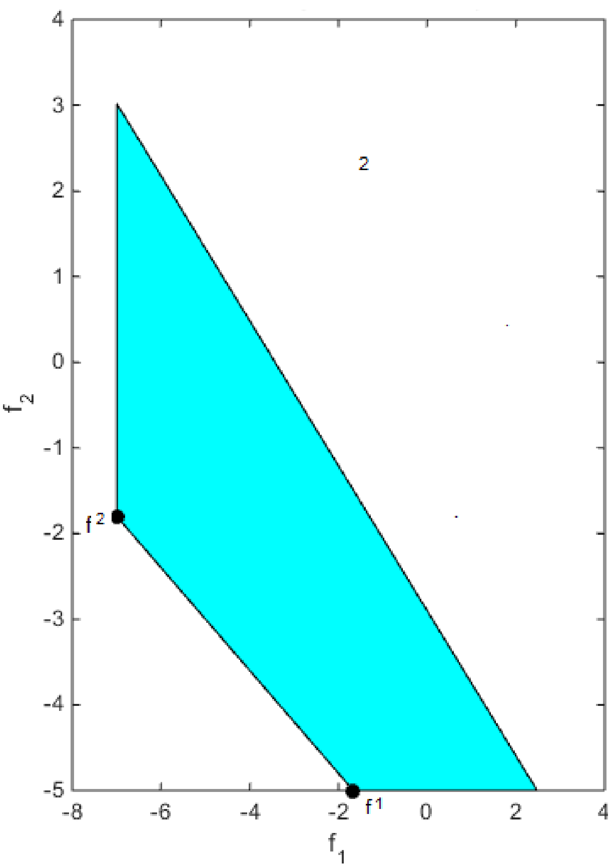

Figure 3. The search path generated by ASIMOLP showing convergence to the efficient frontier.

5. Heuristics and Meta-heuristic Approaches

We have so far discussed the exact solution methods to the problem. We now turn our attention to Heuristics and Meta-heuristic approaches that seek to find good approximations or near efficient solutions to MOLP.

In real-world, MOLP instances are hard to solve. In a worst-case scenario, all vertices of the polyhedron might be efficient, implying that the MOLP problem would be intractable as there might be exponentially many efficient vertices, [85]. It is, therefore, clear that MOLP is intractable in the worst-case, [98]. Moreover, it was noted in [78] and [33] that listing all vertices of a polyhedron is NP-hard, one can also conclude that MOLP is NP-hard since in the worst-case situation all vertices must be computed in order to enumerate the efficient ones. Thus, exact methods are in some cases inefficient and expensive especially when handling realistic instances. Instead of finding efficient and nondominated points,heuristics or meta-heuristics generally find good approximations or near efficient solutions in acceptable computational times. For this reason, they are widely used in multi-objective optimisation (MOO), [98]. It is surprising to note that only a few approximate approaches has been applied to MOLP.

We noted earlier that heuristics or approximate methods have been commonly applied to non-linear and discrete MOO and not so much to MOLP. We stated in [98] that the first to apply a nature-inspired population based evolutionary method to MOLP seems to be Chakraborty [37]. The author applied multi-objective parametric fuzzy programming and NSGA of Srinivas and Deb [127] to an MOLP transportation problem. The MOLP transportation problem was first transformed into a single objective parametric problem with interval parameters. Numerical illustration using a coal energy resource allocation problem show the applicability of the method. In [21] Bagchi applied NSGA [127] to several scheduling problems for which efficient solutions were found. We also noted in [98] that

NSGA-II [48, 49] is an improved version of the NSGA in [127] and it is arguably, the most popular in the context of non-linear multi-objective problems has had tremendous applications in linear MOPs. It was successfully applied in the energy generation expansion planning problem in [76] for which the minimum investment and outage costs were approximated. Similarly, Massobrio [92] applied NSGA-II to the taxi sharing problem in order to determine the minimum cost of journey and delay time from the same location to deferent destinations. The problem was formulated as a bicriterion MOP using a linear aggregation approach to combine the objectives and solved with NSGA-II and greedy heuristics. Numerical results show that NSGA-II outperform the greedy heuristic by achieving significant improvements in both objectives in acceptable computational time, [98].

Sanaz *et al.* [115] proposed a linear multi-objective particle swarm optimisation (LMOPSO) to solve linearly constrained multi-objective optimisation problem (i.e. MOLP). The LMOPSO is designed to explore the feasible region specified by the polyhedron using a linear formulation of particle swarm optimisation (PSO) thereby preserving and guaranteeing the feasibility, convergence and improving the solutions during the search process. The authors presented three feasibility preserving approaches that guarantees the feasibility of solutions by ensuring that any of the generated solution that violate the constraints is pushed back into the feasible region during the search process. The proposed method was applied to a real-world problem with linear constraints which appear in multi-component chemical systems with reference data. The results obtained was compared with the reference data and it show applicability of the proposed method.

Zhang *et al.* [148] introduced the multi-objective beetle antenna search algorithm (MOBAS) by extending the single objective beetle antenna algorithm to solve MOPs. The proposed method was numerically tested on four bench-marked problems in the existing literature and compared with MOPSO [95], NSGA-II [49] and Multi-objective differential evolution algorithm (MODEA) [20], the result obtained show that the proposed method outperforms these well-known methods computationally.

Reyes-Sierra and Coello [109] presented a comprehensive review of MOPSO for MOPs. The survey classifies the methods into five categories namely: Aggregating approaches that combine or transform the MOP into a single objective problem; the lexicographic methods that rank the objective functions according to their order of importance; the sub-population approaches that use sub-population as single objective optimizers thereby exchanging information in order to produce trade-offs among the solutions generated; the Pareto-based approaches that utilize leader selection techniques based on Pareto dominance and finally, the Combined approaches that utilize an adaptive weighted sum PSO where the velocity is modied and the included acceleration term that increases with the number of iterations. The authors recommended further theoretical research on MOPSO and their real-world application so as to attract potential researchers to the field.

Recently in [114] we applied the multi-objective plant propagation algorithm (MOPPA) and NSGA-II [49] for the first time to MOLP and compares their outcomes with those of prominent exact methods. Computational results from a collection of 51 existing MOLP instances suggests that MOPPA compares favourably with four of the most prominent exact methods namely extended multi-objective simplex algorithm (EMSA) [99], affine scaling interior MOLP algorithm (ASIMOLP) [4], Benson's outer-approximation algorithm (BOA) [29] and parametric simplex algorithm (PSA) [111], and returns best nondominated points which are of higher quality than those returned by NSGA-II. However, the nondominated points approximated by NSGA-II are evenly distributed across the nondominated front.

Motahari *et al.* [96]proposed an MOLP model to optimize weighted completion time, transportation cost, and machine idle time for a multi-product system. The model was solved using the e-constraint method for small MOLP instances, While NSGA-II[49] was used to solve large and realistic instances and compared with three other meta-heuristic algorithms.

5.1. Illustration of Nondominated Sorting Genetic Algorithm II

We noted in [114] that a typical population based evolutionary algorithm is NSGA-II [49]. We decided to used this algorithm for illustration because among all the above discussed nature-inspired population based evolutionary methods, NSGA-II is the most popular and known for its capacity to promote the quality of solutions as stated in [75]. NSGA-II [48, 49] is an improved version of the NSGA in [127]. Though NSGA [127] enjoyed patronage in the multi-objective evolutionary community, it was also widely criticized for lacking elitism, high computational cost and the requirement for specifying the sharing parameter. NSGA-II succeeded in solving all the three issues at once by introducing a fast nondominated sorting and tournament selection using the concept of crowding distance, [103]. The pseudo-code of NSGA-II adapted from [145] is given as Algorithm 5.

In order to solve problem (3) with NSGA-II, we use the penalty function method in [93] to handle the constraints. It is worthy to note that, the penalty function method is the most popular constraint handling technique in evolutionary algorithms and many other optimisation frameworks, [93, 141]. It penalizes each objective by reducing its fitness values in proportion to the degree of constraint violation, [126].

We implemented the resulting penalty program in Matlab and use a Matlab implementation of NGSA-II to solve Problem 3. With $x_1 \in [0, 7]$, $x_2 \in [0, 5]$ as variable bounds, a population size of 50, maximum number of runners 5 and the number of generations to perform at 200, mutation rate of 0.02 and crossover rate of 0.08 are chosen. The nondominated front approximated by NSGA-II is shown in Figure 4.

Algorithm 5 Nondominated Sorting Genetic Algorithm II [145]

1: Initialization:
2: ◇ Generate random population
3: ◇ Evaluate objective values
4: ◇ Assign rank (level) based on nondomination
5: ◇ Generate child population
- Tournament selection
- Crossover and mutation

For $i = 1$ to number of generations
6: ◇ Parent and child population are assigned rank based on nondomination
7: ◇ Generate sets of nondominated fronts
8: ◇ Determine the crowding distance between points on each front
9: ◇ Select points based on crowding distance calculation and fill into the parent population until full
10: ◇ Create next generation
11: ◇ Tournament Selection
12: ◇ Crossover and Mutation
13: ◇ Evaluate Objective Values
14: ◇ Increment generation index
End

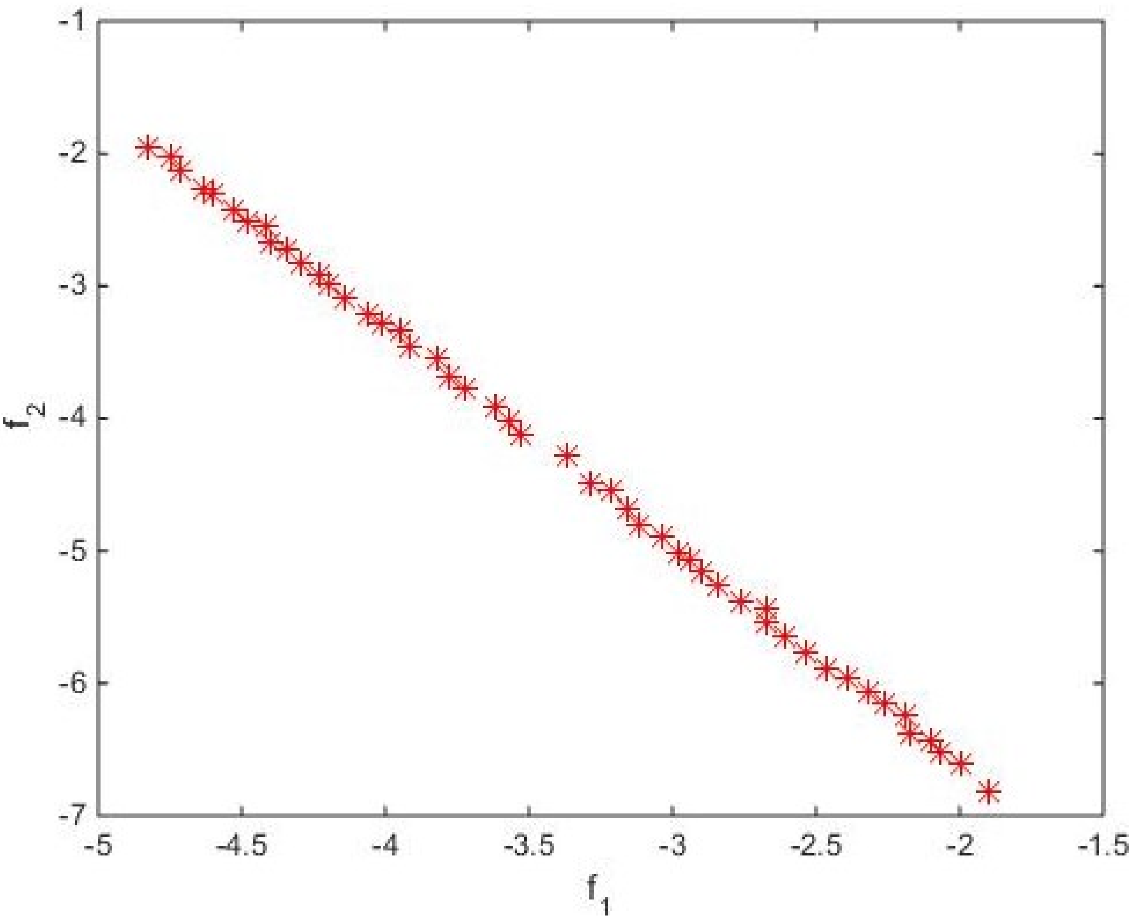

Figure 4. *Nondominated frontier approximated by NSGA-II.*

The fittest end-point which is the first to be listed and is usually assigned an infinite distance value is selected as the best nondominated point, and in this case, it is $f^1 = (-6.82, -1.90)^T$ where $f^1 = (f_1^1, f_2^1)^T \in Y_N$.

6. Interactive Algorithms

6.1. Interactive Simplex-based Algorithms

Having discussed the Non-interactive solution approaches to MOLP, we now turn our attention to the interactive algorithms that seeks to find a most preferred solution to the problem. These methods consists of two main phases: a computational phase where efficient solutions are computed and a dialogue phase where the DM is required to express his or her preferences to guide the solution process.

The earliest to propose an interactive MOLP algorithm seems to be Benayoun *et al.* [24]. The authors presented an interactive MOLP procedure called step method (STEM) to obtain the most preferred solution of the problem. First, a payoff table is constructed in order to obtain the ideal (optimum) solution for each of the objectives under consideration. In each computation phase, a weighted Chebyshev distance to the ideal solution is solved to generate efficient solution presented to the DM to decide if it is most preferred. Otherwise, the DM specifies the maximum or minimum amount he is willing to sacrifice in one objective in order to improve the other objectives. The information supplied by the DM is transformed into additional constraints thereby altering the feasible region and objective values. This process continues until a most preferred solution is obtained.

Zionts and Wallenius [150] presented a practical man-machine interactive method for solving MOLP using a linear utility function rather than payoff table. The authors noted that the STEM algorithm in [24] does not really work well in solving relatively simple problems. The method starts with arbitrary set of weights used to obtain a weighted sum objective function which is then optimized to obtain efficient solution. The DM provides answers to certain questions in order to determine the trade offs used to construct linear approximations of the utility function. Based on the answers provided by the DM, a new set of weights with associated efficient solutions are found. This continues repeatedly until a most preferred solution is found. In a similar paper, Zionts and Wallenius [151] extended their interactive MOLP algorithm in [150] to a class of nonlinear utility functions.

Benson *et al.* [30] applied the STEM algorithm in [24] to solve the citrus rootstock selection problem in Florida [25]. The algorithm first return unsatisfactory results when the payoff table approach was used as suggested in [24]. The authors then used Benson-Sayin global optimization heuristic, [31] which proves to be accurate in solving the problem. It was noted that, the use of appropriate global optimization methods is crucial to the achievement of success in real-world problems and the use of unreliable payoff table method to estimate the nadir (lowest) points in applied MOLP should be discouraged.

Arbel and Oren [13] also presented an interactive approach for MOLP. Unlike the methods in [150] and [151] that used linear and nonlinear utility functions, the authors used AHP to assigned priorities to vertices adjacent to the one representing the current basic solution to possibly improve the objective function values. These priorities were used to obtain an approximate gradient for weighing the objective functions in the next iteration. Here, the DM is not required to provide inputs for determining tradeoff, rather he or she evaluates adjacent vertices for possible improvements until no adjacent vertices are preferred to the current one. The approach produces approximate solutions only.

Steuer [128] suggested an interactive algorithm which utilizes interval criterion weights specified by the DM in the dialogue phase to obtain a subset of efficient solutions. The procedure solved the resulting weighted sum problem in the computation phase to determine a subset of efficient solutions corresponding to the specified weights. This subset of solutions is then presented to the DM to choose the most preferred solution. To further improve the process of obtaining the DM's most preferred solution, Steuer [129] proposed yet another interactive procedure capable of shifting and contrasting the convex cone generated by the gradients of the DM's objectives. This reduction is controlled by the DM's choice of an efficient solution, thereby contracting the criterion cone until it focuses on a small search space that produces the most preferred solution. This approach was necessary to further reduce the subset of efficient solutions which could still be large in his earlier algorithm in [128].

Choi and Kim [39] proposed an interactive algorithm that provides the entire structure of efficient criterion vectors for large problems. The proposed algorithm solves the Tchebycheff error problem to determine the closest point on a face of the feasible region from the reference point and ensures full coverage of the efficient objective vectors, which help to reduce the DM's burden in deriving the most preferred solution.

In [132], Stewart presented an interactive MOLP method for solving MOLP problems. The author noted that, Zionts and Wallenius method in [150] does not specify how an interactive search over faces of the simplex was to be carried out. However, his method avoids the unspecified search and uses a single procedure that requires only pairwise preference statements from the DM.

A Simplified Interactive MOLP (SIMOLP) procedure to assist DMs in determining the most preferred solutions was presented in [108]. Here, the DM is allowed to identify the most preferred solution interactively with minimum inputs, make few choices from a limited number of alternatives and in a relatively few number of steps. The method does not require a payoff or utility function and starts by solving the singled objective LP problems involved to obtain efficient points and their associated images (nondominated criterion vectors) presented to the DM for a review. If the DM is unsatisfied with the solution presented, an LP is solved to obtain an efficient point and its nondominated solution from where the DM chooses the most preferred solution.

Numerical applications show that most preferred solutions can be obtained in a relatively few number of iterations.

Korhonen and Laakso [84] proposed a visual interactive algorithm to solve the problem using an unknown utility function to simulate the DM's behaviour. The approach utilizes aspiration levels or reference direction to determine the direction of improvement that is projected on the efficient frontier to obtain an efficient curve or subset of efficient solutions. This curve reveals the objective function values which are presented to the DM for evaluation. The interface adopted here is based on a static graphic representation, which makes the DM to rely so much on system.

Korhonen and Wallenius [83] improves on their algorithm in [84] by presenting a dynamic version of the reference direction method called Pareto race. This approach allows the DM to take control of the system and can move in any direction on the efficient frontier unlike the method in [84]. The DM preference information changes the direction of search to obtain solutions in a direction that offers variation in the objectives as specified by the DM's preferences. This direction is then projected on the efficient frontier.

Haksever and Ringuest [68] presented an investigation into the computing efficiency of four variants of the SIMOLP procedure of Reeves and Franz [108] for solving large scale problems. Numerical results suggests that solving each successive LP using the optimal basis of the previous problem is an effective way of achieving a reduction in both CPU time and the number of iterations.

Malakooti and Ravindran [91] developed an interactive paired comparison simplex method for solving MOLP problems. The algorithm assumes an unknown DM's utility function and conduct a test for efficiency to reduce the number of efficient extreme points and number of questions the DM is required to answer. This method was compared with the Zionts and Wallenius [150] approach, it was reported that the proposed method is superior in all criteria selected for evaluation.

Dell and Karwan [50] proposed an interactive MOLP (K-D procedure) procedure using a Tchebycheff utility function. This procedure was compared with the method of Zionts and Wallenius [150]. Computational results suggests that the K-D procedure performs slightly better than Zionts and Wallenius method in terms of the quality of solution it returns. However, Zionts and Wallenius method was computationally more efficient than the proposed procedure.

In [89], Lotfi *et al.* presented an interactive procedure for MOLP based on the DM's aspiration levels. The procedure also used a Tchebycheff function and facilitates the achievement of the optimum at a non-extreme point solution. The approach was compared with the SIMOLP procedure of Reeves and Franz [108]. Experimental results show that the proposed method compared favourably in terms of the quality of solutions returned. However, the SIMOLP procedure outperforms the proposed method in terms of computing efficiency.

Michalowski and Szapiro [94] presented a Bi-reference interactive procedure for MOLP. Here, the DM is required to specify his or her worst outcome, from which an ideal outcome is determined. The algorithm then construct an improvement direction from the worst to ideal outcome, leading to a trial solution. A partitioning of the objective functions into that which require improvement, be unchanged and relaxed is done by the DM. This lead to the displacement of outcomes and a new improvement direction is obtained that yields a most preferred solution. The method was compared with the STEM algorithm of Benayoun *et al.* [24] and Zionts and Wallenius [151]. Numerical results show that the Bi-reference procedure is superior to these well-known procedures by generating preferred solutions in a few number of iterations.

Quaddus and Holzman [107] proposed an interactive MOLP algorithm which assumes that the DM has an implicitly known utility function. The approach is similar to that in [150]. The DM also specify a set of weights used to construct a weighted sum objective function which is optimized to obtain a starting efficient point. Here, each objective function is evaluated using the starting efficient point and the solutions presented to the DM to decide if it's satisfactory. If not, the procedure is repeated until a satisfactory or most preferred solution is obtain. The approach is more flexible and require minimum interaction with the DM than that in [150].

A tricriteria LP algorithm (TRIMAP) was introduced in [40]. The algorithm is not aimed at finding the most preferred solution, but rather assist the DM to discard the subsets of efficient solutions which are of no use to him or her. The method is based on a progressive and selective learning of the efficient solutions set, until the DM have adequate knowledge of the efficient set. In each computation phase, a weighted sum problem is solved to obtain efficient solutions presented to the DM. The DM provides lower bounds for the objective values which are graphically transformed into the weight space diagram to visually limit the search scope. During each interaction, the DM provides information to guide the search to unexplored regions of the diagram until he or she have adequate knowledge of the efficient set. Few years later, Climaco and Antunes [41] presented an implementation of their algorithm in [40]. The software is an interactive computational tool to assist the DM to have adequate knowledge of the efficient set. It is written in Pascal and consists of three main phases namely; the weight space decomposition phase, simplex implementation and routine implementation phases. The software is limited to solving problems with only three objectives. During the interactive process, a graphical representation of the weight space with regions corresponding to the efficient vertices and the objective values are presented to the DM for a review. A guided tour on the implementation and how to use the software is also included.

Similarly, Alves *et al.* [3] also introduced an interactive graphical-based computational tool for teaching, research and decision support purposes in MOLP models. The software is equipped with interactive approaches such as: STEM, Pareto race, interval criterion weight and scalarization techniques: weighted sum, reference point and e-constraint techniques as well as a Vector Maximization Algorithm (VMA) that generates all efficient extreme points of the problem. Apart from the VMA which is non-interactive, a brief description of the interactive approaches is also included. Visualization of results as well as the graphical display of regions on the weight space diagram for problems with up to three objectives can also be handled by the software. The program is limited to solving problems with a maximum of six objectives, a hundred constraints and decision variables, [98].

In [66] was presented an MOLP model to measure the value efficiency in Data Envelopment Analysis (DEA). The proposed model was solved using the interactive STEM method in [24] to obtain the value efficiency by selecting the most preferred units based on the DM's choice. Numerical illustration using real world data show the applicability of the method.

6.2. Interactive Interior Point Based Algorithms

Various interactive interior point methods have been suggested to solve the problem. These methods differ in the approach adopted to elicit preference information from the DM to guide the search (or solution) process.

Arbel and Oren [14] first developed an interactive interior point approach to solve the problem. The method used ASPA and assumed that the DM have an implicitly known utility function. The approach generates search directions that are used to approximate the gradient of the utility function. The DM provides his or her preference information that updates the approximated gradient thereby resulting in a sequence of iterates. The iterative process continues and stops at the most preferred solution. Arbel's [9] suggested an interactive path-following Primal-Dual Interior Multiple Objective Linear Programming (PDIMOLP) algorithm. The difference between ASIMOLP and PDIMOLP algorithms is in the way that the interior step directions are generated. While ASIMOLP algorithm is concerned with cost reduction, the PDIMOLP algorithm includes a centring scheme that keeps the current iterate centred in the constraints polytope.

Arbel and Korhonen [11] introduced another interactive algorithm as above, to solve the problem. The algorithm combines the path-following primal-dual process with Achievement Scalarizing Function (ASF) first introduced in [139]. In the dialogue phase, the DM is required to specify his or her aspiration levels of the objectives. The algorithm generates a path from the current iterate to as close as possible to the optimum of the ASF which depends on the DM's aspiration levels. If the DM changes his or her aspiration levels, a new iterate is generated closer to the optimum of the ASF. The process continues until a most preferred solution is found.

Arbel [10] suggested yet another interactive PDIMOLP algorithm for the problem. This approach is quite similar to the algorithm in [14]. The only noticeable difference between the two algorithms is that, the former is based on ASPA, while the latter is based on the path-following primal-dual process.

In a different paper, Arbel and Oren [16] proposed a modification of the path-following primal-dual algorithm to solve the problem. The authors applied AHP to derive priorities used to approximate the gradient of the DM's utility function. Projecting the approximated gradient on the null space of the scaled constraints matrix, provides the combined step direction along which one move from the current iterate to the next. The process continues repeatedly and by taking a full step, the algorithm converges to an efficient solution.

Aghezzaf and Ouaderhman [2] also presented an interactive interior point algorithm for finding the most preferred solution where an implicit utility function is known. The DM provides the trade-offs used to derive an approximate gradient and updates the lower bounds of the objectives. The algorithm generates a sequence of smaller polytopes that shrinks towards the most preferred solution.

In [74], Junior and Lins suggested a complementary interactive algorithm that utilizes ASPA and ASF. The ASF is formulated in such a way that the algorithm follows a path that avoids sudden changes in direction so as to exhibit a win-win property. The DM specifies aspiration levels for each objective and their proportion among other objectives. In each iteration, intermediate solutions with all objective function values improved, are presented to the DM for evaluation. This allows the DM to reach a most preferred solution without making trade-offs between objectives.

Trafalis and Alkahtani [136] proposed an interactive interior point algorithm for MOLP based on the method of analytic centers. In the dialogue phase, the DM supplies his or her trade-offs which are used to obtain a cut in the objective space which induces a corresponding cut in the variable space. The analytic center of the feasible region is constructed and the DM's trade-offs are determined in the analytic center of the objective space, thereby obtaining a trajectory of analytic centers which converges to a most preferred solution.

6.2.1. Interactive Affine Scaling Interior MOLP Algorithm

The Interactive ASIMOLP Algorithm is another examplar algorithm from the interior point class. It is developed in [14]. It works in the decision space and also returns an efficient face of the feasible region, from which the efficient solution of the problem is obtained. The difference between Algorithms 4 and this one is in the assessment of the relative priority vector p which is used to derive the combined step direction vector dx. In Algorithm 4, AHP is used to derive p by filling a comparison matrix, while in here, the implicitly known DM's utility function is used to approximate vector p. At each iteration, the current iterate and boundary point are presented to the DM to choose that which is most preferred.

Algorithm 6 Interactive Affine Scaling Interior MOLP Algorithm

0: Input: A, b, C
1: Initialize: $Choose\ x_0 > 0$, $Stopping\ tolerance\ \sigma\,(0 < \sigma < 1)$, $Step\ size\ factor\ \rho\,(0 < \rho < 1)$, $u(x)$,
$Converged = 0$, $k \leftarrow 0$;
2: while $converged \neq 1$ **do**
3: $k \leftarrow k + 1$
4: $D \leftarrow diag(x_0)$
5: $y_i(k) \leftarrow (AD^2A^T)^{-1}AD^2c_i$
6: $z_i(k) \leftarrow (c_i^T - A^T y_i(k))$
7: $dx_i(k) \leftarrow D^2(z_i(k))$
8: **if** $dx_i(k) \geq 0,\ 1 \leq i \leq q$, **stop**
9: **else**
10: $du \leftarrow [p_i - p_0]$
11: $dv(k) \leftarrow C[\rho\alpha_i dx_i(k)]$
12: $du_x(k) \leftarrow du(dv(k))^{-1}C$
13: $y \leftarrow (AD^2A^T)^{-1}AD^2(du_x(k))^T$
14: $dx(k) \leftarrow D^2(du_x(k))^T - A^T y$
15: $\alpha_i = Min[\frac{-X_i}{dx_i(k)}, \forall dx_i(k) < 0]$
16: $x_i(k+1) \leftarrow x(k) + \rho\alpha_i dx_i(k)$
17: $x_b \leftarrow x(k) + \rho dx(k)$
18: **if** $k > 1$, **do**
19: $du \leftarrow [p_i - p_0,\ p_b - p_0]$
20: $dv(k) \leftarrow C[\rho\alpha_i dx_i(k),\ dx_b]$
21: $du_x(k) \leftarrow du(dv(k))^T(dv(k)(dv(k))^T)^{-1}C$
22: $y \leftarrow (du_x(k))^T - A^T(AD^2A^T)^{-1}AD^2(du_x(k))^T$
23: $dx \leftarrow D^2 y$
24: $x_{new} \leftarrow x(k+1) + \rho dx(k)$
25: $v_{new} \leftarrow C^T x_{new}$
26: $x_b \leftarrow x(k+1) + dx(k)$
27: $dx_b \leftarrow x_b - x_{new}$
28: **else**
29: **if** $\|dx_b\| \leq \sigma$, **stop**
30: **else**
31: $x_0 \leftarrow x_{new}$
32: **Go to step 4**
33: **endif**
34: **endif**
35: **endif**
36: **endwhile**
37 Output: x_{new}
v_{new}

6.2.2. Illustration of Interactive ASIMOLP Algorithm

We have also implemented Interactive ASIMOLP algorithm in Matlab and applied to Problem (3) of section 2.1.3. The efficient solution found is $x^* = (x_1^*, x_2^*)^T = (3.7146, 3.7094)^T$ and the corresponding objective function values are $v^* = (v_1^*, v_2^*)^T = (-3.7146, -3.7094)^T$ respectively. The search path as generated by Algorithm 6 is shown in Figure 5.

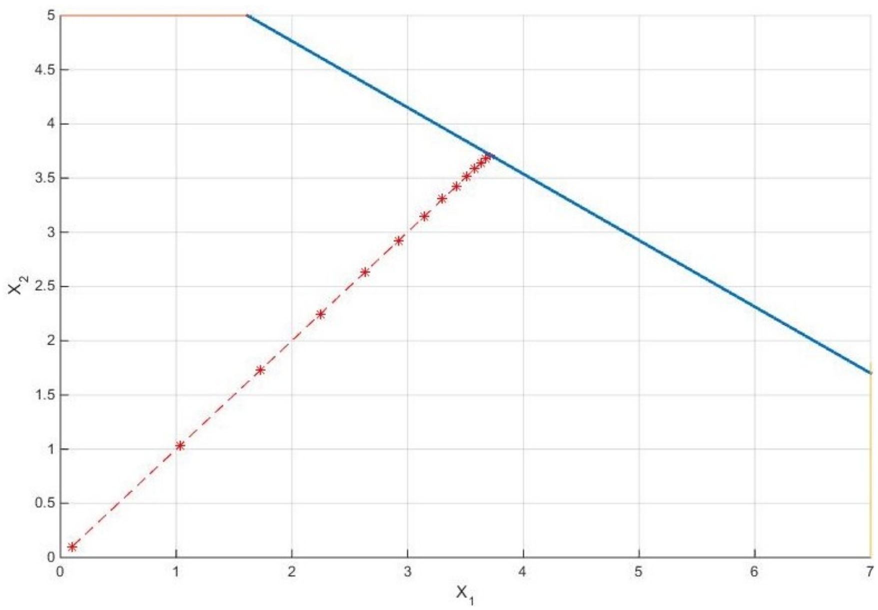

Figure 5. *The search path generated by Interactive ASIMOLP algorithm showing convergence to the efficient frontier.*

Notice that the efficient solutions returned by the simplex-based algorithms are different from those returned by the interior point based methods, for the same problem. This is due to the fact that the simplex-based methods produce vertex solutions and may not be able to generate solutions that lie on a face of the feasible region. Interior point based methods, on the other hand, do not generate vertex solutions; rather, they return efficient solutions that lie on a face of the polytope as can be seen on Figures 3 and 5.

7. A Comprehensive Database of Papers on MOLP

In this section, we present a comprehensive list of all the methods for MOLP reviewed earlier. It is in six tables and placed in appendix A section. The first records the non-interactive simplex-based methods; the second records the interactive simplex-based methods; the objective space based methods are recorded in table three; the fourth records the non-interactive interior point based methods; the fifth is on the interactive interior point based methods and the sixth table records the relevant heuristics and meta-heuristic approaches. Each of these tables is organised chronologically.

As can be seen from the tables, between 1967 and the early 90s, MOLP problems were solved mostly by the simplex-based approach and its variants. Interior point based methods for MOLP appeared in the literature from the early 90s, following the appearance of the Ellipsoid algorithm of Khachiyan [79] and Karmarkar's algorithm [77]. The objective space methods appeared first in the mid 90s.

8. Future Research

Based on our extensive review of the existing literature, it was observed that from inception to date, MOLP problems has been mostly solved by exact methods. It would be worthwhile and of great importance to the potential researchers and practitioners to look at the possibility of solving MOLP problems using heuristic or meta-heuristic approaches. Given that, exact methods are sometime inefficient and very costly, most especially when dealing with realistic MOLP instances.

9. Conclusion

Various real world decision making problems are formulated as MOLP problems. In this article, we have comprehensively reviewed MOLP papers that have appeared since 1964. The survey classifies MOLP algorithms into two broad classes: Non-Interactive and Interactive algorithms. The Non-Interactive include Simplex based, Interior Point based and Objective Space based. The Interactive also include Simplex and Interior point based algorithms. A representative algorithm for the simplex, interior point and objective space based method is given and illustrated on a small size MOLP problem instance. Moreover, a tabulated summary of all algorithms reviewed during the period over the last 50yrs is also included.

As can be seen from tables 1, 2, 3, 4, 5 and 6 which are located on the Appendix A section; between 1964 and the early 90s, MOLP problems were solved mostly by the simplex-based approach and its variants. Interior point based methods for MOLP appeared in the literature from the early 90s, following the appearance of the Ellipsoid algorithm of Khachiyan [79] and Karmarkar's algorithm [77]. The objective space methods appeared first in the mid 90s. They are gaining in notoriety.

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Abbreviations

MOLP	Multi-objective Linear Programming
MCDM	Multiple Criteria Decision Making
MSM	Multicriteria Simplex Method
IPM	Interior Point Method
OSM	Objective Space Method
HMA	Heuristic and Meta-heuristics Approaches
LP	Linear Program
VMA	Vector Maximization Approach
ASIMOLP	Affine Scaling Interior Multi-objective Linear Programming Algorithm
EMSA	Extended Multi-objective Simplex Algorithm
BOA	Benson's Outer Approximation Algorithm
MSA	Multi-criteria Simplex Algorithm
PSA	Parametric Simplex Algorithm
AHP	Analytic Hierarchy Process
ASPA	Affine Scaling Primal Algorithm
EEP	Efficiency Equivalent Polyhedron
MEF	Maximal Efficient Faces
MOO	Multi-objective Optimization
NSGA-II	Nondominated Sorting Genetic Algorithm
LMOPSO	Linear Multi-objective Particle Swarm Optimization
PSO	Particle Swarm Optimization
MOBAS	Multi-objective Beetle Antenna Search Algorithm
MOPSO	Multi-objective Particle Swarm Optimization
MODEA	Multi-objective Differential Evolution Algorithm
MOP	Multi-objective Optimization
MOPPA	Multi-objective Plant Propagation Algorithm
STEM	Step Method
DM	Decision Maker
SIMOLP	Simplified Interactive Multi-objective Linear Programming
K-D	Karwan Dell

TRIMAP	Tricriteria Multi-objective Algorithm
DEA	Data Envelopment Analysis
PDIMOLP	Primal Dual Interior Multi-objective Linear Programming

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Author Contributions

Paschal Bisong Nyiam: Conceptualization, Data curation, Investigation, Writing – original draft, Writing – review & editing

Abdellah Salhi: Funding acquisition, Methodology, Resources, Supervision, Writing – review & editing

Conflicts of Interest

The authors declare no conflicts of interest.

Appendix

Table 1. *Non-interactive Simplex-based methods.*

SN	Author(s)	Year	Title
1	Schonfeld, K. P.	1964	Effizienz und Dualitat in der Aktivitatsanalyse
2	Geoffrion, A. M.	1967	Solving bicriterion mathematical programs
3	Philip, J.	1972	Algorithms for the vector maximization problem
4	Evans & Steuer	1973	A revised simplex method for linear multiple objective programming
5	Belenson & Kapur	1973	An algorithm for solving multicriterion linear programming problems with examples
6	Zeleny, M.	1973	Compromise programming
7	Zeleny, M.	1974	Linear multiobjective programming
8	Yu and Zeleny	1974	The techniques of linear multiple objective programming
9	Yu and Zeleny	1975	The set of all nondominated solutions in linear cases and a multicriteria simplex method
10	Ecker and Kouada	1975	Finding efficient points for linear multiple objective programs
11	Yu and Zeleny	1976	Linear multiparametric programming by multicriteria simplex method
12	Gal, T.	1977	A general method for determining the set of all efficient solutions to a linear vector maximization problem
13	Isermann, H.	1977	The enumeration of the set of all efficient solutions for a linear multiple objective program
14	Philip, J.	1977	Vector optimization at a degenerate vertex
15	Ecker and Kouada	1978	Finding all efficient extreme points for multiple objective linear programs
16	Seiford and Yu	1979	Potential solutions of linear systems: The multicriteria multiple constraint levels program
17	Ecker et al.	1980	Generating all maximal efficient faces for multiobjective linear programs
18	Benson, H. P	1981	Finding an initial efficient extreme point for linear multiobjective program
19	Zeleny, M	1982	Multiple criteria decision making
20	Steuer, R. E.	1986	Multiple criteria optimization: theory, computation & applications
21	Strijbosch et al.	1991	A simplified MOLP algorithm: The MOLP-S procedure
22	Armand and Malivert	1991	Determination of the efficient set in multiobjective linear programming
23	Armand, P.	1993	Finding all maximal efficient faces in multiobjective linear programming
24	Cardoso and Climaco	1994	Efficient frontier scanning in MOLP using a new tool
25	Sayin, S.	1996	An algorithm based on facial decomposition for finding the efficient set in MOLP
26	Ruszczynski and Vanderbei	2003	Frontiers of stochastically nondominated portfolios
27	Yan et al.	2005	Constructing efficient solutions structure of multiobjective linear programming
28	Kim and Thien	2007	Generating all efficient extreme points in multiple objective linear programming problem and its application
29	Ehrgott et al.	2007	Primal-dual simplex method for multiobjective linear programming
30	Kim et al.	2008	Generating all efficient extreme solutions in MOLP and its application to multiplicative programming
31	Pourkarimi et al.	2009	Determining maximal efficient faces in multiobjective linear programming problem
32	Suprajitno, H	2012	Solving multiobjective linear programming problem using interval arithmetic
33	Pandian and Jayalakshmi	2013	Determining efficient solutions to multiple objective linear programming problems
34	Rudloff et al.	2015	A parametric simplex algorithm for linear vector optimization problems An interior point multiple criteria
35	Luc, D. T.	2016	Multiobjective Linear Programming: An Introduction
36	Nyiam and Salhi	2021	A Comparison of Benson's Outer-Approximation Algorithm With an Extended version of Multiobjective Simplex Algorithm
37	Nyiam and Salhi	2021	On the Simplex, Interior Point and Objective Space Approaches to multiple objective Linear Programming
38	Nath and Bhatacharya	2020	Portfolio optimisation in share market using multi-objective Linear Programming
39	Cabrera-Guerrero et al.	2021	Bi-objective optimisation over a set of convex sub-problems

Table 2. *Interactive Simplex-based methods.*

SN	Author(s)	Year	Title
1	Benayoun et al	1971	Linear programming with multiple objective functions
2	Zionts and Wallenius	1976	An interactive programming method for solving the multiple criteria problem
3	Steuer, R. E.	1976	Multiple objective linear programming with interval criterion weights
4	Steuer, R. E.	1977	An interactive multiple objective linear programming procedure
5	Zionts and Wallenius	1983	An interactive multiple objective linear programming method for a class of underlying nonlinear utility functions
6	Reeves and Franz	1985	A simplified interactive multiple objective linear programming procedure
7	Korhonen and Laakso	1986	A visual interactive method for solving the multiple criteria problem
8	Quaddus and Holzman	1986	Imolp: an interactive method for multiple objective linear programs
9	Arbel and Oren	1986	Generating serach directions in MOLP using the analytic hierarchy process
10	Malakooti and Ravindran	1986	Experiments with an interactive paired comparison simplex method for molp problems
11	Stewart, T. J.	1987	An interactive MOLP method based on piecewise linear additive value functions
12	Climaco and Antunes	1987	Trimap - an interactive tricriteria linear programming package
13	Korhonen and Wallenius	1988	A Pareto race
14	Dell and Karwan	1990	An interactive mcdm weight space reduction method utilizing a Tchebycheff utility function
15	Michalowski and Szapiro	1992	A bi-reference procedure for interactive multiple criteria programming
16	Choi and Kim	1994	Approximation of the set of efficient objective vectors for large scale molp
17	Lotfi etal.	1997	Aspiration-based search algorithm for multiple objective linear programming problems: theory and comparative tests
18	Benson etal.	1998	Global optimization in practice: an application to interactive multiple objective linear programming
19	Alves et al.	2015	Interactive MOLP explorer: A graphicalbased computational tool for teaching and decision support in multi-objective linear programming models
20	Gerami J.	2019	An interactive procedure to improve estimate of value efficiency in DEA.

Table 3. *Objective space based methods.*

SN	Author(s)	Year	Title
1	Dauer, J. P.	1987	Analysis of the objective space in MOLP
2	Dauer & Liu	1990	Solving multiple objective linear programs in objective space
3	Dauer & Saleh	1990	Consrtucting the set of efficient objective in multiple objective linear programs
4	Gallagher & Saleh	1995	A representation of an efficient equivalent polyhedron for the objective set of a MOLP
5	Dauer & Gallagher	1996	A combined constraint space, objective space approach for determining high dimensional efficient faces of MOLP
6	Benson, H. P.	1998a	An outer approximation algorithm for generating all efficient extreme points in the outcome set of MOLP problem
7	Benson, H. P.	1998b	Further analysis of an outcome set-based algorithm for MOLP
8	Benson, H. P.	1998c	Hybrid approach for solving multiple objective linear programs in outcome space
9	Benson & Sun	2002	A weight set decomposition algorithm for finding all extreme points in the outcome set of a MOLP
10	Kim, N. T. B.	2002	Efficiency equivalent polyhedra for feasible set of multiple objective linear programming
11	Yan *et al.*	2005	Constructing efficient solutions structure of multiobjective linear programming
12	Tantawy, S.	2007	Detecting nondominated extreme points for MOLP
13	Shao & Ehrgott	2008a	Approximately solving MOLPs in objective space and an application in radiotherapy treatment planning
14	Shao & Ehrgott	2008b	Approximating the nondominated set of an MOLP by approximately solving its dual problem
15	Heyde & Lohne	2008	Geometric duality in MOLP
16	Heyde *et al.*	2009	Set-valued duality theory for MOLP & application to mathematical finance
17	Foroughi & Jafari	2009	A modified method for construction efficient solutions structure of MOLP
18	Burton & Ozlen	2010	Projective geometry and the outer approximation algorithm for MOLP
19	Ehrgott *et al.*	2011	An approximation algorithm for convex MOLP problem
20	Ehrgott *et al.*	2012	A dual variant of Benson's outer approximation algorithm for MOLP
21	Lohne, A.	2012	Vector optimization with infimum and supremum
22	Csirmaz, L.	2013	Using multiobjective optimization to map the entropy region of four random variables.
23	Hamel *et al.*	2014	Benson type algorithms for linear vector optimization & applications
24	Lohne *et al.*	2014	Primal and dual approximation algorithms for convex vector optimization problems
25	Shao & Ehrgott	2015	Primal and dual MOLP algorithms for linear multiplicative programming
26	Dörfler, D.	2021	A Benson-type Algorithm for bounded convex vector optimisation problems with vertex selection

Table 4. *Non-nteractive Interior point based methods.*

SN	Author(s)	Year	Title
1	Abhyankar et al.	1990	Efficient faces of polytopes: interior point algorithms, parametrization of algebraic varieties and multiple objective optimization
2	Arbel, A.	1993a	An interior multiobjective linear programming algorithm
3	Arbel, A.	1993b	A weighted-gradient approach to MOLP problems using the analytic hierarchy process
4	Arbel and Oren	1994	A modification of Karmarkar's algorithm to multiple objective linear programming problems
5	Arbel, A.	1994a	Anchoring points and cones of opportunities in interior multiple objective linear porgramming
6	Arbel, A.	1994b	Using efficient anchoring points for generating search directions in interior MOLP
7	Arbel, A.	1994c	Interior point methods for multiobjective linear programming problems: In multiple criteria decision making
8	Wen and Weng	1998	An interior algorithm for solving multiobjective linear programming problem
9	Bufardi, A.	1999	Multicriteria decision making: and advances in MCDM models, algorithms, theory, and applications
10	Arbel and Korhonen	2001	Using objective values to start multiple objective linear programming algorithms
11	Weng and Wen	2001	An interior point algorithm for solving linear optimization over the efficient set problems
12	Zhong and Shi	2001	An interior point approach for solving mc2 linear programming problems
13	Arbel and Sadka	2005	Weighted euclidean centers
14	Lin et al.	2006	On the modified interior point algorithm for solving MOLP problems
15	Fliege, J.	2006	An efficient interior point method for convex multicriteria optimization problems

Table 5. *Interactive Interior point based methods.*

SN	Author(s)	Year	Title
1	Arbel and Oren	1993	Generating interior search directions for multiobjective linear programming
2	Arbel, A.	1995	An interior multiple objective primal-dual linear programming algorithm using efficient anchoring points
3	Arbel and Oren	1996	Using approximate gradients in developing an interactive primal-dual MOLP algorithm
4	Arbel & Korhonen	1996	Using aspiration levels in an interior primal-dual multiple objective linear programming algorithm
5	Arbel, A.	1997	An interior multiobjective primal dual linear programming algorithm based on approximated gradients & efficient anchor points
6	Trafalis and Alkahtani	1999	An interactive analytic centre trade off cutting plane algorithm for multiobjective linear programming
7	Junior and Lins	2009	A win-win approach to multiple objective linear programming problems
8	Aghezzaf and Ouaderhmann	2001	An interactive interior point algorithm for multiple objective linear programming problems

Table 6. *Heuristics and Meta-heuristic Approaches.*

SN	Author(s)	Year	Title
1	Sanaz et al.	2006	Linear Multi-objective particle swarm optimisation
2	Reyes-Sierra et al.	2006	Multi-objective particle swarm optimisation A Survey of the state-of-the-art
3	Kannan et al.	2009	Application of NSGA-II to generation expansion planning
4	Chakraborty M. R.	2010	A Parametric Approach and Genetic Algorithm for MOLP with imprecise parameters
5	Massobrio et al.	2015	Multi-objective taxi sharing optimisation using NSGA-II evolutionary algorithm
6	Zhang et al.	2020	Multi-objective Beetle Antennae Search Algorithm
7	Nyiam & Salhi	2023	Application of the Plant Propagation Algorithm & NSGA-II to Multiple objective linear programming
8	Motahari et al.	2023	A MOLP model for scheduling part families & designing a group layout in cellular manufacturing systems

Table 7. *Characteristics of Algorithm 1-6.*

	Criteria for Evaluation	
Algorithms	**Computational Efficiency**	**Quality of Solution returned**
MSA	Computationally less efficient for realistic instances	Return high quality efficient solutions
PSA	Computationally very efficient	Return subset of high quality efficient solutions
BOA	Computationally very efficient	Return all high quality nondominated solutions
ASIMOLP	Computationally more efficient than MSA, PSA and BOA	Return only one less quality efficient solutions and one nondominated point
NSGA-II	Computationally very efficient	Return near efficient or approximate solutions
Interactive ASIMOLP	Computationally more efficient than MSA, PSA and BOA	Return only one less quality efficient solutions and one nondominated point

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