

# The Genesis of a Theorem in the Galois Theory of $p$ -Extensions of $\mathbb{Q}$ with Restricted Tame Ramification

John Labute

Department of Mathematics and Statistics, McGill University Montreal, Quebec, Canada

## Email address:

labute@math.mcgill.ca

## To cite this article:

John Labute. (2024). The Genesis of a Theorem in the Galois Theory of  $p$ -Extensions of  $\mathbb{Q}$  with Restricted Tame Ramification. *Pure and Applied Mathematics Journal*, 13(4), 59-65. <https://doi.org/10.11648/j.pamj.20241304.12>

**Received:** 26 July 2024; **Accepted:** 20 August 2024; **Published:** 26 August 2024

**Abstract:** This article traces the genesis of a theorem that gives for the first time examples of the Galois group  $G_S$  of the maximal  $p$ -extension of  $\mathbb{Q}$ , unramified outside a finite set of primes not containing an odd  $p$ , that are of cohomological dimension 2 if the primes in  $S$  satisfy a certain linking condition. Because the ramification is tame the pro- $p$ -group  $G_S$  has all of its derived factors finite which is a strong finiteness condition on  $G_S$ . The paper starts with a question of Serre on one relator pro- $p$ -groups and then a detour to discrete groups where the notion of strong freeness for a sequence of homogeneous Lie elements is given and a criterion for strong freeness is established. These notions are then carried over to pro- $p$ -groups where the linking condition on the primes of  $S$  is translated into a cohomological criterion for a pro- $p$ -group to have cohomological dimension 2. An analysis is given of the work of Koch where he gives a weaker criterion for a pro- $p$ -group to have cohomological dimension 2. A connection is made with this work of Koch and that of the author which would have been sufficient to prove the fact that  $G_S$  was of cohomological dimension 2 for certain sets  $S$  had it been applied to investigate whether the linking condition was true for certain sets  $S$ . It is not known if the cohomological dimension of  $G_S$  is 2 if  $S$  does not satisfy this linking condition.

**Keywords:** Pro- $p$ -group, Cohomology, Galois Group,  $p$ -extension, Tame Ramification, Lie Algebra, Mild Group, Mild Pro- $p$ -group, Linking Number

## 1. Introduction

Let  $p$  be an odd prime and let  $G_S$  be the Galois group of the maximal  $p$ -extension of  $\mathbb{Q}$  which is unramified outside of a finite set of primes  $S$  of cardinality  $m$ , not containing  $p$ . Not much was known about these groups; all that was known was that, by the Golod-Shafarevich Theorem, they were infinite if  $m \geq 4$ . These groups remain mysterious in general. In this paper the evolution of Theorem 2.1, which was unexpected by researchers in the field, is traced from its inception to its discovery. The aim is to see why this discovery was not made much earlier.

## 2. The Theorem

Let  $p$  be an odd prime and let  $S = \{q_1, \dots, q_m\}$  be a set of primes  $q_i \equiv 1 \pmod p$ . Let  $g_i$  be a primitive root mod  $q_i$  and let

the linking number  $\ell_{ij} \in \mathbb{F}_p$  be defined by

$$q_i \equiv g_j^{-\ell_{ij}} \pmod{q_j}.$$

If  $g$  is another primitive root mod  $q_j$  then  $\ell_{ij}$  is replaced by  $c_j \ell_{ij}$  for some  $c_j$ .

**Theorem 2.1.** Suppose that  $m$  is even and that

1.  $\ell_{ij} = 0$  if  $i, j$  are both odd,
2.  $\ell_{12}\ell_{23} \cdots \ell_{m-1,m}\ell_{m1} - \ell_{21}\ell_{32} \cdots \ell_{m,m-1}\ell_{1m} \neq 0$ .

Then  $G_S$  is of cohomological dimension 2.

**Example 2.1.** For  $S = \{7, 19, 61, 163\}$  and  $p = 3$  the non-zero  $\ell_{ij}$  are

$$\ell_{12} = \ell_{21} = \ell_{14} = \ell_{23} = \ell_{24} = \ell_{34} = 1, \ell_{43} = \ell_{41} = -1.$$

Conditions (a) and (b) are satisfied, so  $\text{cd}(G_S) = 2$  and this gives the first example of a free pro- $p$ -group whose cohomological dimension is 2. The most general statement of Theorem 1 which covers the case  $m$  is odd is elegantly

formulated by Alexander Schmidt:

**Theorem 2.2.** Let  $G$  be a finitely generated pro- $p$ -group such that  $H^1(G, \mathbb{Z}/p\mathbb{Z})$  is the direct sum of non-trivial subspaces  $U, V$  and such that the cup product

$$H^1(G, \mathbb{Z}/p\mathbb{Z}) \otimes H^1(G, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^2(G, \mathbb{Z}/p\mathbb{Z})$$

1. is trivial on  $U \otimes U$  and
  2. maps  $U \otimes V$  surjectively onto  $H^2(G, \mathbb{Z}/p\mathbb{Z})$ ,
- then  $G$  is of cohomological dimension 2.

### 3. It All Began with a Question of Serre

In a 1963 Séminaire Bourbaki Lecture [2] Serre proposed the following question where  $F$  is a finitely generated free pro- $p$ -group and  $F_2 = F^p[F, F]$ . :

“Soit  $r \in F_2$ , et soit  $G_r = F/(r)$ . Peut-on étendre à  $G_r$  les résultats démontrés par Lyndon [1] dans le cas discret? En particulier, si  $r$  n’est pas une puissance  $p$ -ième, est-il vrai que  $G_r$  est de dimension cohomologique 2?”

In the Fall of 1964 Serre was at Harvard and gave a course on “Lie Groups and Lie Algebras which gave an introduction to the the work of Lazard on filtrations of groups in his course “Groups and Lie Algebras”. In a private lecture he gave a proof on the Elimination Theorem for groups and Lie algebras which will play a decisive role in the proof of Theorem 1. This work is very much influenced by the interplay between groups and Lie algebras. Sometimes, a group theory question can be solved if the corresponding Lie algebra question and the result is strong enough, it can be pulled back to a proof of the original group theory question. This is the case here, at least partially so. The following is a sketch of an attempt to solve his question with this in mind. Unfortunately the proof had a gap which was discovered after the result was conveyed to Serre. It was only when a detailed proof was attempted that it was discovered that there was a natural boundary inherent in the methodology. This was embarrassing having to tell Serre this especially when he said he told Tate that his question had been solved positively.

Let  $R = (r)$  and let  $M = R/[R, R]$  where  $[R, R]$  is the subgroup of  $R$  generated by the commutators  $[x, y] = x^{-1}y^{-1}xy$  with  $x, y \in R$ . Then by a result of Brumer [3] we have  $\text{cd}(G_r) = 2$  if and only if  $M$ , viewed as a module over the completed algebra  $\mathbb{Z}_p[[G]]$  of  $G$ , is a free module of rank 1 generated by the image of  $r$  in  $M$ . In [4] we showed that this was true if  $r$  was not “too close” to a  $p$ -th power. More precisely, let  $(F_n)$  be the filtration of  $F$  defined by  $F_1 = F$  and  $F_{n+1} = F^p[F, F_n]$ , also known as the descending  $p$ -central series of  $F$ , and let  $e$  be largest with  $r \in F_e$ ; we have  $e < \infty$  if  $r \neq 1$ , since the intersection of the subgroups  $F_n$  is 1. We show that  $\text{cd}(G_r) = 2$  if  $r$  is not a  $p$ -th power mod  $F_{e+1}$ .

The filtration  $(F_n)$  has two important properties:

1.  $[F_n, F_m] \subset F_{n+m}$ ,
2.  $F_n^p \subset F_{n+1}$ .

Let  $\text{gr}_n(G)$  be the abelian group  $G_n/G_{n+1}$ , which is denoted additively. A Lie bracket is introduced on  $\text{gr}(G) = \bigoplus \text{gr}_n(G)$  using the commutator operation as follows: let

$\xi_n \in \text{gr}_n(G), \eta_m \in \text{gr}_m(G)$  be the the images of  $x_n \in G_n, y_m \in G_m$  respectively. Then, letting  $[\xi_n, \eta_m]$  be the image in  $\text{gr}_{n+m}$  of  $[x_n, y_m] \in G_{n+m}$ , one gets a Lie bracket on the graded  $\mathbb{F}_p$ -module  $\text{gr}(G)$ . The  $p$ -th power operator on  $G$  induces the structure of a graded  $\mathbb{F}_p[\pi]$ -algebra on  $\text{gr}(G)$ ; namely, if  $x \in F_n$  and  $\xi$  is its image in  $\text{gr}_n(G)$  then  $\pi\xi$  is the image of  $x^p$  in  $\text{gr}_{n+1}(G)$ . If  $F = F(x_1, \dots, x_d)$  is the free group on  $x_1, \dots, x_d$  and  $\xi_i$  is the image of  $x_i$  in  $\text{gr}_1(F)$  then  $L = \text{gr}(F)$  is the free Lie algebra over  $\mathbb{F}_p[\pi]$  on  $\xi_1, \dots, \xi_d$ .

Let  $\rho$  be the image of  $r$  in  $\text{gr}_e(F)$ ; this element is called the *initial form* of  $r$ . Then  $\rho$  not a multiple of  $\pi$  is the same as saying  $r$  is not a  $p$ -th power modulo  $F_{e+1}$ . Let  $\mathfrak{r}$  be the ideal of  $L$  generated by  $\rho$  and let  $\mathfrak{m} = \mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ . Let  $\mathfrak{g} = L/\mathfrak{r}$  and let  $U_{\mathfrak{g}}$  be the enveloping algebra of  $\mathfrak{g}$ . Then  $\mathfrak{m}$  is a  $U_{\mathfrak{g}}$  module via the adjoint representation. Then the corresponding Lie algebra question would be to show that  $\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$  was a free  $U_{\mathfrak{g}}$  module of rank 1 generated by the image of  $\rho$ . The proof of this this required a proof that  $\mathfrak{g}$  was torsion free as an  $\mathbb{F}_p[\pi]$  module which would entail the same is true for  $U_{\mathfrak{g}}$  and hence by the Birkhoff-Witt Theorem that  $U_{\mathfrak{g}}$  has no zero divisors. It would be then a straightforward exercise using the exact sequence

$$\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}] \rightarrow I_{\mathfrak{g}} \rightarrow U_{\mathfrak{g}} \rightarrow \mathbb{F}_p[\pi] \rightarrow 0$$

where  $I_{\mathfrak{g}} \cong U_{\mathfrak{g}}^d$  is the augmentation ideal of  $U_{\mathfrak{g}}$ . This yields the resolution by free  $U_{\mathfrak{g}}$  modules

$$0 \rightarrow \mathfrak{r}/[\mathfrak{r}, \mathfrak{r}] \rightarrow I_{\mathfrak{g}} \rightarrow U_{\mathfrak{g}} \rightarrow \mathbb{F}_p[\pi] \rightarrow 0$$

showing that  $\text{cd}(\mathfrak{g}) = 2$ .

To show that this lifts to  $G = F/R$  one has to show that  $\text{gr}(G) = L/(\rho)$  or, equivalently that  $\text{gr}(R) = (\rho)$  where  $R_n = F \cap F_n$ . The proof of this utilizes suitable Lazard filtrations of  $F$  and iterative applications of Birkhoff-Witt; it is much too lengthy to be even sketched here. The same goes for the proof of  $L/(\rho)$  being torsion free if  $\rho$  is not a multiple of  $\pi$ . The details can be found in the work of the author [4].

This proof generalizes to the case of several relators in the work of the author [11] but the linear independence of the initial forms of the relators over  $\mathbb{F}_p[\pi]$  is not enough to prove the result. One needs to assume that  $U_{\mathfrak{g}}$  is a torsion free  $\mathbb{F}_p[\pi]$  module and that  $\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$  is a free  $U_{\mathfrak{g}}$  module on the images of the initial forms of the relators. Such relators are called *strongly free*. A pro- $p$ -group with a strongly free presentation is called *mild*.

As it turned out, the question of Serre had a negative answer; for example, in the case  $r = x^p[x^p, y]$  as was shown by Guildenhuis [6] since  $x^p = 1$  in  $G_r$  and  $r$  is not a  $p$ -th power in  $F$ . One still does not have a criterion for deciding whether the group  $G_r$  has finite or infinite cohomological dimension.

### 4. A Fortituitous Detour to Discrete Groups

In this section  $F$  will be a free discrete group of rank  $m$ . We let  $(F_n)$  be the filtration of  $F$  defined by  $F_1 = F, F_{n+1} =$

$[F, F_n]$ , also known as the descending central series of  $F$ . The graded Lie algebra  $\text{gr}(F)$  is defined as above. It is a free Lie algebra over  $\mathbb{Z}$ . Let  $r \in F$  and let  $G = F/(r)$ . Suppose  $r \in F_e$ ,  $r \notin F_{e+1}$  with  $e \geq 2$ . Let  $\rho$  be the image of  $r$  in  $\text{gr}_e(F)$ , the initial form of  $r$ . If  $\rho$  is not a proper multiple then Waldinger in [5] showed that  $\text{gr}_n(G)$  is a free  $\mathbb{Z}$  module for  $e \leq n \leq 3e$  and gave formulae for the ranks as a partial answer to a question of Magnus who asked if  $\text{gr}(G)$  was torsion free for  $G = F/(r_1, \dots, r_d)$  if the initial forms of the  $r_i$  were linearly independent.

$$ng_n = \sum_{d|n} \mu(n/d) \left[ \sum_{0 \leq i \leq n/d} (-1)^i \frac{1}{d+i-ei} \binom{d+i-ei}{i} m^{d-ei} \right].$$

In particular  $g_n$  depends only on  $n, e$  and  $m$ .

In the work of the author [11], these results were extended to the case of several relators provided that their initial forms  $\rho_1, \dots, \rho_d$  satisfied the following two conditions

1. The enveloping algebra  $U$  of  $\text{gr}(F)/(\rho_1, \dots, \rho_d)$  is a torsion free  $\mathbb{Z}$  module;
2. If  $\tau = (\rho_1, \dots, \rho_d)$  then  $\tau/[\tau, \tau]$  is a free  $U$  module with basis the the images of  $\rho_1, \dots, \rho_d$ .

Such a sequence is also called strongly free. A method for constructing such sequences is given using the Elimination Theorem for Lie algebras. This method would prove decisive in the proof of Theorem 2.1.

**Theorem 4.1** (Elimination Theorem). Let  $K$  be a commutative ring and let  $L = L(X)$  be the free Lie algebra over  $K$  on the set  $X$ . Let  $S$  be a subset of  $X$  and let  $\mathfrak{s}$  be the ideal of  $L(X)$  generated by  $X - S$ . Let  $W$  be the enveloping algebra of  $L(S)$  and let  $M(S)$  be the submonoid of  $W$  generated by  $S$ . Then  $\mathfrak{s}$  is the free Lie algebra over  $K$  on the elements

$$T = \{\text{ad}(m)(x) \mid x \in X - S, m \in M(S)\}.$$

**Corollary 4.2.** The  $W$  module  $\mathfrak{s}/[\mathfrak{s}, \mathfrak{s}]$  is a free module over  $W$  with basis the image of  $T$ .

**Theorem 4.3** (Criterion for Strong Freeness). Let  $\rho_1, \dots, \rho_d$  be elements of  $\mathfrak{s}$  such that the elements

$$T_1 = \{\text{ad}(m)(\rho_j) \mid m \in S, 0 \leq j \leq d\}$$

are part of a basis of  $\mathfrak{s}$ . Then  $\rho_1, \dots, \rho_n$  is a strongly free sequence.

*Proof* The ideal  $\tau$  of  $L$  generated by  $\rho_1, \dots, \rho_d$  is generated as an ideal of  $\mathfrak{s}$  by the set  $T_1$ . Since  $T_1$  is part of a basis of  $\mathfrak{s}$  the Corollary to the Elimination Theorem says that  $\tau/[\tau, \tau]$  is a free module over the enveloping algebra  $V$  of  $\mathfrak{s}/\tau$  with basis the image of  $T_1$ . The exact sequence

$$0 \rightarrow \mathfrak{s}/\tau \rightarrow U/\tau \rightarrow L(S) \rightarrow 0$$

In the work of the author [7] it is shown that in fact  $\text{gr}(G) = \text{gr}(F)/(\rho)$  is a free  $\mathbb{Z}$  module if  $\rho$  is not a proper multiple and that the Poincaré series of its enveloping algebra was

$$\frac{1}{1 - mt + t^e} = \prod_{n \geq 1} \frac{1}{(1 - t^n)^{g_n}}$$

where  $g_n$  is the rank of  $\text{gr}_n(G)$ . A straightforward calculation yields the formula

splits which implies that

$$U = V \otimes W = \bigoplus_{m \in M(S)} Vm.$$

This implies that  $U$  is a free  $K$  module. To show that  $\tau/[\tau, \tau]$  is a free module let  $\bar{\rho}_i$  be the image of  $\rho_i$  in  $\tau/[\tau, \tau]$  and suppose that

$$\sum_i u_i \rho_i = 0 \text{ with } u_i \in U.$$

Then  $u_i = \sum_j v_{ij} m_j$  with  $u_i \in U$ ,  $v_{ij} \in V$  which implies that

$$\sum_i u_i \bar{\rho}_i = \sum_{i,j} v_{ij} \text{ad}(m_j)(\bar{\rho}_i) = 0.$$

But this implies  $v_{ij} = 0$  since  $\tau/[\tau, \tau]$  is a free  $V$  module with basis  $T_1$ .

The Lie algebras  $L/\tau$  constructed in this way are of cohomological dimension  $\leq 2$ .

An important example of this is

$$[\xi_1, \xi_2], [\xi_2, \xi_3], [\xi_3, \xi_4], \dots, [\xi_{m-1}, \xi_m],$$

where  $X = \{x_1, \dots, x_m\}$  and  $S = \{x_1, x_3, x_5, \dots\}$  so  $X - T = \{x_2, x_4, x_6, \dots\}$ .

In the work of Anick [12], he gave the name *mild group* to a discrete group with defining relators whose initial forms are strongly free and gave many examples using his criterion of *combinatorial freeness* to prove mildness. The most important of these is the fundamental group  $G$  of the complement of a *pure braid link*  $\mathcal{L}$  in  $S^3$  which is obtained from a pure braid by identifying the top and bottom of each strand. If  $m$  is the number of strands of the braid then  $G$  has the presentation  $G = F/(r_1, \dots, r_{m-1})$ , where  $r_i = [x_i, y_i]$  with

$$y_i = x_i^{a_i} \prod_{j=1}^m x_j^{a_{ij}} \text{ mod } [F, F].$$

so that

$$r_i = \prod_{j=1}^n [x_i, x_j]^{a_{ij}} \text{ mod } [F, [F, F]].$$

The matrix  $(a_{ij})$  is a symmetric matrix with zero diagonal because  $a_{ij}$  is the *linking number* between the  $i$ -th and  $j$ -th unknot of the link. We assume that the matrix  $a_{ij}$  has no zero rows so that the image  $\rho_i$  in  $\text{gr}_2(F)$  is non-zero. Then

$$\rho_i = \sum_{j=1}^m a_{ij}[\xi_i, \xi_j],$$

where  $\xi_i$  is the image of  $x_i$  in  $\text{gr}_2(F)$ . Anick uses a weighted graph associated to the matrix  $(a_{ij})$  to give a criterion for the strong freeness of the sequence  $\rho_1, \dots, \rho_{m-1}$ ; note that  $\sum \rho_i = 0$ . This graph, which Anick calls a linking diagram, has as vertices the set  $\{\xi_1, \dots, \xi_m\}$  with  $\xi_i, \xi_j$  being joined if  $a_{ij} \neq 0$ ; in this case, the weight being  $a_{ij}$ . The graph is connected mod  $p$  if and only if there is a spanning subtree whose vertices are not congruent to zero modulo  $p$  where  $p$  can be any prime.

Anick then shows that if the linking diagram of the matrix  $(a_{ij})$ , or of the link  $\mathcal{L}$ , is connected mod  $p$  then the sequence  $\rho_1, \dots, \rho_i$  is strongly free mod  $p$  and strongly free if it is connected mod  $p$  for any prime  $p$ .

## 5. Back to Pro- $p$ -groups

In [9] Koch uses Lazard filtrations of the completed group algebra  $\mathbb{F}_p[[F]]$  to show that if  $G = F(x_1, \dots, x_m)/(r_1, \dots, r_d)$  then  $\text{cd}(G) = 2$  if the initial forms of the relators form a strongly free sequence of Lie elements in the free Lie subalgebra  $\mathfrak{L}$  of  $\text{gr}(F)$  on  $\xi, \dots, \xi_m$ , the initial forms of  $x_1, \dots, x_m$ . The  $\mathbb{F}_p$ -algebra  $\mathfrak{A} = \text{gr}(F)$  is the free associative algebra over  $\mathbb{F}_p$  on the elements  $\xi_1, \dots, \xi_m$ . For Koch, strong freeness means that, if  $\mathfrak{R}$  is the ideal of  $\mathfrak{A}$  generated by  $\rho_1, \dots, \rho_d$  and  $\mathfrak{I}$  is the augmentation ideal of  $\mathfrak{A}$ , then  $\mathfrak{A}/\mathfrak{I}\mathfrak{R}$  is a free  $\mathfrak{A}/\mathfrak{R}$  module on the images of the  $\rho_i$ . But  $\mathfrak{B} = \mathfrak{A}/\mathfrak{R}$  is the enveloping algebra of  $\mathfrak{g} = \mathfrak{L}/\mathfrak{r}$ , where  $\mathfrak{r}$  is the ideal of  $\mathfrak{L}$  generated by the  $\rho_i$  and  $\mathfrak{R}/\mathfrak{I}\mathfrak{R}$  is isomorphic to  $\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ , the isomorphism being induced by the inclusion  $\mathfrak{r} \subset \mathfrak{R}$ . So his definition is the same as the one given above but this connection was not made in his work [9].

Koch also gives a criterion for strong freeness. To describe it, let

$$\rho_i = \sum_{j=1}^d a_{ij} \xi_j$$

where  $a_{ij} = \partial_j \rho_i \in \mathfrak{A}$  (Fox derivative). The free commutative, associative  $\mathbb{F}_p$ -algebra  $\tilde{\mathfrak{A}}$  on  $\xi_1, \dots, \xi_m$  is naturally a quotient of  $\mathfrak{A}$ ; we identify  $\xi$  with its image in  $\tilde{\mathfrak{A}}$ . Koch's Criterion is that the rank of the  $d \times m$  matrix  $M_K = (\tilde{a}_{ij})$  be equal to  $d$ . If Koch's Criterion holds, one has  $d < m$  because of the relations

$$\sum_{j=1}^d \partial_j \rho_i \xi_j = 0, \quad 1 \leq i \leq m$$

which shows that the columns of  $M_K$  are linearly dependent.

For the relators  $\rho_1 = [\xi_1, \xi_2], \rho_2 = [\xi_2, \xi_3], \rho_3 = [\xi_3, \xi_4]$

where  $m = 4$

$$M_K = \begin{bmatrix} -\xi_2 & \xi_1 & 0 & 0 \\ 0 & -\xi_3 & \xi_2 & 0 \\ 0 & 0 & -\xi_4 & \xi_3 \end{bmatrix}$$

which is of rank 3 and so the relators are strongly free.

In the work of Koch [8], using local and global classfield theory, he gives a presentation for the Galois group  $G_S$  of the maximal  $p$ -extension of  $\mathbb{Q}$  which is unramified outside a finite set  $S$  of primes which is analogous to that of the fundamental group of the complement of a tame link in  $S_3$ . If  $S = \{q_1, \dots, q_m\}$  with  $q_i \equiv 1 \pmod p$ , he shows that  $G_S = F(x_1, \dots, x_m)/(r_1, \dots, r_m)$  where

$$r_i = x_i^{q_i-1} \prod_{j \neq i} [x_i, x_j]^{\ell_{ij}} \pmod{F_3}$$

where for  $j \leq m$ ,  $\ell_{ij}$  is the image in  $\mathbb{Z}/p\mathbb{Z}$  of any integer  $r$  with

$$q_i \equiv g_i^{-r} \pmod{q_i}$$

where  $g_i$  is a primitive root mod  $q_i$ .

If  $S = \{q_1, \dots, q_m, p\}$  the presentation is the same except for one additional variable  $x_{m+1}$  but with  $\ell_{ij}$  defined as before for  $i, j \leq m$  while for  $i \leq m, j = m+1$ ,  $\ell_{ij}$  is defined by

$$q_i \equiv (1+p)^{-\ell_{ij}} \pmod{p}.$$

In both cases there is a linking diagram associated to the matrix  $(\ell_{ij})$ . There is a striking similarity between the latter presentation and that of the complement of a pure braid link in  $S^3$ . In the Galois case one has  $\text{cd}(G_S) = 2$  which is proven in the work of Brumer [3]; but this is not always the case for the link group.

The first presentation is strikingly different since  $\ell_{ij} \neq \ell_{ji}$  in general, the group has the same number of generators as relators and  $G$  is fab. If  $\rho_i$  be the image of  $r_i$  in  $\text{gr}_2(F)$

$$\rho_i = c_i \pi \xi_i + \sum_{j \neq i} \ell_{ij} [\xi_i, \xi_j]$$

where  $c_i = (q_i - 1)/p$ . Reducing mod  $\pi$ ,

$$\bar{\rho}_i = \sum_{j \neq i} \ell_{ij} [\xi_i, \xi_j]$$

in the free lie algebra over  $\mathbb{F}_p$  on the  $\xi_i$ , where  $\xi_i$  is the image of  $x_i$  in  $\text{gr}_1(F)$ . The relators  $(\rho_i)$  are strongly free iff the relators  $(\bar{\rho}_i)$  are strongly free.

At this point in time there was not even one example of a pro- $p$ -group  $G$  with  $G/[G, G]$  finite and  $\text{cd}(G) = 2$ . Even worse the Galois group in question has all of its derived factors finite because by class field theory, the maximal abelian  $p$ -extension of a number field which is unramified outside a finite set of primes not divisible by  $p$  is of finite degree since the ramification is tame for such primes. In this case the sentiment was that such a group must have torsion, so could not have finite cohomological dimension.

## 6. The Lightning Bolt Hits

While on leave at Western University in London, Ontario in the Fall of 2004, after discussing with Jan Mináč criteria for strong freeness of the relators in the Koch presentation for  $G_S$  with  $p \in S$ , it suddenly became apparent while reviewing the criteria for strong freeness of the author [11] that, if  $m$  was even and  $\geq 4$ , one could prove that the relators

$$\rho_1 = [\xi_1, \xi_2], \rho_2 = [\xi_2, \xi_3] \dots \rho_{m-1} = [\xi_{m-1}, \xi_m], \rho_m = [\xi_m, \xi_1]$$

are strongly free by the Elimination Theorem with  $K = \mathbb{F}_p$  and  $S = \{\xi_i | i \text{ odd}\}$ . Note in this case, the linking diagram is a circuit. This meant that one could prove that the relators

$$x_1^p[x_1, x_2], x_2^p[x_2, x_3], \dots, x_{m-1}^p[x_{m-1}, x_m], x_m^p[x_m, x_1]$$

are also strongly free giving the first example of a pro- $p$ -group  $G$  of cohomological dimension 2 with  $G/[G, G]$  finite.

Motivated by this a search was made for circuits in the linking diagram of the Galois group of the maximal  $p$ -extension unramified outside  $S = \{q_1, \dots, q_m\}$  with  $m = 4$ ,  $p = 3$ . For  $q_1 = 7$ ,  $q_2 = 19$ ,  $q_3 = 61$ ,  $q_4 = 163$  and corresponding primitive roots 2, 2, 2, 3, the following circuit

$$[\xi_1, \xi_2], [\xi_2, \xi_3], [\xi_3, \xi_4], [\xi_4, \xi_1]$$

was found in the linking diagram of its presentation (mod  $\pi$ )

$$\begin{aligned} \bar{\rho}_1 &= [\xi_1, \xi_2] + [\xi_1, \xi_4] \\ \bar{\rho}_2 &= [\xi_2, \xi_1] + [\xi_2, \xi_3] + [\xi_2, \xi_4] \\ \bar{\rho}_3 &= [\xi_3, \xi_4] \\ \bar{\rho}_4 &= -[\xi_4, \xi_1] - [\xi_4, \xi_3]. \end{aligned}$$

To prove that theses relators are strongly free it is enough, by the criterion for strong freeness, to prove that they are part of a basis of  $\mathfrak{s} = (\xi_2, \xi_4)$ . But because the Lie agebras are graded over  $\mathbb{F}_p$ , it is enough to prove the linear independence of their images  $\bar{\rho}_i$  in  $\mathfrak{s}/[\mathfrak{s}, \mathfrak{s}]$  are linearly independent. Since the  $\bar{\rho}_i$  lie in the subspace  $\mathfrak{T}$  spanned by the linearly independent elements

$$[\xi_1, \xi_2], [\xi_2, \xi_3], [\xi_3, \xi_4], [\xi_4, \xi_1].$$

one just has to prove that the rank of the matrix of the  $\bar{\rho}_i$  with respect to this basis is 4. But this matrix

$$M_L = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

whose determinant is 1 which gives the result.

The proof in the general case of Theorem 2.1 is quite similar. The Koch presentation of  $G_S$  has  $d = m$  and the relators are

$$r_i = x_i^{q_i-1} \prod_{j \neq i} [x_i, x_j]^{\ell_{ij}} w_i$$

with  $w_i$  in the third term of the descending  $p$ -central series of  $F$ . This implies that the initial forms of the relators modulo  $\pi$  are

$$\rho_i = \sum_{j \neq i} \ell_{ij} [\xi_i, \xi_j].$$

If the conditions of Theorem 1 are satisfied, that  $\rho_1, \dots, \rho_m$  is a strongly free sequence follows from Corollary 7 if we take  $S$  to be the  $\xi_i$  with  $i$  even. Indeed, if one indexes the columns of the matrix  $M_L$  by  $(1, 2), (2, 3), \dots, (m-1, m), (1, m)$  one gets

$$M_L = \begin{bmatrix} \ell_{12} & 0 & 0 & \dots & 0 & \ell_{1m} \\ -\ell_{21} & \ell_{23} & 0 & \dots & 0 & 0 \\ 0 & -\ell_{32} & \ell_{34} & \dots & 0 & 0 \\ 0 & 0 & -\ell_{43} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \ell_{m,m-1} & 0 \\ 0 & 0 & 0 & \dots & -\ell_{m,m-1} & -\ell_{m1} \end{bmatrix},$$

whose determinant is  $\ell_{12}\ell_{23} \dots \ell_{m-1,m}\ell_{m1} - \ell_{21}\ell_{32} \dots \ell_{m,m-1}\ell_{1m}$ .

## 7. Unravelling the Statement of Theorem 2.2

Let  $G$  be a finitely generated pro- $p$ -group. The cohomology group  $H^i(G, \mathbb{Z}/p\mathbb{Z})$  will be denoted by  $H^i(G)$ ; it is a vector space over the finite field  $\mathbb{F}_p$ . The pro- $p$ -group  $G$  is said to be of cohomological dimension  $n$  if  $H^n(G) \neq 0$  and  $H^i(G) = 0$  for  $i > n$ . The cohomological dimension of  $G$  is said to be infinite if no such  $n$  exists. The cohomological dimension of  $G$  is denoted by  $\text{cd}(G)$ . The cohomological dimension of a free pro- $p$ -group is 1. In [4] we show that If  $\dim H^2(G) = 1$  and the cup-product

$$H^1(G) \otimes H^1(G) \rightarrow H^2(G)$$

is a non-degenerate bilinear form then  $\text{cd}(G) = 2$ . This is the case when  $G$  is the Galois group of the maximal  $p$ -extension of  $\mathbb{Q}_p(\zeta_p)$ , where  $\mathbb{Q}_p$  is the  $p$ -adic number field and  $\zeta_p$  is a primitive  $p$ -th root of unity.

If  $F = F(x_1, \dots, x_d)$  is the free pro- $p$ -group on  $x_1, \dots, x_d$  and

$$1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$$

is a presentation of  $G$  there is an exact sequence

$$0 \rightarrow H^1(G) \rightarrow H^1(F) \rightarrow H^1(R)^F \rightarrow H^2(G) \rightarrow H^2(F),$$

where  $H^1(G)$  is the dual of  $G/G^p[G, G]$ . Here  $[G, G]$  is the closed subgroup of  $G$  generated by the commutators  $[x, y] = x^{-1}y^{-1}xy$  with  $x, y$  in  $G$ . This implies that  $\dim H^1(G)$  is the minimal number of generators of  $G$ . If  $\dim H^1(F) = \dim H^1(G)$  the presentation is called minimal, in which case  $H^2(G) = H^1(R)^F = H^1(R/R^p[R, F])$ . We let  $\xi_i$  be the image of  $x_i$  in  $F/F^p[F, F] = G/G^p[G, G]$  and  $\chi_1, \dots, \chi_d \in$

$H^1(G) = H^1(F)$  the basis of  $H^1(F) = H^1(G)$  dual to  $\xi_1, \dots, \xi_d$ . Thus, for a minimal presentation,  $\dim H^2(G)$  is the minimal number of elements of  $R$  that generate  $R$  as a closed normal subgroup of  $R$ . The presentation is minimal if and only if  $R \leq F^p[F, F]$ .

Let  $r_1, \dots, r_m$  be elements of  $F^p[F, F]$ , let  $R = \langle r_1, \dots, r_m \rangle$  be the closed normal subgroup of  $F$  generated by  $r_1, \dots, r_m$  and let  $G = F/R$ . We have

$$r_k = \prod_{j=1}^d x_j^{p c_{kj}} \prod_{1 \leq i < j \leq d} [x_i, x_j]^{a_{ijk}} s_k$$

with  $s_k \in [F, [F, F]]$  and  $c_{kj}, a_{ijk} \in \mathbb{Z}_p$ . Let  $\bar{r}_k$  be the image of  $r_k$  in  $R/R^p[R, F] = H^2(G)^*$ , the dual of  $H^2(G)$ . Then  $\bar{r}_1, \dots, \bar{r}_m$  generate  $H^2(G)^*$  so that  $m \geq \dim H^2(G)$  which is  $\geq 1$  if  $G$  is not a free pro- $p$ -group. An important basic fact is that  $\bar{r}_k(\chi_i \cup \chi_j) = \bar{a}_{ijk}$ , the image of  $a_{ijk}$  in  $\mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p$ . Dualizing the cup product mapping

$$\phi : H^1(G) \otimes H^1(G) \rightarrow H^2(G),$$

gives, when  $p \neq 2$ , a mapping

$$\phi^* : H^2(G)^* \rightarrow \bigwedge^2 W = \bigoplus_{1 \leq i < j \leq d} \mathbb{F}_p \xi_i \wedge \xi_j$$

where  $W = F/F^p[F, F]$ . Then  $\langle \chi_i \cup \chi_j, \bar{r}_k \rangle = \rho_k(\chi_i \cup \chi_j) = \bar{a}_{ijk}$ . But  $\chi_i \cup \chi_j = \phi(\chi_i \otimes \chi_j)$  so that  $\bar{a}_{ijk} = \langle \chi_i \otimes \chi_j, \phi^*(\bar{r}_k) \rangle$  which implies that

$$\phi^*(\bar{r}_k) = \sum_{1 \leq i < j \leq d} \bar{a}_{ijk} \xi_i \wedge \xi_j.$$

If  $H^1(G)$  is the direct sum of non-trivial subspaces  $U, V$  then, after a possible change of basis,

$$U = \bigoplus_{i \in A} \mathbb{F}_p \chi_i \quad V = \bigoplus_{i \in B} \mathbb{F}_p \chi_i,$$

where  $A, B$  is a partition of  $\{1, \dots, d\}$ . If  $\psi$  is the restriction of  $\phi$  to  $U \otimes V$ , then  $\psi$  maps  $U \otimes V$  surjectively onto  $H^2(G)$  if and only if

$$\psi^* : H^2(G)^* \rightarrow Z = \bigoplus_{i \in A, j \in B} \mathbb{F}_p \xi_i \wedge \xi_j.$$

is injective; note that

$$\psi^*(\bar{r}_k) = \sum_{i \in A, j \in B} a_{ijk} \xi_i \wedge \xi_j.$$

If  $\sigma$  is the projection of  $\bigwedge^2 W$  onto  $Z$  then  $\psi^* = \sigma \phi^*$  so that  $\psi^*$  is injective if and only if  $\phi^*$  is injective, the latter being equivalent to  $m$  being equal to the rank of the matrix  $M_L$  whose columns are indexed by the pairs  $(i, j)$  with  $i \in A, j \in B$  and whose entry in the  $k$ -th row and  $(i, j)$ -th column is  $\bar{a}_{ijk}$ . Thus, to prove Theorem 2.2, one has to prove that

$\text{cd}(G) = 2$  if the following two conditions

- (a)  $\bar{a}_{ijk} = 0$  when  $i, j \in A$ ,
- (b) the rank of  $M_L$  is  $m$ ,

hold for the elements

$$\rho_k = \sum_{1 \leq i < j \leq d} \bar{a}_{ijk} [\xi_i, \xi_j],$$

where  $\bigwedge^2 W$  can be identified with the space of degree 2 elements of  $L$  with  $\xi_i \wedge \xi_j$  corresponding to  $[\xi_i, \xi_j]$ . But, by the criterion for strong freeness, the sequence  $\rho_1, \dots, \rho_m$  is strongly free and hence  $\text{cd}(G) = 2$ . Note that that rank  $M_L = m$  implies  $\dim H^2(G) = m$ .

## 8. Conclusion

It appears that the apparent lack of an important application of the criteria for strong freeness found in Labute [11] and the the sentiment that a pro- $p$ -group whose derived factors are finite could not be of cohomological dimension 2 was responsible for the delay in the discovery of Theorem 1. Schmidt [14] extended Theorem 1 to the case of global fields using Theorem 2.. Mináč and the author [16] extended these to the case  $p = 2$ . Forré [15] gave an independent treatment which covered the case  $p = 2$ . In J. Gartner [17], using higher Massey products, extended Theorem 2 to obtain mild pro- $p$ -groups defined by relators of arbitrary degree. There is a large body of research on mild pro- $p$  groups to which references can be found in the bibliography of Mináč et al [18].

## ORCID

0009-0006-9971-0468 (John Labute)

## Acknowledgments

This paper was written at the El Monte Eco Lodge in Mindo, Ecuador. The author wishes to thank Mariela, Tom and their staff and guides for providing a stimulating and restful environment.

## Author Contributions

John Labute is the sole author. The author read and approved the final manuscript.

## Conflicts of Interest

The author declares no conflicts of interest.

---

## References

- [1] R. Lyndon, *Cohomology theory of groups with a single defining relation*, Annals of Math., Series 2, t. 52, (1950), 650-665. <https://doi.org/10.2307/1969440>
- [2] Jean-Pierre Serre, *Structure de certains pro-p-groupes*, Séminaire Bourbaki, 252, (1963), 1-11.
- [3] A. Brumer, *Pseudocompact algebras, profinite groups and class formations*, Journal of Algebra, 4, (1966), 442-470. [https://doi.org/10.1016/0021-8693\(66\)90034-2](https://doi.org/10.1016/0021-8693(66)90034-2)
- [4] J. P. Labute, *Algèbres de Lie et pro-p-groupes définis par une seule relation*, Invent. Math. 4, (1967), 142-158. <https://doi.org/10.1007/BF01425247>
- [5] H. V. Waldinger, *On extending Witt's formula*, J. Algebra 5, 41-58 (1967). [https://doi.org/10.106/0021-8693\(67\)900245](https://doi.org/10.106/0021-8693(67)900245)
- [6] D. Gildenhuys, *On pro-p-groups with a single defining relation*, Invent. Math. 5 (1968), 357-366. <https://doi.org/10.1017/BF01389782>
- [7] J. P. Labute, *On the descending central series of groups with a single defining relation*, J. Algebra 14, 16-23 (1970). [https://doi.org/10.1016/0021-8693\(70\)90130-4](https://doi.org/10.1016/0021-8693(70)90130-4)
- [8] H. Koch, *Galois Theory of p-extensions*. Springer-Verlag, Berlin, 0222, xiv+190pp. <https://doi.org/10.101007/978-3-04967-9>
- [9] H. Koch, *Über pro-p-gruppen der kohomologischen dimension 2*, Math. Nachr. 78 (1977), 285-289. <https://doi.org/10.1002/mana.19770780125>
- [10] D. Anick, *Non-commutative algebras and their Hilbert series*, J. Algebra 78 (1982), 120-140. [https://doi.org/10.1016/0021-869336\(82\)90104-1](https://doi.org/10.1016/0021-869336(82)90104-1)
- [11] J. Labute, *The Determination of the Lie algebra associated to the central series of a group*, Trans. Amer. Math. Soc. 288 (1985), 51-57. <https://doi.org/10.2307/2000425>
- [12] D. Anick, *Inert sets and the Lie algebra associated to a group*, J. Algebra 111 (1987), 154-165. [https://doi.org/10.1016/0021-8693\(87\)90245-8](https://doi.org/10.1016/0021-8693(87)90245-8)
- [13] J. Labute, *Mild pro-p-groups and Galois groups of p-extensions of  $\mathbb{Q}$* , J. Reine Angew. Math. 596 (2006), 115-130. <https://doi.org/10.1515/crelle.2006.058>
- [14] A. Schmidt, *Über pro-p-fundamentalgruppen markierte arithmetischer curven*, J. Reine Angew. Math., 640 (2010), 203-235. <https://doi.org/10.1515/crelle.210.025>
- [15] P. Forré, *Strongly free sequences and pro-p-groups of cohomological dimension 2*, J. Reine Angew. Math. 658, (2011). <https://doi.org/10.1515/crelle.2011.067>
- [16] J. Labute, J. Mináč, *Mild pro-2-groups and 2-extensions of  $\mathbb{Q}$  with restricted ramification*, J. Alg, 332 (2011), 136-158. <https://doi.org/10.1016/j.algebra.2011.01.019>
- [17] J. Gärtner, *Higher Massey products in the cohomology of mild pro-p-groups*, J. Alg., 422 (2015)788-822. <https://doi.org/10.1016/j.algebra.2014.07.023>
- [18] J. Mináč, F. Passini, C. Quadrelli, D.T. Nguyễn, *Mild pro-p-groups and the Koszulity Conjectures*, Expositiones Mathematicae 40. <https://doi.org/10.1016/j.exmath.2022.03.004>