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Prime Incidence Analysis Under Bounded Integer-Line Isometries

Ioannis Papadakis *

Independent Researcher, Charlotte, United States of America

Abstract

For $N \geq 3$, let $\mathcal{L}(N)$ denote the primes in $[3, 2N-3]$. The binary Goldbach condition at $2N$ is equivalent to positive prime incidence under reflection about the midpoint N. This paper studies prime incidence under two bounded integer-line isometries: (1) displaced reflection (reflected branch) and (2) translation (translated branch). In both cases, the prime and its image must remain in $\mathcal{L}(N)$. For either branch, the Maximal Displacement Envelope (MDE) is the largest even displacement through which incidence remains positive at every even displacement, including zero; if incidence fails at zero, that branch has no MDE. When both branch envelopes exist, their minimum defines the combined MDE(N). The Prime Incidence Conjecture (PIC) asserts that MDE(N) exists for every $N \geq 3$ and that MDE$(N)/N \to 2$ as $N \to \infty$. At zero displacement, reflected incidence is precisely the binary Goldbach condition. Both branches admit single-parameter arithmetic formulations through the least prime realizing a prescribed even prime difference and the least symmetric prime-pair displacement about a midpoint. Exhaustive computation through $N = 3 \times 10^7$ determines the branch-specific and combined envelopes over the full admissible displacement range and shows that the normalized first-failure displacements trend toward the normalized endpoint $2 - 6/N$ over the computed range. Established analytic estimates further imply that, for every fixed $0 < \varepsilon < 2$ and $A > 0$, the number of even displacements $s \leq (2-\varepsilon)N$ at which at least one branch fails is $O_{A,\varepsilon}(N(\log N)^{-A})$ for all sufficiently large N.

Keywords

Binary Goldbach Conjecture, Prime Incidence, Prime Incidence Conjecture, Displacement, Reflection, Translation, Maximal Displacement Envelope, Left–Right Partition Table

1. Introduction

The binary Goldbach conjecture (bGC) states that every even integer $n \geq 6$ is the sum of two odd primes [1]. Montgomery and Vaughan proved that bGC holds for all but a set of even integers of asymptotic density zero [2], and bGC has been computationally verified through 4×10^{18} [3]. No proof is currently known that covers every even integer. Bhowmik and Grimmelt reviewed advances in Goldbach exceptional-set estimates and derived an explicit major-arc formula, while Tsuda established small-gap estimates between primes in Goldbach representations for almost all even integers [4, 5].

Earlier work introduced the Left–Right Partition Table (LRPT), a symmetric representation of the odd decompositions of $2N$, and used it to study Goldbach-pair structure and hypothetical failure states [6, 7].

For a given integer N, the LRPT lists the odd pairs (x, y) satisfying $x + y = 2N$ and $3 \leq x \leq y \leq 2N-3$, with the left-hand entries increasing and the corresponding right-hand entries decreasing. A row is called prime–prime (PP) when

*Correspondence: Ioannis Papadakis (ypap@primactaglobal.com)

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both entries are prime. Figure 1 shows the transition from LRPT(19) to LRPT(20), providing an example of the LRPT representation and its evolution as N increases.

LRPT(19)

LHS	**RHS**
3	35
5	33
7	**31**
9	29
11	27
13	25
15	23
17	21
19	**19**

$\Longrightarrow$

LRPT(20)

LHS	**RHS**
3	**37**
5	35
7	33
9	31
11	**29**
13	27
15	25
17	**23**
19	21

Figure 1. *Evolution from* LRPT(19) *to* LRPT(20). *Bold entries mark PP rows.*

At state $N = 19$, $(7, 31)$ is the only distinct PP row, while $(19, 19)$ is a diagonal PP row. After the transition to $N = 20$, the primes already represented at state 19 form the new PP rows $(11, 29)$ and $(17, 23)$, while the new top-right entry (TRE) 37 forms the PP row $(3, 37)$. Thus LRPT evolution can change prime-pair incidence both by changing pairings among values already represented and by introducing a new value at the upper boundary. Appendix III describes additional properties of this evolution.

The connection with bGC follows directly from the LRPT pairing rule. A PP row $p + q = 2N$ satisfies

$$q = 2N - p,$$

so Goldbach pairing is a reflection about the midpoint N. The binary Goldbach condition at state N is therefore positive prime incidence under this zero-displacement reflection.

For an even displacement s, displacing the reflection gives

$$p \mapsto 2N + s - p.$$

If both primes remain in the finite LRPT domain, the corresponding pair satisfies

$$p + q = 2N + s.$$

This defines the reflected branch of the Prime Incidence Conjecture, denoted PIC$(-)$.

The other isometry type of the integer line is translation. For the same even displacement,

$$p \mapsto p + s,$$

and positive incidence requires two primes in the finite domain whose difference is s. This defines the translated branch, denoted PIC$(+)$.

The common displacement range of the two maps is

$$0 \le s \le 2N - 6, \qquad s \text{ even}.$$

For each branch, the Maximal Displacement Envelope (MDE) is the largest even displacement s such that incidence is positive at every even $t \in [0, s]$. The combined envelope is the smaller of the reflected and translated branch envelopes. PIC states that this combined envelope exists for every $N \ge 3$ and satisfies

$$\frac{\mathrm{MDE}(N)}{N} \to 2 \qquad (N \to \infty).$$

For $s = 0$, i.e. at zero displacement, PIC$(-)$ is equivalent to the binary Goldbach condition at $2N$.

The two incidence branches admit single-parameter formulations. For positive even s, translated incidence is determined by the least prime $g^*(s)$ for which $g^*(s) + s$ is prime. Reflected incidence is determined by the least symmetric displacement $R_{\min}(m)$ of a prime pair about the midpoint $m = N + s/2$. These quantities also determine the branch-specific MDEs and permit computational verification without testing every pair (N, s) independently. Section 3 develops these relations and computes the envelopes through $N = 3 \times 10^7$ over the entire finite-domain displacement range.

Section 4 formulates the translated branch as a prime-difference problem and the reflected branch as a bounded Goldbach representation problem, and derives exceptional-set bounds for both. For every fixed $0 < \varepsilon < 2$, established analytic estimates imply that, for sufficiently large N, the number of even displacements

$$0 \le s \le (2 - \varepsilon)N$$

for which at least one PIC branch fails is

$$O_{A,\varepsilon}\left(N(\log N)^{-A}\right)$$

for every $A > 0$. Thus, for every fixed $0 < \varepsilon < 2$, the proportion of even displacements $s \le (2-\varepsilon)N$ at which at least one branch fails tends to zero faster than any fixed negative power of $\log N$.

The remaining sections of the paper are organized as follows. Section 2 defines prime incidence under bounded reflections and translations, introduces the branch-specific and combined MDEs, and states PIC. Section 5 gives the conclusions. Appendix I compares two empirical models for the logarithmically smoothed normalized reflected-branch gap from its conjectured asymptotic limit. Appendix II gives autocorrelation, self-convolution, and Fourier representations of the two PIC incidence counts. Appendix III contains additional results on LRPT evolution and prime-pair formation.

2. Prime Incidence Under Bounded Isometries

The Goldbach reflection introduced in Section 1 provides the starting point for the two incidence conditions studied in this paper. This section first illustrates the reflected and translated maps at a single LRPT state, then defines the incidence counts and the branch-specific Maximal Displacement Envelopes, and states the Prime Incidence Conjecture.

2.1. Translated and Reflected Prime Incidence: A Numerical Example

Consider the state $N = 20$ with the corresponding LRPT integer domain given by

$$W_{20} = [3, 37]$$

and

$$\mathcal{L}(20) = \{3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37\}.$$

For $s = 4$, the displaced reflection is

$$p \mapsto 44 - p$$

with prime incidences including $(7, 37)$ and $(13, 31)$. Translation by the same displacement is

$$p \mapsto p + 4,$$

with prime incidences including $(3, 7)$ and $(13, 17)$.

The common finite-domain displacement range at $N = 20$ is

$$0 \le s \le 34, \qquad s \text{ even}.$$

Table 1 shows selected values across this range.

Table 1. *Selected reflected and translated prime incidences at $N = 20$.*

s	reflected prime pair	$\Sigma^{-}_{20}(s)$	translated prime pair	$\Sigma^{+}_{20}(s)$
0	$(3, 37)$	6	$(3, 3)$	11
4	$(7, 37)$	4	$(3, 7)$	4
22	$(31, 31)$	1	$(7, 29)$	1
24	none	0	$(5, 29)$	3
34	$(37, 37)$	1	$(3, 37)$	1

The reflected incidence first vanishes at $s = 24$, although it is positive again at $s = 34$. The largest even displacement through which reflected incidence remains positive is 22. Translated incidence is positive at every even displacement through 34. The corresponding branch-specific envelopes are therefore

$$\mathrm{MDE}_{-}(20) = 22, \qquad \mathrm{MDE}_{+}(20) = 34,$$

and the combined envelope is

$$\mathrm{MDE}(20) = 22.$$

Normalizing by N gives

$$\frac{\mathrm{MDE}_{-}(20)}{20} = 1.1, \qquad \frac{\mathrm{MDE}_{+}(20)}{20} = 1.7,$$
$$\frac{\mathrm{MDE}(20)}{20} = 1.1.$$

This example shows that the MDE can differ from the largest displacement at which an individual positive incidence occurs.

2.2. Bounded LRPT Domain and Incidence Maps

For an integer $N \geq 3$, the represented LRPT domain is

$$W_N = [3, 2N-3],$$

and its finite prime set is

$$\mathcal{L}(N) = \mathbb{P} \cap W_N.$$

The top-right entry (TRE) is

$$\mathrm{TRE}(N) = 2N - 3.$$

For an even displacement $s \geq 0$, define the displaced reflection by

$$\sigma_{-,s}(p) = 2N + s - p$$

and the translation by

$$\sigma_{+,s}(p) = p + s.$$

Both maps are isometries of the integer line under the usual distance $d(a,b) = |a-b|$. The term *bounded* refers to the restriction

$$p, \sigma_{\pm,s}(p) \in W_N.$$

For the reflected map, this restriction gives

$$s + 3 \leq p \leq 2N - 3.$$

Define

$$\Sigma_N^-(s) = \#\left\{p \in \mathcal{L}(N) : 2N + s - p \in \mathcal{L}(N)\right\}.$$

Thus $\Sigma_N^-(s)$ counts the primes $p \in \mathcal{L}(N)$ for which $2N+s-p$ also belongs to $\mathcal{L}(N)$.

For the translated map, the finite-domain restriction gives

$$3 \leq p \leq 2N - 3 - s.$$

Define

$$\Sigma_N^+(s) = \#\left\{p \in \mathcal{L}(N) : p + s \in \mathcal{L}(N)\right\}.$$

Thus $\Sigma_N^+(s)$ counts the primes $p \in \mathcal{L}(N)$ for which $p + s$ also belongs to $\mathcal{L}(N)$.

A nonempty finite-domain range is possible for either map only when

$$s \leq 2N - 6.$$

The common displacement range studied here is therefore

$$0 \leq s \leq 2N - 6, \qquad s \text{ even}.$$

At zero displacement,

$$\sigma_{-,0}(p) = 2N - p.$$

Hence

$$\Sigma_N^-(0) > 0$$

if and only if there are primes $p, q \in W_N$ satisfying

$$p + q = 2N.$$

The zero-displacement reflected condition is therefore the binary Goldbach condition at $2N$.

For $s > 0$, reflected incidence is equivalent to the existence of primes p, q satisfying

$$p + q = 2N + s, \qquad p, q \in [s + 3, 2N - 3] \cap \mathbb{P}.$$

The reflected branch therefore retains the prime-sum structure of the Goldbach condition while shifting the even target from $2N$ to $2N + s$.

2.3. Maximal Displacement Envelopes and the Prime Incidence Conjecture

For each N, define the translated-branch Maximal Displacement Envelope, when the set below is nonempty, by

$$\mathrm{MDE}_+(N) = \max\Big\{s \in 2\mathbb{Z}_{\geq 0} : s \leq 2N - 6, \\ \Sigma_N^+(t) > 0 \text{ for all } t \in 2\mathbb{Z}_{\geq 0} \cap [0, s]\Big\}.$$

Define the reflected-branch Maximal Displacement Envelope analogously:

$$\mathrm{MDE}_-(N) = \max\Big\{s \in 2\mathbb{Z}_{\geq 0} : s \leq 2N - 6, \\ \Sigma_N^-(t) > 0 \text{ for all } t \in 2\mathbb{Z}_{\geq 0} \cap [0, s]\Big\}.$$

When both branch-specific envelopes exist, define the combined Maximal Displacement Envelope by

$$\mathrm{MDE}(N) = \min\left\{\mathrm{MDE}_+(N), \mathrm{MDE}_-(N)\right\}.$$

Conjecture 2.1. *For every integer $N \geq 3$, $\mathrm{MDE}(N)$ exists and*

$$\lim_{N \to \infty} \frac{\mathrm{MDE}(N)}{N} = 2.$$

This is the Prime Incidence Conjecture (PIC). The reflected and translated incidence conditions are denoted by PIC$(-)$ and PIC$(+)$, respectively. Since existence of $\mathrm{MDE}_-(N)$ requires

$$\Sigma_N^-(0) > 0,$$

the binary Goldbach conjecture is the zero-displacement reflected case of PIC.

The limit in Conjecture 2.1 states that the normalized gap between the combined MDE and the finite-domain endpoint tends to zero as $N \to \infty$. Section 3 expresses the two branch-specific envelopes through single-parameter arithmetic quantities and uses those formulations for computational verification.

3. Single-Parameter Formulations and Computational Analysis of PIC

The incidence counts in Section 2 depend on both N and s. Each branch, however, is determined by a single arithmetic quantity. These formulations determine the first failing displacement at each state and therefore the branch-specific and combined Maximal Displacement Envelopes.

3.1. Single-Parameter Formulation of PIC(+)

At $s = 0$, translated incidence is automatic:

$$\Sigma_N^+(0) = \#\mathcal{L}(N) \geq 1.$$

For positive even s, define

$$g^*(s) = \min\{p \in \mathbb{P} : p + s \in \mathbb{P}\},$$

with $g^*(s) = +\infty$ if no such prime exists. The prime 2 cannot attain this minimum, because $2 + s > 2$ is even. Hence every finite value of $g^*(s)$ is at least 3.

At state N,

$$\Sigma_N^+(s) > 0 \quad \Longleftrightarrow \quad g^*(s) + s \leq 2N - 3.$$

Since $g^*(s) + s$ and $2N - 3$ are odd when $g^*(s)$ is finite,

$$\Sigma_N^+(s) = 0 \quad \Longleftrightarrow \quad g^*(s) + s \geq 2N - 1.$$

Thus one value of $g^*(s)$ determines translated incidence at every state for which $0 < s \leq 2N - 6$.

If $g^*(s) = 3$, translated incidence is positive at every state for which $s \leq 2N - 6$. If $g^*(s) \geq 5$, the last state at which translation by s fails is

$$N_+(s) = \frac{s + g^*(s) + 1}{2},$$

and at that state

$$2N_+(s) - s = g^*(s) + 1.$$

For positive even s, the finite values of $g^*(s)$ coincide with OEIS A020483 at index $s/2$ [8].

3.2. Single-Parameter Formulation of PIC(−)

For reflected incidence, set

$$m = N + \frac{s}{2}.$$

A reflected prime pair satisfies

$$p + q = 2m.$$

Write

$$p = m - r, \qquad q = m + r,$$

and define

$$R_{\min}(m) = \min\{r \in \mathbb{Z}_{\geq 0} : m - r \in \mathbb{P},\ m + r \in \mathbb{P},\ m - r \geq 3\},$$

with $R_{\min}(m) = +\infty$ when no such pair exists.

Since

$$p = m - r, \qquad q = m + r,$$

the reflected pair lies in $W_N = [3, 2N - 3]$ when

$$m - r \geq 3 \qquad \text{and} \qquad m + r \leq 2N - 3.$$

When $R_{\min}(m)$ is finite, the first condition is already included in its definition. The remaining finite-domain requirement is therefore

$$m + R_{\min}(m) \leq 2N - 3,$$

or equivalently,

$$R_{\min}(m) \leq 2N - m - 3.$$

Since $R_{\min}(m) = +\infty$ when no symmetric prime pair exists, this condition is also equivalent to positive reflected incidence:

$$\Sigma_N^-(s) > 0 \quad \Longleftrightarrow \quad R_{\min}(m) \leq 2N - m - 3.$$

When $R_{\min}(m)$ is finite, the primes

$$m - R_{\min}(m) \qquad \text{and} \qquad m + R_{\min}(m)$$

are odd, so $R_{\min}(m)$ has parity opposite to m. Since

$$2N - m - 3 \equiv m + 1 \pmod 2,$$

$2N - m - 3$ also has parity opposite to m. Hence $R_{\min}(m)$ and $2N - m - 3$ have the same parity. Therefore, since

$$\Sigma_N^-(s) = 0 \quad \Longleftrightarrow \quad R_{\min}(m) > 2N - m - 3,$$

and $R_{\min}(m)$ and $2N - m - 3$ have the same parity, this is equivalent to

$$R_{\min}(m) \geq 2N - m - 1.$$

Since $R_{\min}(m)$ is a nonnegative integer, $R_{\min}(m) < 2$ means either $R_{\min}(m) = 0$, giving the diagonal prime pair (m, m), or $R_{\min}(m) = 1$, giving the twin-prime pair $(m-1, m+1)$. In either case, reflected incidence is positive at every state associated with m for which $0 \le s \le 2N-6$.

If $R_{\min}(m) \ge 2$, the last such state at which reflected incidence fails is

$$N_-(m) = \frac{m + R_{\min}(m) + 1}{2}.$$

The corresponding displacement is

$$s_-(m) = 2\big(m - N_-(m)\big) = m - R_{\min}(m) - 1,$$

and at that state

$$2N_-(m) - s_-(m) = 2R_{\min}(m) + 2.$$

For $m \ge 3$, the finite values of $R_{\min}(m)$ coincide with OEIS A047160 [9]; the present definition uses $+\infty$, rather than -1, when no such prime pair exists.

3.3. First Failures and the MDE

At state N, a translated failure is an even displacement

$$s \in 2\mathbb{Z}_{\ge 2}, \qquad s \le 2N-6,$$

for which

$$\Sigma_N^+(s) = 0.$$

Equivalently,

$$g^*(s) + s \ge 2N - 1.$$

When such a displacement exists, define the first translated failure by

$$F_+(N) = \min\Big\{s \in 2\mathbb{Z}_{\ge 2} : s \le 2N-6, \; g^*(s) + s \ge 2N - 1\Big\}.$$

Then

$$\mathrm{MDE}_+(N) = F_+(N) - 2.$$

If no translated failure occurs for $0 \le s \le 2N-6$, then

$$\mathrm{MDE}_+(N) = 2N - 6.$$

For the reflected branch, when a failure occurs, define

$$F_-(N) = \min\Big\{s \in 2\mathbb{Z}_{\ge 0} : s \le 2N-6, \; R_{\min}\Big(N + \frac{s}{2}\Big) \ge N - \frac{s}{2} - 1\Big\}.$$

For either branch, if no failure occurs over $0 \le s \le 2N-6$, set

$$F_\pm(N) = +\infty.$$

Define the first combined failure by

$$F(N) = \min\{F_+(N), F_-(N)\}.$$

If $F_-(N) = 0$, then $\mathrm{MDE}_-(N)$ does not exist. If $F_-(N) \ge 2$, then reflected incidence is positive at every even displacement through $F_-(N) - 2$ and fails at $F_-(N)$. Hence

$$\mathrm{MDE}_-(N) = F_-(N) - 2.$$

If no reflected failure occurs for $0 \le s \le 2N-6$, then

$$\mathrm{MDE}_-(N) = 2N - 6.$$

Whenever both branch-specific envelopes exist,

$$\mathrm{MDE}(N) = \min\{\mathrm{MDE}_+(N), \mathrm{MDE}_-(N)\}.$$

The formulas above reduce the computation of the MDEs to the single-parameter quantities $g^*(s)$ and $R_{\min}(m)$.

3.4. Computational Verification for $N \le 3 \times 10^7$

The computation evaluated the full finite-domain displacement range

$$0 \le s \le 2N - 6$$

for every

$$3 \le N \le 3 \times 10^7.$$

The corresponding single-parameter ranges are

$$2 \le s \le 6 \times 10^7 - 6$$

for the translated branch and

$$3 \le m \le 6 \times 10^7 - 3$$

for the reflected branch. Primality information was generated by an Eratosthenes sieve, and the preceding conditions were used to locate the first failure of each branch. As a consistency check, these conditions reproduce the incidence values and resulting branch-specific MDEs for the example $N = 20$ in Table 1.

The combined MDE exists at every tested state. Only six states have no failure in either branch over the entire range $0 \le s \le 2N-6$, the largest being $N = 56$. At every other tested state, the reflected branch has a failing displacement. Among the 29,859,539 states at which both branches fail, the reflected branch fails strictly first in every case; no ties are observed. Figure 2 shows the normalized branch-specific and combined MDEs over the computed range.

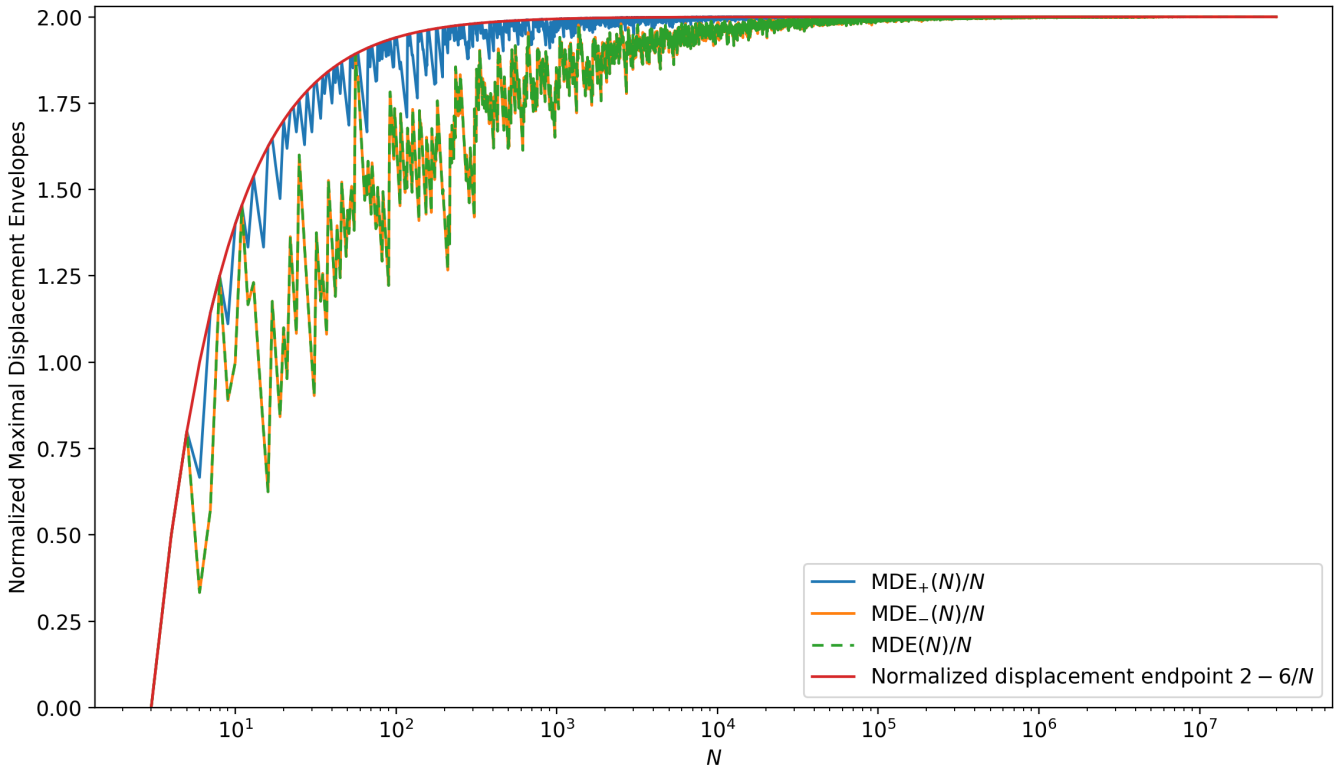


Figure 2. *Normalized branch-specific and combined Maximal Displacement Envelopes* $\mathrm{MDE}_+(N)/N$, $\mathrm{MDE}_-(N)/N$, *and* $\mathrm{MDE}(N)/N$ *over the computed range. The upper reference curve* $2 - 6/N$ *is the normalized endpoint of the displacement range. Throughout the computed range,* $\mathrm{MDE}(N)/N = \mathrm{MDE}_-(N)/N$, *so the combined and reflected normalized envelopes overlap.*

The persistent ordering

$$\mathrm{MDE}_-(N) < \mathrm{MDE}_+(N)$$

observed over the computed range is consistent with the different prime-density regimes sampled by the two branches near the maximal admissible displacement $s = 2N - 6$.

Let

$$B = (2N - 6) - s,$$

so that B is the distance from the displacement endpoint. Then

$$s = 2N - 6 - B.$$

For the translated branch, the bounded-domain condition gives

$$3 \le p \le B + 3,$$

while

$$p + s \in [2N - 3 - B,\ 2N - 3].$$

Thus translated incidence pairs a prime from the lower interval

$$[3, B + 3]$$

with a prime from the upper interval

$$[2N - 3 - B,\ 2N - 3].$$

For the reflected branch,

$$p + q = 4N - 6 - B,$$

and both primes must lie in

$$[2N - 3 - B,\ 2N - 3].$$

The following comparison is heuristic. It uses the approximate local prime density $1/\log x$ to estimate the typical number of available prime pairings in the two branches. It is intended to explain their different incidence behavior near the displacement endpoint, not to predict the extreme first-failure locations that determine the MDE.

For

$$1 \ll B \ll N,$$

the prime density in the lower interval is approximately $1/\log B$, while the prime density in the upper interval is approximately

$1/\log N$. For the translated branch, the expected incidence count is of order

$$\frac{B}{\log B \log N}$$

and for the reflected branch, the expected incidence count is of order

$$\frac{B}{(\log N)^2}. \tag{1}$$

Since $B < N$,

$$\log B < \log N,$$

so the translated branch has the larger expected incidence count at the same distance B from the endpoint. It is therefore expected to remain positive closer to $s = 2N - 6$, consistent with the observed ordering

$$\mathrm{MDE}_-(N) < \mathrm{MDE}_+(N).$$

Appendix I further analyzes the reflected-branch computational data by comparing a two-parameter power-law model aN^b with the log-corrected inverse model $c(\log N)^k/N$ for the logarithmically smoothed normalized reflected-branch gap from its conjectured asymptotic limit. Both fitted gaps tend to 0 as $N \to \infty$, and therefore the corresponding fitted normalized reflected MDEs tend to the asymptotic value 2.

4. Equivalent Formulations and Analytic Results for PIC

Section 3 expressed the two incidence branches through $g^*(s)$ and $R_{\min}(m)$. The present section formulates the translated branch as a prime-difference problem with fixed difference s and the reflected branch as a bounded Goldbach representation problem, and derives an exceptional-set bound for displacements at which either branch has zero incidence. For every fixed $0 < \varepsilon < 2$, the even displacement range considered is

$$0 \le s \le (2 - \varepsilon)N.$$

4.1. The Translated Branch as a Prime-Difference Problem

For $X \ge 5$, define

$$\mathcal{D}(X) = \{q - p : p, q \in \mathbb{P},\ 3 \le p < q \le X\}.$$

For every positive even $s \le 2N - 6$,

$$\Sigma_N^+(s) > 0 \iff s \in \mathcal{D}(2N - 3).$$

Equivalently,

$$\Sigma_N^+(s) > 0 \iff g^*(s) + s \le 2N - 3.$$

This places the translated branch among classical problems on fixed prime differences. Maillet's conjecture asserts that every positive even integer occurs as the difference of two primes [10]. PIC implies Maillet's conjecture. Under PIC,

$$\frac{\mathrm{MDE}(N)}{N} \to 2,$$

so $\mathrm{MDE}(N) \to \infty$. Hence, for any fixed positive even s, one has $s \le \mathrm{MDE}(N)$ for all sufficiently large N, and therefore

$$\Sigma_N^+(s) > 0.$$

4.2. Exceptional-Set Bound for PIC(+) Failures

Let Λ denote the von Mangoldt function and define

$$T(X, k) = \sum_{k < n \le X} \Lambda(n)\Lambda(n - k).$$

For even k, define

$$\mathfrak{S}_2(k) = 2C_2 \prod_{\substack{\ell \mid k \\ \ell > 2}} \frac{\ell - 1}{\ell - 2}, \qquad C_2 = \prod_{\ell > 2} \left(1 - \frac{1}{(\ell - 1)^2}\right),$$

and set $\mathfrak{S}_2(k) = 0$ for odd k. For every positive even k,

$$\mathfrak{S}_2(k) \ge 2C_2.$$

Lavrik's mean-square estimate [11, 12] gives, for any fixed $B > 0$,

$$\sum_{1 \le k \le X} |T(X, k) - \mathfrak{S}_2(k)(X - k)|^2 = O_B\left(X^3 (\log X)^{-B}\right). \tag{2}$$

Proposition 4.1. *For any fixed $\eta > 0$ and $A > 0$, the number of positive even integers $s \le X$ satisfying*

$$g^*(s) > \eta s$$

is

$$O_{A,\eta}\left(X(\log X)^{-A}\right).$$

Proof. Consider a dyadic interval

$$Y < s \le 2Y$$

and set

$$\rho = \min\left\{\frac{\eta}{4}, \frac{1}{4}\right\}, \qquad H = \rho Y.$$

Partition $(Y, 2Y]$ into at most

$$\left\lceil \frac{Y}{H} \right\rceil = \left\lceil \frac{1}{\rho} \right\rceil$$

blocks of length at most H. For a block with left endpoint Y_j, define

$$X_j = Y_j + 2H.$$

Since every s in the block satisfies

$$Y_j < s \le Y_j + H,$$

we have

$$X_j - s = Y_j + 2H - s,$$

and therefore

$$H \le X_j - s < 2H. \tag{3}$$

For even s, the main term $\mathfrak{S}_2(s)(X_j - s)$ in (2) satisfies, by (3),

$$\mathfrak{S}_2(s)(X_j - s) \ge 2C_2H. \tag{4}$$

Since each block lies in $(Y, 2Y]$, its left endpoint satisfies

$$Y < Y_j < 2Y.$$

Also, $H \le Y/4$, so

$$X_j = Y_j + 2H < 2Y + \frac{Y}{2} = \frac{5Y}{2}.$$

Hence

$$Y < X_j < \frac{5Y}{2}.$$

Applying (2) with the cutoff X_j gives

$$\sum_{1 \le k \le X_j} \left|T(X_j, k) - \mathfrak{S}_2(k)(X_j - k)\right|^2 = O_B\left(X_j^3 (\log X_j)^{-B}\right).$$

Since $Y < X_j < 5Y/2$, the right-hand side is

$$O_B\left(Y^3 (\log Y)^{-B}\right).$$

Define an even s as exceptional in the block if

$$\left|T(X_j, s) - \mathfrak{S}_2(s)(X_j - s)\right| \ge C_2H.$$

If E_j denotes the number of exceptional s in the block, then each such s contributes at least $(C_2H)^2$ to the preceding mean-square sum. Hence

$$E_j (C_2H)^2 = O_B\left(Y^3 (\log Y)^{-B}\right).$$

Since $H = \rho Y$,

$$E_j = O_{B,\eta}\left(Y (\log Y)^{-B}\right).$$

Thus, for a nonexceptional even s,

$$\left|T(X_j, s) - \mathfrak{S}_2(s)(X_j - s)\right| < C_2H.$$

Hence, by (4),

$$T(X_j, s) > \mathfrak{S}_2(s)(X_j - s) - C_2H \ge C_2H.$$

To deduce the existence of a prime pair from this lower bound, we must exclude terms in which either argument of Λ is a proper prime power. There are $O(X_j^{1/2})$ proper prime powers up to X_j, and each term contributes at most $(\log X_j)^2$. Hence their total contribution is

$$O\left(X_j^{1/2} (\log X_j)^2\right) = O\left(Y^{1/2} (\log Y)^2\right) = o(H).$$

Therefore, for sufficiently large Y, the lower bound $T(X_j, s) > C_2H$ cannot be accounted for entirely by proper prime powers. Hence at least one prime pair $p, p + s$ satisfies

$$p \le X_j - s < 2H.$$

By the choice of ρ,

$$p < 2H \le \eta s,$$

so

$$g^*(s) \le \eta s.$$

Summing over the at most $\lceil 1/\rho \rceil$ blocks, choosing $B > A + 2$, and then summing over dyadic intervals proves the result. □

We now apply Proposition 4.1 to the PIC(+) incidence condition in order to show that, for every fixed $0 < \varepsilon < 2$, failures in the range

$$0 \le s \le (2 - \varepsilon)N$$

form an exceptional set of order

$$O_{A,\varepsilon}\left(N (\log N)^{-A}\right).$$

Let $0 < \varepsilon < 2$ and define

$$\eta = \frac{\varepsilon}{2(2 - \varepsilon)}.$$

If

$$s \le (2 - \varepsilon)N,$$

then

$$2N - s \ge \frac{\varepsilon}{2 - \varepsilon} s.$$

Outside the exceptional set of Proposition 4.1,

$$g^*(s) \le \eta s.$$

Since

$$\eta = \frac{\varepsilon}{2(2 - \varepsilon)},$$

we have

$$\frac{\varepsilon}{2 - \varepsilon} = 2\eta.$$

Therefore, for every s satisfying

$$\eta s \geq 3,$$

we have

$$\eta s \leq \frac{\varepsilon}{2-\varepsilon} s - 3.$$

Equivalently,

$$s \geq \frac{3}{\eta} = \frac{6(2-\varepsilon)}{\varepsilon}.$$

Hence, outside the exceptional set of Proposition 4.1, every positive even s satisfying

$$\frac{6(2-\varepsilon)}{\varepsilon} \leq s \leq (2-\varepsilon)N$$

also satisfies

$$g^*(s) \leq \frac{\varepsilon}{2-\varepsilon} s - 3 \leq 2N - s - 3.$$

Thus

$$g^*(s) + s \leq 2N - 3,$$

so

$$\Sigma_N^+(s) > 0.$$

The remaining positive even values

$$s < \frac{6(2-\varepsilon)}{\varepsilon}$$

form a finite set depending only on ε and may be included in the exceptional set. Therefore the number of PIC(+) failures in the range

$$0 \leq s \leq (2-\varepsilon)N$$

is

$$O_{A,\varepsilon}\left(N(\log N)^{-A}\right).$$

4.3. Exceptional-Set Bound for PIC(-) Failures

Coppola and Laporta [13], in Theorem 1 and its subsequent corollary, proved that, for arbitrary fixed positive constants θ and A, if

$$Y^{5/8+\theta} < U < Y, \tag{5}$$

then, apart from

$$O_{A,\theta}\left(Y(\log Y)^{-A}\right) \tag{6}$$

exceptions among even targets $M = 2m$ in a dyadic interval $Y < M < 2Y$, there are primes p_1, p_2 satisfying

$$M = p_1 + p_2, \qquad |p_i - m| < U \quad (i = 1, 2). \tag{7}$$

Taking

$$\theta = \frac{1}{16}, \qquad U = Y^{3/4},$$

gives

$$Y^{5/8+\theta} = Y^{11/16} < Y^{3/4} < Y,$$

so the condition (5) is satisfied.

Proposition 4.2. *For any fixed $\delta > 0$ and $A > 0$, the number of even integers $M = 2m \leq X$ for which there is no prime representation*

$$M = p_1 + p_2$$

satisfying

$$|p_i - m| < \delta m \qquad (i = 1, 2) \tag{8}$$

is

$$O_{A,\delta}\left(X(\log X)^{-A}\right).$$

Proof. Consider a dyadic interval

$$Y < M < 2Y.$$

With

$$U = Y^{3/4},$$

and $m = M/2 > Y/2$,

$$\frac{U}{m} < \frac{Y^{3/4}}{Y/2} = 2Y^{-1/4}. \tag{9}$$

For fixed $\delta > 0$, if

$$Y > \left(\frac{2}{\delta}\right)^4, \tag{10}$$

then (9) gives

$$\frac{U}{m} < \delta,$$

and therefore

$$U < \delta m. \tag{11}$$

For every nonexceptional M in the Coppola–Laporta estimate, (7) and (11) therefore give

$$|p_i - m| < U < \delta m \qquad (i = 1, 2).$$

Hence every failure of (8) in a dyadic interval satisfying (10) belongs to the Coppola–Laporta exceptional set. By (6), the number of such failures in that interval is

$$O_{A,\delta}\left(Y(\log Y)^{-A}\right). \tag{12}$$

To sum (12) over $M \leq X$, first consider the dyadic intervals with $Y \geq X^{1/2}$. On these intervals,

$$\log Y \geq \frac{1}{2} \log X,$$

and the sum of their lengths Y is $O(X)$. Their total exceptional count is therefore

$$O_{A,\delta}\left(X(\log X)^{-A}\right).$$

The remaining even integers satisfy $M \le X^{1/2}$ and contribute at most $O(X^{1/2})$ additional exceptions. Since

$$X^{1/2} = O_A\Big(X(\log X)^{-A}\Big),$$

the total number of exceptions up to X is

$$O_{A,\delta}\Big(X(\log X)^{-A}\Big).$$

□

We now apply Proposition 4.2 to the PIC$(-)$ incidence condition in order to show that, for every fixed $0 < \varepsilon < 2$, failures in the range

$$0 \le s \le (2-\varepsilon)N$$

form an exceptional set of order

$$O_{A,\varepsilon}\Big(N(\log N)^{-A}\Big).$$

Let

$$\delta = \frac{\varepsilon}{2(4-\varepsilon)},$$

so that

$$\frac{\varepsilon}{4-\varepsilon} = 2\delta. \tag{13}$$

For an even displacement

$$0 \le s \le (2-\varepsilon)N,$$

let

$$m = N + \frac{s}{2}, \qquad M = 2m = 2N + s.$$

The displacement bound gives

$$m = N + \frac{s}{2} \le N + \frac{2-\varepsilon}{2}N = \frac{4-\varepsilon}{2}N.$$

Therefore

$$N \ge \frac{2m}{4-\varepsilon},$$

and hence

$$2N - m \ge \frac{\varepsilon}{4-\varepsilon}m = 2\delta m, \tag{14}$$

where the final equality follows from (13).

Also,

$$2N \le M = 2N + s \le (4-\varepsilon)N.$$

Apply Proposition 4.2 with

$$X = (4-\varepsilon)N.$$

Outside an exceptional set of

$$O_{A,\varepsilon}\Big(N(\log N)^{-A}\Big) \tag{15}$$

even values of M, there are primes p_1, p_2 satisfying

$$p_1 + p_2 = 2m, \qquad |p_i - m| < \delta m.$$

Ordering $p_1 \le p_2$ and writing

$$p_1 = m - r, \qquad p_2 = m + r,$$

gives

$$r < \delta m.$$

For $M \ge 6$, both primes are odd, so $p_1 \ge 3$ and the pair is admissible in the definition of $R_{\min}(m)$. Consequently,

$$R_{\min}(m) < \delta m. \tag{16}$$

Since $m \ge N$, for

$$N \ge \frac{3}{\delta} = \frac{6(4-\varepsilon)}{\varepsilon},$$

we have

$$\delta m \ge 3.$$

Therefore

$$\delta m \le 2\delta m - 3. \tag{17}$$

Combining (16), (17), and (14) gives

$$R_{\min}(m) < \delta m \le 2\delta m - 3 \le 2N - m - 3.$$

Hence

$$R_{\min}(m) \le 2N - m - 3,$$

and by the reflected-incidence condition,

$$\Sigma_N^-(s) > 0.$$

Finally, the map

$$s \longmapsto M = 2N + s$$

is injective. Therefore, the number of even displacements

$$0 \le s \le (2-\varepsilon)N$$

for which $\Sigma_N^-(s) = 0$ is bounded by the exceptional count (15). Therefore, for every fixed $0 < \varepsilon < 2$ and $A > 0$, for all sufficiently large N, the number of PIC$(-)$ failures in this range is

$$O_{A,\varepsilon}\Big(N(\log N)^{-A}\Big).$$

Remark 4.3. The Coppola–Laporta estimate also gives a quantitative almost-all bound for $R_{\min}(m)$, the least distance from the midpoint m to a symmetric prime pair $m - r, m + r$. Fix $0 < \eta < 3/8$. For every $A > 0$, all but

$$O_{A,\eta}\Big(X(\log X)^{-A}\Big)$$

even targets $M = 2m \le X$ satisfy

$$R_{\min}(m) \le C_\eta m^{5/8+\eta}, \tag{18}$$

where $C_\eta > 0$ depends only on η. Hence

$$\frac{R_{\min}(m)}{m} \to 0$$

outside an exceptional set of density zero.

4.4. Mathematical Equivalence of PIC Through Arithmetic Decomposition

The single-parameter formulations developed above also provide an arithmetic decomposition of PIC. Specifically, Conjecture 2.1 is equivalent to the conjunction of the following four conditions:

(i) the binary Goldbach condition holds for every $N \ge 3$;

(ii) $g^*(s) < \infty$ for every positive even s;

(iii)
$$g^*(s) = o(s) \qquad (s \to \infty,\ s \text{ even});$$

(iv)
$$R_{\min}(m) = o(m) \qquad (m \to \infty).$$

Condition (ii) is Maillet's conjecture in the notation used here.

Necessity. Suppose PIC holds. Translated incidence at $s = 0$ is automatic, whereas existence of the reflected MDE requires

$$\Sigma_N^-(0) > 0.$$

Thus existence of the combined MDE for every $N \ge 3$ is equivalent to the binary Goldbach condition at every state, giving condition (i).

Moreover,

$$\frac{\mathrm{MDE}(N)}{N} \to 2$$

implies $\mathrm{MDE}(N) \to \infty$. Hence every fixed positive even s eventually lies below the combined MDE, so

$$\Sigma_N^+(s) > 0$$

for sufficiently large N. By the translated single-parameter formulation, this requires $g^*(s) < \infty$, giving condition (ii).

If condition (iii) failed, there would exist $c > 0$ and positive even integers $s_k \to \infty$ such that

$$g^*(s_k) \ge c s_k.$$

At the last state at which translation by s_k fails,

$$N_k = N_+(s_k) = \frac{s_k + g^*(s_k) + 1}{2}.$$

Since translated incidence fails at s_k,

$$\mathrm{MDE}(N_k) < s_k,$$

and therefore

$$\frac{\mathrm{MDE}(N_k)}{N_k} < \frac{2s_k}{s_k + g^*(s_k) + 1} \le \frac{2}{1+c} < 2,$$

contradicting the PIC limit. Hence

$$g^*(s) = o(s).$$

Similarly, if condition (iv) failed, there would exist a constant $0 < c < 1$ and integers $m_k \to \infty$ such that

$$R_{\min}(m_k) \ge c m_k.$$

At the last state associated with m_k at which reflected incidence fails,

$$N_k = N_-(m_k) = \frac{m_k + R_{\min}(m_k) + 1}{2},$$

and the corresponding failing displacement is

$$s_-(m_k) = m_k - R_{\min}(m_k) - 1.$$

Hence

$$\mathrm{MDE}(N_k) < s_-(m_k),$$

so

$$\frac{\mathrm{MDE}(N_k)}{N_k} < \frac{2\big(m_k - R_{\min}(m_k) - 1\big)}{m_k + R_{\min}(m_k) + 1} \le \frac{2(1-c)}{1+c} < 2.$$

This again contradicts the PIC limit, and therefore

$$R_{\min}(m) = o(m).$$

Sufficiency. Conversely, assume conditions (i)–(iv), and fix $0 < \varepsilon < 2$. For the translated branch, set

$$\eta = \frac{\varepsilon}{2(2-\varepsilon)}.$$

By condition (iii), for all sufficiently large positive even s,

$$g^*(s) \le \eta s.$$

If

$$s \le (2-\varepsilon)N,$$

then

$$g^*(s) + s \le (1+\eta)(2-\varepsilon)N = \left(2 - \frac{\varepsilon}{2}\right)N.$$

For sufficiently large N,

$$\left(2 - \frac{\varepsilon}{2}\right)N \le 2N - 3,$$

so

$$\Sigma_N^+(s) > 0.$$

The finitely many smaller positive even values of s are also covered for sufficiently large N, because condition (ii) makes each corresponding value $g^*(s)$ finite. Thus translated incidence is positive throughout

$$0 \le s \le (2-\varepsilon)N$$

for all sufficiently large N.

For the reflected branch, set

$$\delta = \frac{\varepsilon}{2(4-\varepsilon)}.$$

For

$$m = N + \frac{s}{2}, \qquad s \le (2-\varepsilon)N,$$

the preceding reflected-window calculation gives

$$2N - m \ge \frac{\varepsilon}{4-\varepsilon} m = 2\delta m.$$

By condition (iv), for all sufficiently large m,

$$R_{\min}(m) \le \delta m.$$

Since $m \ge N$, sufficiently large N also gives

$$\delta m \ge 3.$$

Consequently,

$$R_{\min}(m) \le \delta m \le 2\delta m - 3 \le 2N - m - 3,$$

and therefore

$$\Sigma_N^-(s) > 0$$

throughout the same displacement range.

Condition (i) guarantees existence of the reflected MDE for every $N \ge 3$, while the translated MDE always exists because $\Sigma_N^+(0) > 0$. Hence the combined MDE exists for every $N \ge 3$. For every fixed $\varepsilon > 0$, the preceding bounds imply, for all sufficiently large N,

$$\mathrm{MDE}(N) \ge (2-\varepsilon)N - 2.$$

Together with the finite-domain bound

$$\mathrm{MDE}(N) \le 2N - 6,$$

this gives

$$2 - \varepsilon - \frac{2}{N} \le \frac{\mathrm{MDE}(N)}{N} \le 2 - \frac{6}{N}.$$

Letting $N \to \infty$ and then $\varepsilon \to 0$ yields

$$\frac{\mathrm{MDE}(N)}{N} \to 2.$$

The decomposition therefore separates PIC into two universal existence requirements, conditions (i)–(ii), and two asymptotic sublinear-growth requirements, conditions (iii)–(iv). The exceptional-set results proved in the preceding subsections are almost-all counterparts of the latter two conditions: for every fixed positive relative bound, violations are confined to exceptional sets of asymptotic density zero. PIC requires the corresponding behavior to hold for every admissible displacement, rather than for almost all admissible displacements.

4.5. Common Exceptional-Set Bound for PIC(-) and PIC(+)

Corollary 4.4. *For every fixed* $0 < \varepsilon < 2$ *and* $A > 0$*, and every sufficiently large* N*, the number of even displacements*

$$0 \le s \le (2-\varepsilon)N$$

at which at least one PIC branch has zero incidence is

$$O_{A,\varepsilon}\left(N(\log N)^{-A}\right).$$

Proof. The preceding results give

$$O_{A,\varepsilon}\left(N(\log N)^{-A}\right)$$

failures for each branch. The union of the two exceptional sets has the same order. □

The corollary is an exceptional-set statement in the displacement variable s: for every fixed $0 < \varepsilon < 2$, the proportion of even displacements

$$0 \le s \le (2-\varepsilon)N$$

at which at least one PIC branch has zero incidence tends to zero as $N \to \infty$. It does not exclude a failure at any particular displacement, including $s = 0$, nor does it determine the location $F(N)$ of the first combined incidence failure, which determines the MDE. The present asymptotic estimates do not provide an explicit numerical threshold for "sufficiently large N."

5. Conclusions

PIC reframes the binary Goldbach condition as the zero-displacement reflected case of a broader problem of prime incidence under bounded reflections and translations over all even displacements, and considers whether

$$\frac{\mathrm{MDE}(N)}{N} \to 2.$$

Analytically, for every fixed $0 < \varepsilon < 2$ and $A > 0$, and all sufficiently large N, the number of even displacements

$$0 \le s \le (2-\varepsilon)N$$

at which at least one branch has zero incidence is

$$O_{A,\varepsilon}\left(N(\log N)^{-A}\right).$$

Thus such displacements have asymptotic density zero in every range

$$0 \le s \le (2-\varepsilon)N, \qquad 0 < \varepsilon < 2.$$

These bounds estimate the number of failing displacements but not the location of the first one, which determines the MDE. A stronger result would be to show that, for every fixed $0 < \varepsilon < 2$, both incidence branches are positive throughout

$$0 \le s \le (2-\varepsilon)N$$

for all sufficiently large N. Since this range includes $s = 0$, such a result would in particular establish the binary Goldbach condition for all sufficiently large even integers. It would also give

$$\mathrm{MDE}(N) \ge (2-\varepsilon)N - 2,$$

while the finite displacement range always gives

$$\mathrm{MDE}(N) \le 2N - 6.$$

Hence

$$2 - \varepsilon - \frac{2}{N} \le \frac{\mathrm{MDE}(N)}{N} \le 2 - \frac{6}{N}.$$

Since $\varepsilon > 0$ is arbitrary, this would establish

$$\frac{\mathrm{MDE}(N)}{N} \to 2.$$

Computationally, the combined MDE exists for every tested state $3 \le N \le 3 \times 10^7$ over the full even displacement range $0 \le s \le 2N - 6$. Throughout the computed range, $\mathrm{MDE}(N) = \mathrm{MDE}_{-}(N)$, and whenever both branches fail, the reflected branch fails first, with no ties.

Future research includes determining the asymptotic behavior of $\mathrm{MDE}(N)/N$; obtaining sharper bounds on the growth of $g^*(s)$ and $R_{\min}(m)$; determining whether the reflected branch sets the combined MDE beyond the computed range; extending the analytic estimates to displacements increasingly close to $2N-6$; and extending the normalized MDE computation beyond $N = 3 \times 10^7$.

Abbreviations

bGC	Binary Goldbach Conjecture
LHS	Left-hand Side
LRPT	Left–Right Partition Table
MDE	Maximal Displacement Envelope
PIC	Prime Incidence Conjecture
PP	Prime–prime Row
RHS	Right-hand Side
TRE	Top-right Entry

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Author Contributions

Ioannis Papadakis: Conceptualization, Formal Analysis, Methodology, Software, Writing – original draft

Data Availability Statement

The Python code, computational scripts, numerical search algorithms, and numerical data used in this study are available upon request from the corresponding author.

Conflicts of Interest

The author declares no conflicts of interest.

Appendix

Appendix I: Empirical Model Comparison for $2 - \mathrm{MDE}_{-}(N)/N$

The computational results show that the reflected branch determines the combined MDE throughout the computed range. To examine the finite-range decay of the normalized reflected-branch gap from its conjectured asymptotic limit, define

$$Y(N) = 2 - \frac{\mathrm{MDE}_{-}(N)}{N}.$$

To reduce the effect of N-to-N arithmetic fluctuations on the data-fitting procedure, $Y(N)$ is smoothed on a logarithmic N scale. The interval $3 \le N \le 3 \times 10^7$ is divided into 20 equal-width bins per decade in $\log_{10} N$, beginning at $\log_{10} 3$. Empty bins are omitted. The arithmetic mean of $Y(N)$ is computed within each remaining bin, and a centered moving average over five successive nonempty bins is then applied to the resulting sequence. Each smoothed value is assigned to the geometric center, in N, of its central bin. Denote the resulting smoothed gap by $\overline{Y}(N)$.

As a benchmark, consider the two-parameter power-law model

$$\overline{Y}(N) = aN^b.$$

Linear regression of $\log \overline{Y}(N)$ on $\log N$ over the full smoothed range gives

$$\overline{Y}(N) \approx 9.794995\, N^{-0.644758}, \qquad R^2_{\log \overline{Y}} = 0.980245.$$

Table 2 shows that the fitted exponent is not stable across successive subranges of the same smoothed data.

Table 2. *Power-law exponent fitted over successive ranges of the smoothed normalized reflected-branch gap from its conjectured asymptotic limit.*

Fit range	Fitted exponent b
$10^2 \le N \le 10^4$	−0.534301
$10^3 \le N \le 10^5$	−0.689397
$10^4 \le N \le 10^6$	−0.764473
$10^5 \le N \le 10^7$	−0.811085
$10^6 \le N \le 3 \times 10^7$	−0.834715

The systematic variation of b across these ranges does not support a scale-independent power-law exponent over the computed domain. We therefore compare the power law with the two-parameter model

$$\overline{Y}(N) = c\,\frac{(\log N)^k}{N}, \tag{19}$$

where log denotes the natural logarithm. Equivalently,

$$\log\bigl(N\overline{Y}(N)\bigr) = \log c + k \log\log N,$$

so that c and k are obtained by linear regression of $\log(N\overline{Y}(N))$ on $\log\log N$.

Over the full smoothed range, this fit gives

$$\overline{Y}(N) \approx 0.908192\,\frac{(\log N)^{2.729814}}{N}, \qquad R^2_{\log \overline{Y}} = 0.997552.$$

Restricting the fit to $N \ge 10^3$ gives

$$c = 1.022018, \qquad k = 2.693833, \qquad R^2_{\log \overline{Y}} = 0.999884.$$

For the model in (19), the local log–log slope is

$$\frac{d \log \overline{Y}}{d \log N} = -1 + \frac{k}{\log N}.$$

Thus its effective power-law slope becomes more negative as N increases and approaches -1, consistent with the direction of the subrange exponent drift in Table 2.

Figure 3 compares the smoothed normalized reflected-branch gap with the two fitted models.

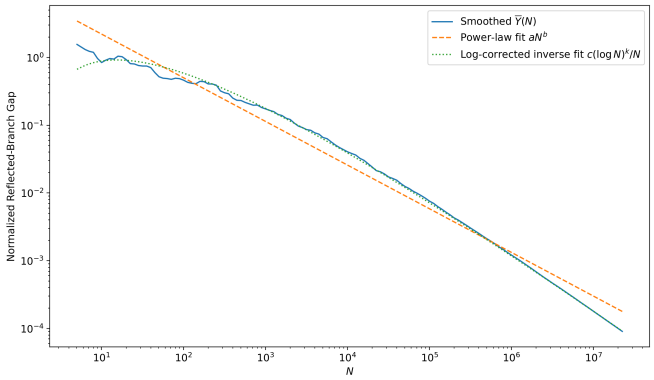


Figure 3. *Logarithmically smoothed normalized reflected-branch gap from its conjectured asymptotic limit, $\overline{Y}(N)$, over the computed range, together with the two-parameter fits aN^b and $c(\log N)^k/N$.*

Over the computed range, the log-corrected inverse model provides a closer fit to the log-transformed smoothed data than the power law with the same number of fitted parameters. The value $k \approx 2.7$ is an empirical finite-range parameter; no theoretical significance or extrapolation beyond $N = 3 \times 10^7$ is claimed. Both fitted models tend to zero if extrapolated, so the comparison concerns the finite-range shape of the observed smoothed gap and does not verify the asymptotic limit asserted by Conjecture 2.1.

Remark I.1. Equation (18) gives the almost-all asymptotic bound $R_{\min}(m)/m \to 0$ outside an exceptional set of density zero. At zero displacement, $m = N$. For sufficiently large nonexceptional N, the Coppola–Laporta estimate supplies a distinct symmetric prime pair, so the normalized Minimum Distinct Radius $\mathrm{MDR}(N)/N$, introduced and studied in the FMA framework [7], also tends to 0 outside an exceptional set of density zero. The empirical results raise the stronger question of the rate at which these normalized quantities tend to zero. Near the reflected displacement endpoint, the endpoint-density heuristic gives expected incidence of order $B/(\log N)^2$ when the admissible reflected interval has size B, as in (1). As future work, the normalized MDR data could be analyzed using a polylogarithmic fitting model analogous to that in Figure 3.

The empirical form

$$\overline{Y}(N) \approx c\,\frac{(\log N)^k}{N}$$

suggests a possible polylogarithmic scale for the reflected-branch finite-domain endpoint gap. Since

$$(2N-6) - \mathrm{MDE}_-(N) = NY(N) - 6,$$

an open question is whether such a scale can be established analytically or related to a more precise characterization of the first reflected-failure displacement $F_-(N)$.

Appendix II: Autocorrelation, Convolution, and Fourier Representations of Prime Incidence

The two PIC incidence counts admit autocorrelation, self-convolution, and Fourier representations.

Let

$$f_N : \mathbb{Z} \to \{0,1\}$$

be the indicator function of the finite prime set $\mathcal{L}(N)$:

$$f_N(n) = \begin{cases} 1, & n \in \mathcal{L}(N), \\ 0, & n \notin \mathcal{L}(N). \end{cases}$$

Since $f_N(k) = 0$ for every $k \notin \mathcal{L}(N)$, the products $f_N(n)f_N(n+s)$ and $f_N(n)f_N(2N+s-n)$ are nonzero only when both arguments belong to $\mathcal{L}(N)$.

II.1. Translated Incidence as Autocorrelation

For an even displacement $s \geq 0$,

$$\Sigma_N^+(s) = \sum_{n\in\mathbb{Z}} f_N(n)f_N(n+s).$$

Thus the translated incidence count is the discrete autocorrelation of f_N at displacement s.

Define

$$\widehat{f_N}(\theta) = \sum_{n\in\mathbb{Z}} f_N(n)e^{-in\theta}.$$

Fourier inversion gives

$$\Sigma_N^+(s) = \frac{1}{2\pi}\int_0^{2\pi} |\widehat{f_N}(\theta)|^2 e^{is\theta}\,d\theta.$$

II.2. Reflected Incidence as Self-Convolution

For reflected incidence,

$$\Sigma_N^-(s) = \sum_{n\in\mathbb{Z}} f_N(n)f_N(2N+s-n).$$

With

$$(f*g)(t) = \sum_{n\in\mathbb{Z}} f(n)g(t-n),$$

this becomes

$$\Sigma_N^-(s) = (f_N * f_N)(2N+s).$$

Fourier inversion gives

$$\Sigma_N^-(s) = \frac{1}{2\pi}\int_0^{2\pi} \widehat{f_N}(\theta)^2 e^{i(2N+s)\theta}\,d\theta.$$

Thus translated incidence is represented by the autocorrelation coefficients of f_N, while reflected incidence is represented by the self-convolution coefficients of f_N.

The above Fourier representations provide an exact discrete spectral formulation of PIC incidence. Exact harmonic encodings of primes and Goldbach pairings were developed in [14]. Whether those specific harmonic constructions can be related to the spectral coefficients governing the present incidence counts is an open question. More generally, one may ask how many Fourier components, or what spectral bandwidth, is required to recover the first-failure displacement $F_\pm(N)$ and hence the combined boundary $F(N)$.

Appendix III: LRPT Evolution and Additive Prime-Pair Incidence

This appendix examines how LRPT evolution changes additive prime-pair incidence through two mechanisms: entry of new primes at the top-right boundary and new pairings among primes already represented. It also examines subsequent PP formation using only primes represented at the starting state.

III.1. LRPT Evolution and Represented Arithmetic Information

If q is an RHS entry of both $\mathrm{LRPT}(N)$ and $\mathrm{LRPT}(N+t)$, its paired LHS value at state $N+t$ is

$$r_q(t) = 2(N+t) - q = r_q(0) + 2t.$$

Each transition $N \mapsto N+1$ therefore increases the LHS value paired with q by 2. If q is prime, a PP row occurs at a state for which the corresponding LHS value is also prime.

Figure 4 illustrates this LRPT transition and the resulting PP alignment.

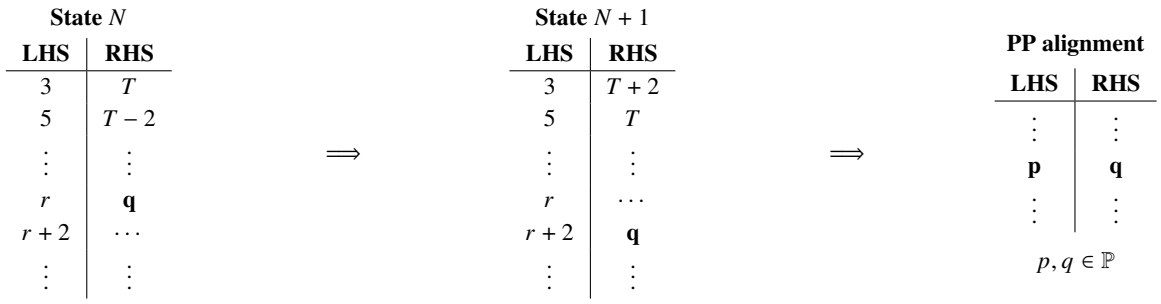


Figure 4. *Evolution of an RHS prime q under an LRPT transition. Here $T = 2N - 3$, and r is the LHS value paired with q at state N. When $N \mapsto N + 1$, the paired LHS value increases by 2. A PP row occurs when the paired LHS value is prime.*

For LRPT(N_0), the prime/composite status of each odd integer in W_{N_0} is part of the arithmetic information represented at state N_0. Under LRPT evolution, each value

$$2(N_0 + t) - 3, \qquad t \geq 1,$$

enters as a new TRE, and its prime/composite status is not represented in LRPT(N_0). By contrast, for $t \geq 1$, the positions in LRPT$(N_0 + t)$ of values already in W_{N_0} are determined by LRPT evolution.

This distinction concerns only the prime/composite status pattern represented in LRPT(N_0), not whether the primality of larger integers can be determined from other arithmetic information.

III.2. TRE Replenishment

A second aspect of LRPT evolution is the rate at which a new prime can enter through the top-right boundary. Let

$$T = \mathrm{TRE}(N) = 2N - 3$$

and let $q(T)$ be the least prime strictly greater than T. Define the TRE replenishment transition count by

$$\tau_{\mathrm{TRE}}(N) = \frac{q(T) - T}{2}.$$

Since both T and $q(T)$ are odd, $\tau_{\mathrm{TRE}}(N)$ is the number of LRPT transitions required for the TRE to reach $q(T)$.

For $N \geq 14$, we have $T \geq 25$, and Nagura's theorem gives a prime q satisfying

$$T < q < \frac{6}{5}T.$$

The least prime $q(T) > T$ therefore satisfies

$$q(T) - T < \frac{T}{5},$$

and division by the LRPT step size 2 gives

$$\tau_{\mathrm{TRE}}(N) < \frac{2N - 3}{10} < \frac{N}{5}.$$

For $N = 10, 11, 12, 13$, the transition counts are, respectively,

$$1, 2, 1, 3.$$

The value $N = 13$ gives

$$\tau_{\mathrm{TRE}}(13) = 3 > \frac{13}{5},$$

so the uniform bound obtained above is

$$\tau_{\mathrm{TRE}}(N) < \frac{N}{5}$$

for every $N \geq 14$.

This replenishment bound also limits the length of a hypothetical consecutive run of states with no PP row. If $PP(N) = 0$, then every TRE encountered before the next prime TRE must be composite; hence, with $\ell_{\mathrm{PP}}(N)$ denoting the number of consecutive states beginning at N for which $PP = 0$,

$$\ell_{\mathrm{PP}}(N) \leq \tau_{\mathrm{TRE}}(N).$$

Therefore, for $N \geq 14$,

$$\ell_{\mathrm{PP}}(N) < \frac{N}{5}.$$

The Baker–Harman–Pintz bound on prime gaps implies that the next prime above a sufficiently large threshold T lies at distance

$$O(T^{0.525})$$

[15]. Consequently,

$$\tau_{\mathrm{TRE}}(N) = O(N^{0.525}), \qquad \frac{\tau_{\mathrm{TRE}}(N)}{N} \to 0,$$

so

$$\tau_{\mathrm{TRE}}(N) = o(N).$$

III.3. PP Formation from Primes Represented at the Starting State

TRE replenishment is not the only way a later LRPT state can acquire a PP row. A PP row may also arise from a new

pairing of primes whose prime status was already represented at the starting state.

If $\mathrm{PP}(N) = 0$, then N is composite, since a prime N would make the diagonal row (N, N) PP. Let

$$P < N < Q$$

be the consecutive odd primes immediately below and above such a composite N, and set

$$M = \frac{P+Q}{2}.$$

The pair (P, Q) is PP at state M, while the diagonal pair (Q, Q) is PP at state Q.

Define

$$B = \begin{cases} M, & N \le M, \\ Q, & N > M. \end{cases}$$

Then $B \ge N$ and $\mathrm{PP}(B) > 0$. If $N < M$, then $N \ge P + 1$, so

$$M - N \le M - (P+1) = \frac{Q-P}{2} - 1.$$

If $N > M$, then $N \ge M + 1$, so

$$Q - N \le Q - (M+1) = \frac{Q-P}{2} - 1.$$

The case $N = M$ gives $B - N = 0$. Thus this construction yields the explicit bound

$$0 \le B - N \le \frac{Q-P}{2} - 1.$$

For $N \ge 25$, Nagura's theorem gives a prime in $(N, 6N/5)$ [16], so the least prime $Q > N$ satisfies

$$Q < \frac{6N}{5} < 2N - 3.$$

The same inequality $Q < 2N - 3$ is verified directly for $10 \le N \le 24$. Since also $P < N$, both bounding primes satisfy

$$P, Q \in \mathcal{L}(N).$$

The PP row at state B is consequently formed entirely from primes whose prime status was already represented at the starting state: it is (P, Q) when $N \le M$ and (Q, Q) when $N > M$.

This construction gives an explicit upper bound on the displacement to a later PP state formed from starting-state primes. It does not claim that B is the earliest such state.

Remark III.1. A stronger prime-presence property also holds: for every integer $N \ge 10$, each of the four intervals

$$\left[3, \frac{N}{2}\right], \qquad \left[\frac{N}{2}, N\right], \qquad \left[N, \frac{3N}{2}\right], \qquad \left[\frac{3N}{2}, 2N-3\right]$$

contains a prime. The first contains 3. For $N \ge 26$, Bertrand's postulate applied to $\lfloor N/2 \rfloor$ gives a prime in $(N/2, N)$, while Nagura's theorem applied to N and $3N/2$ gives primes in $(N, 6N/5)$ and $(3N/2, 9N/5)$, respectively; the latter lies below $2N - 3$ for $N > 15$ [16]. The remaining finite cases $10 \le N \le 25$ are verified directly.

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