

The Ordered Rectangular Band of Reflexive Generalized Inverses of a Matrix

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Abstract: This article studies the algebraic structure of the set of all reflexive generalized inverses of a real matrix using a sandwich-type binary operation and examines the compatibility of certain matrix order relations with this operation. The concept of generalized inverses arises when dealing with singular or rectangular matrices, where a standard inverse does not exist. The study shows that equipped with a sandwich operation, the entire set of reflexive generalized inverses of a matrix forms the structure of a particular type of semigroup, known as a rectangular band. Further, several algebraic properties of this semigroup are investigated in detail. In particular, the compatibility of certain well-known matrix order relations, namely Sussman's order and Mitsch's order, with the sandwich operation is examined. It is shown that these order relations are preserved under the defined operation, which enables the semigroup of reflexive generalized inverses to be viewed naturally as an ordered matrix semigroup. The results obtained in this study contribute to a deeper understanding of the relationship between generalized inverse theory, semigroup structures, and matrix partial orders, thereby providing a useful framework for further research on algebraic and order-theoretic properties of generalized inverses and their applications in diverse fields.

Keywords: Partial Order, Ordered Semigroup, Rectangular Band, Sussman's Order, Mitch's Order

1. Preliminaries

The general linear group $GL_n(F)$ where the field F is $\mathbb{R}[\mathbb{C}]$ is very fundamental in Linear Algebra. However if we include singular matrices, the structure is still well studied and it is the interesting class $M_n(F)$ of Von Neumann regular rings. A ring R is von Neumann regular, if any $x \in R$ admits a generalized inverse, i.e., there is an $x' \in R$ such that

$$xx'x = x \quad \text{and} \quad x'xx' = x'.$$

The generalized inverses of a matrix are an extension of the idea of a matrix inverse, which has some, but not all the properties of an ordinary inverse (and which reduces to the usual inverse when A is non singular). The concept of generalized inverses was first mentioned by Fredholm in 1903 who defined it for an integral operator. Later E H Moore defined a unique inverse for every finite square and rectangular matrix with entries in the complex numbers. An extensive

study on generalized inverses have been done by many authors like Ben Israel, Greville, C Radhakrishna Rao, Adetunde I A, Hanifa Zekraoui (see [1-4]).

We consider the semigroup of all reflexive generalized inverses of a matrix, discuss some properties of this semigroup and explore some compatible partial orders. We denote the set of all matrices of order $m \times n$ over the real field \mathbb{R} , by $M_{m \times n}(\mathbb{R})$.

Definition 1.1. For $A \in M_{m \times n}(\mathbb{R})$; consider the following equations:

1. $AGA = A$.
2. $GAG = G$.
3. $(AG)^* = AG$.
4. $(GA)^* = GA$, where $G \in M_{n \times m}(\mathbb{R})$ and G^* denotes the conjugate transpose of G .

The solutions $G \in M_{n \times m}(\mathbb{R})$ of equation (1) are called the *inner inverses* or $\{1\}$ -inverses of A . Note that every matrix A including $A = 0$, has at least one generalized inverse G , unless

A^{-1} exists, in which case $G = A^{-1}$, there are infinitely many such inverses G of A .

The solutions G of equations (1) and (2) are called textitreflexive(reciprocal) generalized inverses/semi inverses or $\{1, 2\}$ -inverses of A .

The matrix G satisfying (1),(2) and (3) is the textitweak generalized inverse of A .

The solution of all the four equations is called the generalized reciprocal or textitMoore Penrose inverse of A which is unique.

Besides these, many generalized inverses like Group inverse, Drazin inverses *etc.* have been defined and characterised so far.

A partial order ' \leq ' on a semigroup (S, \star) is binary relation which is reflexive, antisymmetric and transitive with respect to the binary composition, ' \star '.

An textitordered semigroup (S, \star, \leq) is a semigroup (S, \star) together with a compatible partial order, ie, a partial order \leq on S such that for any $a, b, c \in S$,

$$a \leq b \Rightarrow (a \star c) \leq (b \star c) \quad \text{and} \quad (c \star a) \leq (c \star b).$$

or, equivalently, $a \leq b$ and $c \leq d \Rightarrow (a \star c) \leq (b \star d) \forall a, b, c, d \in S$.

Example 1.1. Given a set S , $(P(S), \cup)$, where the power set $P(S)$ of S with the associative binary operation ' \cup ' is a semigroup and inclusion is a partial order on $(P(S), \cup)$.

Definition 1.2. (Mitsch' order) The order ' ρ' ' on a semigroup (S, \star) , given by

$$a \rho b \Leftrightarrow a = p \star a = p \star b = a \star q = b \star q \quad (1)$$

for some $p, q \in S$, is known as the Mitsch' order on (S, \star) .

Definition 1.3. (Sussman's order) (cf.[5]) The order relation ' $<$ ' on the semigroup (S, \star) given by

$$a < b \Leftrightarrow a \star a = a \star b = b \star a \text{ for } a, b \in (S, \star). \quad (2)$$

is termed as Sussman's order.

Definition 1.4. A semigroup (S, \star) is a band if all its elements are idempotent, i.e., $a \star a = a^2 = a \forall a \in S$ and a textitrectangular band if S is a band in which $a \star b \star a = a \forall a, b \in S$.

The set of all idempotents in a semigroup (S, \star) is denoted by $E(S)$. When S is a band, $S = E(S)$.

Definition 1.5. (cf.[6]) A semigroup (S, \star) is

1. weakly cancellative if $a \star x = a \star y$ and $x \star a = y \star a \Rightarrow x = y$ (or, $a \star x = a \star y$ and $x \star b = y \star b \Rightarrow x = y) \forall a, x, y \in S$.
2. conditionally commutative if $a \star b = b \star a \Rightarrow a \star x \star b = b \star x \star a \forall x \in S$.
3. quasi separative if

$$a \star a = a \star b = b \star a = b \star b \Rightarrow a = b \forall a, b \in S. \quad (3)$$

4. weakly separative if for $a, b \in S$,

$$a \star s \star a = a \star s \star b = b \star s \star a = b \star s \star b \quad (4)$$

$$\forall s \in S \Rightarrow a = b.$$

5. medial if $x \star a \star b \star y = x \star b \star a \star y \forall x, y, a, b \in S$.
6. E-inversive if for all $s \in S$, there exists an $x \in S$ such that $s \star x \in E(S)$.

Definition 1.6. A regular semigroup (S, \star) is said to be orthodox if the set of all idempotents, $E(S)$ is a subsemigroup of (S, \star) .

Definition 1.7. An element $x \in (S, \star)$ is a mid unit if $a \star x \star b = a \star b \forall a, b \in S$.

Ordered Matrix Semigroups

It is well known that the collection of all matrices of order n over the real field \mathbb{R} , denoted by $M_n(\mathbb{R})$ is a semigroup under the usual multiplication of matrices. Several partial orders have been defined on this matrix semigroup. Some of them are Conrad's partial order, Mitsch's order, Baksalary's left star and right star orders, Minus partial order, Lowener partial order, Sharp order. But many of them fail to be compatible with respect to the usual matrix multiplication. However, in [7], it is shown that $M_n(\mathbb{R})$ is an ordered matrix semigroup under

1. Conrad's order given by

$$A \rho B \Leftrightarrow ASA = ASB = BSA \forall S \in M_n(\mathbb{R}).$$

2. Left star partial order:

$$A(\leq)_\star B \Leftrightarrow A^*A = A^*B \text{ and } R(A) \subseteq R(B).$$

where A^* stands for the conjugate transpose of A and $R(A)$ is the range space of A .

3. Right star partial orders given by

$$A(\leq)^* B \Leftrightarrow AA^* = BA^* \text{ and } R(A^*) \subseteq R(B^*).$$

2. The Semigroup of Reflexive Generalized Inverses

Consider $A \in M_{m \times n}(\mathbb{R})$. A reflexive generalized inverse of A is the solution $G \in M_{n \times m}(\mathbb{R})$ of the equations $AGA = A$ and $GAG = G$. Note that there may be infinitely many such solutions G .

Example 2.1. For the 3×4 matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}.$$

then the 4×3 matrices

$$\mathbf{G}_1 = \begin{pmatrix} 3/2 & -1/2 & 0 \\ 5/4 & -1/4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

and

$$\mathbf{G}_2 = \begin{pmatrix} 0 & -5/2 & 3/2 \\ 0 & 9/4 & -5/4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

are reflexive generalized inverses.

We denote the set of all reflexive generalized inverses of an $m \times n$ matrix $A \in M_{m \times n}(\mathbb{R})$ by $A\{1, 2\}$. Define a sandwich operation ' \star ' on $A\{1, 2\}$ as

$$G_1 \star G_2 = G_1 A G_2 \quad \forall G_1, G_2 \in A\{1, 2\}. \quad (5)$$

Lemma 2.1. The set $A\{1, 2\}$ with the binary operation \star defined by equation (5) is a semigroup written as $(A\{1, 2\}, \star)$.

Proof: $A\{1, 2\} = \{G \in M_{n \times m}(\mathbb{R}) / AGA = A \text{ and } GAG = G\}$.

Clearly, $A(G_1 \star G_2)A = A(G_1 A G_2)A = A$ and

$$\begin{aligned} (G_1 \star G_2)A(G_1 \star G_2) &= (G_1 A G_2)A(G_1 A G_2) \\ &= G_1 A G_2 \\ &= G_1 \star G_2 \end{aligned}$$

Thus $G_1 \star G_2 \in A\{1, 2\}$ and ' \star ' is well defined on $A\{1, 2\}$.

Also, $G_1 \star (G_2 \star G_3) = G_1 \star (G_2 A G_3) = (G_1 A G_2) A G_3 = (G_1 \star G_2) \star G_3$ which shows that \star is associative and hence $A\{1, 2\}$ is a semigroup with respect to the operation \star .

Properties of the Semigroup $(A\{1, 2\}, \star)$

1. $A\{1, 2\}$ is regular.
2. $A\{1, 2\}$ is a rectangular band.
Proof: For every $G_1, G_2 \in A\{1, 2\}$, $G_1^2 = G_1 \star G_1 = G_1 A G_1 = G_1$, every element is idempotent. Also, $G_1 \star G_2 \star G_1 = G_1 A G_2 A G_1 = G_1 A G_1 = G_1$ and hence is a rectangular band.
3. $A\{1, 2\}$ is an orthodox semigroup.
Since every element of $A\{1, 2\}$ is idempotent, $A\{1, 2\} = E(A\{1, 2\})$. Hence the result.
4. $A\{1, 2\}$ is E-inversive.
For $G_1 \in A\{1, 2\}$, $G_1 \star G_1 = G_1 \in E(A\{1, 2\}) = A\{1, 2\}$.
5. $A\{1, 2\}$ is weakly cancellative.
Proof: For $G, G_1, G_2 \in A\{1, 2\}$, suppose $G \star G_1 = G \star G_2$ and $G_1 \star G = G_2 \star G$. Then $G A G_1 = G A G_2$ and $G_1 A G = G_2 A G \Rightarrow A G_1 = A G_2$ and $G_1 A = G_2 A$.
 $G_1 = G_1 A G_1 = G_1 (A G_1) = G_1 (A G_2) = (G_1 A) G_2 = (G_2 A) G_2 = G_2$.

6. $A\{1, 2\}$ is weakly separative.

Proof: Suppose for $G_1, G_2 \in A\{1, 2\}$,

$$\begin{aligned} G_1 \star S \star G_1 &= G_1 \star S \star G_2 \\ &= G_2 \star S \star G_1 \\ &= G_2 \star S \star G_2 \quad \forall S \in A\{1, 2\}. \end{aligned}$$

As $G_1 \star S \star G_1 = G_1 A S A G_1 = G_1 (A S A) G_1 = G_1 A G_1$, we get

$$\begin{aligned} G_1 A G_1 &= G_1 A G_2 = G_2 A G_1 = G_2 A G_2 \\ \implies G_1 &= G_2. \end{aligned}$$

7. $A\{1, 2\}$ is quasi separative as

$$\begin{aligned} G_1 \star G_1 &= G_1 \star G_2 = G_2 \star G_1 = G_2 \star G_2 \\ \implies G_1 \star G_1 &= G_2 \star G_2. \end{aligned}$$

i.e., $G_1 = G_2 \quad \forall G_1, G_2 \in A\{1, 2\}$.

8. $A\{1, 2\}$ is conditionally commutative.

If $G_1 \star G_2 = G_2 \star G_1$, then $G_1 \star X \star G_2 = G_2 \star X \star G_1 \quad \forall X \in A\{1, 2\}$.

9. Every element in $A\{1, 2\}$ is a mid unit.

For every $G, G_1, G_2 \in A\{1, 2\}$, we have

$$\begin{aligned} G_1 \star G \star G_2 &= G_1 A G A G_2 \\ &= G_1 (A G A) G_2 \\ &= G_1 A G_2 \\ &= G_1 \star G_2. \end{aligned}$$

This means that every G is a mid unit.

10. $A\{1, 2\}$ is a medial semigroup as

$$X \star G_1 \star G_2 \star Y = X \star G_2 \star G_1 \star Y. \quad \forall X, Y, G_1, G_2 \in \{1, 2\}.$$

11. $A\{1, 2\}$ is normal as

$$G \star G_1 \star G_2 \star G = G \star G_2 \star G_1 \star G \quad \forall G, G_1, G_2 \in A\{1, 2\}.$$

3. Partial Orders on the Semigroup

$$(A\{1, 2\}, \star)$$

3.1. Mitsch's Partial Order

The Mitsch's order relation on $A\{1, 2\}$ is defined as follows:-
For $G_1, G_2 \in A\{1, 2\}$,

$$G_1 \rho G_2 \Leftrightarrow G_1 = P \star G_1 = P \star G_2 = G_1 \star Q = G_2 \star Q$$

for some $P, Q \in A\{1, 2\}$. i.e.,

$$\begin{aligned}
 G_1 &= P_1AG_1 \\
 G_1\rho G_2 &\implies \begin{aligned} &= P_1AG_2 \\ &= G_1AQ_1 \\ &= G_2AQ_1 \end{aligned} \quad (6)
 \end{aligned}$$

Lemma 3.1. $(A\{1, 2\}, \star)$ is an ordered semigroup under Mitch's order relation.

From equation (6),

$$G_1\rho G_2 \Leftrightarrow G_1 = P_1AG_1 = P_1AG_2 = G_1AQ_1 = G_2AQ_1$$

for some $P, Q \in A\{1, 2\}$. By taking $P_1 = Q_1 = G_1$, $G_1\rho G_1$, that is ρ is reflexive.

Suppose that $G_1\rho G_2$ and $G_2\rho G_1$. Then,

$$G_1\rho G_2 \Rightarrow G_1 = P_1AG_1 = P_1AG_2 = G_1AQ_1 = G_2AQ_1.$$

and

$$\begin{aligned}
 G_2 &= P_2AG_2 \\
 G_2\rho G_1 &\implies \begin{aligned} &= P_2AG_1 \\ &= G_1AQ_2 \\ &= G_2AQ_2 \end{aligned} \quad (7)
 \end{aligned}$$

Using equations(6) and (7), we get

$$\begin{aligned}
 G_1 &= G_1AG_1 \\
 &= (G_2AQ_1)A(P_1AG_2) \\
 &= G_2(AQ_1A)P_1AG_2 \\
 &= G_2(AP_1A)G_2 \\
 &= G_2AG_2 = G_2.
 \end{aligned}$$

For transitivity, suppose that $G_1\rho G_2$ and $G_2\rho G_3$. i.e.,

$$G_1 = P_1AG_1 = P_1AG_2 = G_1AQ_1 = G_2AQ_1$$

and

$$\begin{aligned}
 G_2 &= P_2AG_2 \\
 &= P_2AG_3 \\
 &= G_2AQ_2 \\
 &= G_3AQ_2.
 \end{aligned} \quad (8)$$

Now,

$$G_1 = P_1AG_2 = P_1A(P_2AG_3) = P_1AG_3$$

again,

$$\begin{aligned}
 G_1 &= G_2AQ_1 \\
 &= (G_3AQ_2)AQ_1 \\
 &= G_3(AQ_2A)Q_1 \\
 &= G_3AQ_1.
 \end{aligned}$$

Taking $P_1 = P_3$ and $Q_1 = Q_3$, we get

$$\begin{aligned}
 G_1 &= P_3AG_1 = P_3AG_3 = G_1AQ_3 = G_3AQ_3 \\
 &\implies G_1\rho G_3.
 \end{aligned}$$

Next to see that ρ is compatible with respect to \star . Let $G_1\rho G_2$ and $G_3\rho G_4$.

Then equation(6) gives,

$$G_1 = P_1AG_1 = P_1AG_2 = G_1AQ_1 = G_2AQ_1.$$

and

$$\begin{aligned}
 G_3 &= P_2AG_3 \\
 &= P_2AG_4 \\
 &= G_3AQ_2 \\
 &= G_4AQ_2.
 \end{aligned} \quad (9)$$

for some $P_2, Q_2 \in A\{1, 2\}$.

Using equations (6) and (9), we obtain :

$$G_1AG_3 = (P_1AG_1)AG_3 = P_1A(G_1AG_3)$$

$$G_1AG_3 = (P_1AG_2)A(P_2AG_4) = P_1A(G_2AG_4)$$

Again,

$$G_1AG_3 = G_1A(G_3AQ_2) = (G_1AG_3)AQ_2$$

$$G_1AG_3 = (G_2AQ_1)A(G_4AQ_2) = (G_2AG_4)AQ_2$$

Taking $P_1 = P$ and $Q_2 = Q$, we have

$$\begin{aligned}
 G_1AG_3 &= PA(G_1AG_3) \\
 &= PA(G_2AG_4) \\
 &= (G_1AG_3)AQ \\
 &= (G_2AG_4)AQ.
 \end{aligned}$$

$$\text{i.e., } (G_1AG_3)\rho(G_2AG_4).$$

thus ρ is a compatible partial order on $(A\{1, 2\}, \star)$.

3.2. Sussman' Partial Order

For $G_1, G_2 \in A\{1, 2\}$.

$$G_1 < G_2 \Leftrightarrow G_1 \star G_1 = G_1 \star G_2 = G_2 \star G_1.$$

i.e.,

$$G_1 < G_2 \Leftrightarrow G_1AG_1 = G_1AG_2 = G_2AG_1. \quad (10)$$

Lemma 3.2. $(A\{1, 2\}, \star)$ is an ordered semigroup under Sussman's order relation.

Proof: It is obvious that $G_1 = G_1AG_1 \Rightarrow G_1 < G_1$ hence ' $<$ ' is reflexive.

To prove the antisymmetry, suppose that $G_1 < G_2$ and $G_2 < G_1$.

$$G_1AG_1 = G_1AG_2 = G_2AG_1$$

and

$$G_2AG_2 = G_2AG_1 = G_1AG_2.$$

From these equations, we get

$$G_1AG_1 = G_2AG_2 \Rightarrow G_1 = G_2.$$

Next suppose that $G_1 < G_2$ and $G_2 < G_3$. Then

$$G_1AG_1 = G_1AG_2 = G_2AG_1$$

and

$$G_2 < G_3 \Rightarrow G_2AG_2 = G_2AG_3 = G_3AG_2 \quad (11)$$

From (10), we have

$$\begin{aligned} G_1AG_1 &= G_1AG_2 \\ &= G_1A(G_2AG_2) \\ &= G_1A(G_2AG_3) \\ &= G_1AG_3 \end{aligned}$$

and

$$\begin{aligned} G_1AG_1 &= G_2AG_1 \\ &= (G_2AG_2)AG_1 \\ &= (G_3AG_2)AG_1 \\ &= G_3AG_1. \end{aligned}$$

i.e.

$$G_1AG_1 = G_1AG_3 = G_3AG_1 \Leftrightarrow G_1 < G_3.$$

Thus ' $<$ ' is transitive. Now, to prove the compatibility of ' $<$ ', suppose that $G_1 < G_2$ and $G_3 < G_4$. Then, $G_1AG_1 = G_1AG_2 = G_2AG_1$ and

$$G_3 < G_4 \Rightarrow G_3AG_3 = G_3AG_4 = G_4AG_3. \quad (12)$$

We have to show that

$$(G_1 \star G_3) < (G_2 \star G_4)$$

$$\begin{aligned} \Leftrightarrow (G_1AG_3)A(G_1AG_3) &= (G_1AG_3)A(G_2AG_4) \\ &= (G_2AG_4)A(G_1AG_3). \end{aligned}$$

i.e.,

$$G_1AG_3 = G_1AG_4 = G_2AG_3. \quad (13)$$

Using equations (10) and (12), we have the following:

$$G_1AG_3 = G_1A(G_3AG_3) = G_1(A(G_3A)G_4) = G_1AG_4$$

and

$$G_1AG_3 = (G_1AG_1)AG_3 = (G_2AG_1)AG_3 = G_2AG_3.$$

Thus equation (13) is verified. Hence the result.

4. conclusion

Ordered semigroups have many applications, especially in the theory of computer arithmetic, error-correcting codes and the like. There are many contributors - Greville, Ben Israel, Radhakrishna Rao, and K. Mitra who have introduced new classes of generalized inverses. The calculus of generalized inverses provides an elegant tool for areas like linear estimation, multivariable analysis, ANOVA etc. In this paper, we introduced two ordered semigroup structures for the reflexive generalized inverses of a rectangular matrix using two famous matrix order relations, viz., Sussman's and Mitch's order relations.

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Author Contributions

Sathi P H: Conceptualization, Methodology, Writing - original draft, Resources

Romeo P G: Supervision, Validation, Writing - review & editing

Conflicts of Interest

The authors declare no conflicts of interest.

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