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Research Article

# On the Socle of Finite Primitive Permutation Groups Having Frobenious Structure

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## Abstract

The nilpotentcy class for the Frobenius was determined based on the structure theorem. The socle of the groups were observed to be regular normal and elementary abelian such features were the conditions for the nilpotency classes, as they were the basis on which the socle of these groups constructed were nilpotent of some classes or order. The socle of the nilpotent groups whose structures is in conformity with D were classified based on the classification scheme for the finite primitive groups in relation to socle type. The socle type described in the classification scheme was in condition (1) was in line with the structure of D, as such it pave way in determining the socle with nilpotency class having same or similar structure with D. Further investigations showed that Frobenious group's were 2-transitive and the structure of D gave the conditions it being regular elementary abelian and so is nilpotent. It was observed the stabiliser of the groups in a finite primitive groups were paramount in the determination of the socle of the groups, as such much attention was given to the stabilizer of each group under consideration in a quest to determine the socle and the nilpotency class. The other conditions for the classification of finite primitive groups based of the socle type were not given much attention as it could give the needed condition for the existence of nilpotency class of the groups, as groups of such types were either almost simple, diagonal, product or twisted wreath product type. Therefore finite primitive group's under those conditions which could not give the expected nilpotency class and order were not give much attention. The degree of homogeneity was not given much priority as the article intended to discuss only the socle type and it nilpotency class or order.

## Keywords

Frobenius, Groups, Socle, Nilpotency, Finite, Abelian, Regular

## 1. Introduction

The structure of a finite Frobenius group was as a result of the work carried out on primitive groups by [11] and was further investigated by Burnside [2]. He showed that a finite Frobenius group has proper nontrivial characteristic subgroups. Therefore, this idea played a key role in the study of finite 2-transitivee permutation groups as in the work (2 (The proof of the work of [11] was geared toward the use of character theory. The major work on this group was carried out by [2]. He gave the outline of the structure for finite Frobenius but in the real sense, the structure of the Frobenius group was described by Frobenius in paper [11], Zassenhaus and Thompson as stated in [10].

Let $G$ be a finite Frobenius group and with point stabilizer such that;

$$\text{(D)}\ K = \{\, x \epsilon G \mid x = 1 \text{ or } fix(x) = \emptyset \,\}$$

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Then $D$ is a subgroup of $G$ and is nilpotent, also for any prime, $p$, for which the Sylow p- subgroup of $G$ has a point stabilizer which is cyclic. Throughout we denote $D$ as the Frobenius subgroup.

The statement for the structure of finite Frobenius groups as given in D was due to work of the work of Frobenius in [11] which was later generalised in the work of [18]. It follows that a permutation group which is a Frobenius group based on the above conditions has a regular abelian subgroup and the point stabilizer has only one element of order 2. He further state that if H is a primitive group but not regular and, also $\frac{3}{2}$-transitive, then $H$ is normal in $G$,

Therefore, every finite primitive group has a unique minimal normal subgroup which is regular abelian and simple and also isomorphic to each other. Also [13] worked on finite p-groups with a Frobenius group of automorphism, whose kernel is a cyclic p- group. In his work, he defined a Frobenius group as a finite normal subgroup which is non-trivial. We intend in this paper to determine the socle of primitive with a subgroup satisfying the condition defined in D, with a view to determine subgroup generated by minimal normal subgroups with a structure satisfying $D$ that is the socle of the group. The classification scheme for finite primitive based on the O'Nan-Scott theorem rest basically on the socle type of the finite primitive groups. The notion of k- homogeneity was carried out by [12] for k<5 where the geometric approach was employed. Further investigation were due to the work of [8] and [16] in which the set were partitioned into tabloids. The notion of the 0-miinimal structures in the work of [16] was as a result of [5] on aspects of infinite permutation groups.

## 2. Preliminary Result

Theorem 2.1: Let $G$ be a finite group which is not regular and $H$ a subgroup with $G_\alpha$ a point stabilizer of $G$. Therefore $G$ can be written as $G = HG_\alpha$.

Basically this theorem could be found in the work carried out in [3], [7] and notablly the work of [15].

Theorem 2.2: Let $G$ be a finite Frobenius group and $G_\alpha$ be a point stabilizer and let D be defined, then the following hold.

1. $D$ is a subgroup of $G$ which is normal and regular in $G$.
2. For each odd prime, $p$, the Sylow p-subgroup of $G_\alpha$ is cyclic.
3. $D$ is nilpotent.

The structure theorem for the Frobenius group is the necessary and sufficient condition for a subgroup to have a structure of $D$ of a Frobenius group. The subgroup are fix point free. Suppose $G$ is transitive then $D$ is a derangement except for the identity subgroup. With these we take the following.

Remark 2.3:

In the case that $G_\alpha$ is not soluble, then $G$ has only one abelian composition factor which is $A_5$.

The development of the structure theory were borne out of the idea of [9].

We now state a theorem which gives us a condition for the existence of a Frobenius group with a point stabilizer having only one element of even order. This assertion only satisfy the condition in which $p = 2$ exactly, since it is the only prime number which is also even.

The next result is given in the relation to the concept of reugular abelean groups defined in [14].

Theorem 2.4: Let $G$ be a finite Frobenius group of degree n, and $D$ a subgroup. If $G_\alpha$ has even order then $D$ is a regular normal abelian subgroup of $G$ and $G_\alpha$ has exactly one element of order 2.

Proof:

For $G_\alpha$ to have even order it imply that it have elements of order of two. Let $T$ be the $G$ −conjugacy class containing these elements. Since the point stabilizers are disjoint, each of the point stabilizer contains at least one element from $T$and $|T| \geq n$. Consider the cycle decomposition of an element $t \in T$suth tht $t$ has one cycle of length 1 and $\frac{n-2}{2}$ −cycle of length 2. Since no nontrivial elment of $G$ has more than one fixe point, no two elements from $G$ can contain the same $2$ −cycle. There are exactly $\frac{n(n-1)}{2}$, $2$ −cycles in $sym(\Omega)$, and so we coclude that $\frac{|T|(n-1)}{2} \leq \frac{n(n-1)}{2}$ and hence $|T| \leq n$. But $|T| \geq n$, therefore $|T| = n, and\ every$ $2$ −cycle occurs inone of the element of $T$. In particular, each point stabilizer contains exactly one element from $T$, and $T$ contains all elements of order $2$ in $G$.

We suppose that $st \in D$ for any $s, t \in T$ then we assume that $fix(st) = \Phi$ for if not then any $\beta \epsilon \Omega$ we have $fix(st) = \beta$ which contrry to the definition in $D$ and we may have$\beta^t = \beta^{(st)t} = \beta$for any distinct $s$ and $t$, and so eithr $(\beta\ \beta^s) = (\beta \beta^t)$ is a $2$ −cycle in both $s$ and $t$. But the case is not possible. Therefore $st \epsilon D$ as asserted.

$fix(t) \epsilon T\ then\ tT \subseteq D$, and since both have size $n$ we onclude that $tT = D$ in particular $1 \epsilon D$ and fore, $DD^{-1} \subseteq TT \subseteq D$, and so $D$isa subgroup and further $D\ is$ abelian.

The next result is due to the idea of a statement of result in [10]. In another development the study of regular abelian group's were captured in the work of [4] and [6]

Theorem 2.5: Let $G$ be a $2$ −transitive Frobenius group, and $D$ a subgroup. Then suppose that either $G$ is finite or the point stabilizer $G_\alpha$ is abelian, then $D$ is regular normal abelian subgroup of $G$ in which each nontrivial element has same order.

Proof:

Suppose $|\Omega| = n$ then $|D| = n$ for if $G$ is $2$ −transitive then $|G_\alpha|$ divides $n$ −1 therefore $G$ is a Frobenius group if for any $u \neq 1$ and $u \in D$ then $C_G(u) \subseteq D$ then $|G: C_G(u)| \geq n - 1$. Therefore $u$ has atleast one conjugate element in $G$. On the other hand each conjugate element of $u$ is cearly a nontrivial element from $D$ then we conclude that $C_G(u) = D$. Thus we have shown that $D$ is a subgroup and ach element of $D$ lies in the centre of $D$. Hence $D$ is elementary abelian $p$ −group and so $D$ is regular and normal. Further based on Theorem 2.4 $D$ is abelian also since $G$ is Frobenius then for any $s,t \in D$ and $\alpha,\beta\epsilon\Omega$ then $(\alpha\beta)^t = (\beta\alpha)$, therefore, for any $(t \neq 1)$, it imply $t^2 = 1$, and so $t$ fixes both $\alpha$ and $\beta$. Conversely, if $G$ is Frobenius then every element x$x, y \in G$ imply $x, y^{-1} \in D$ and so $G$ is of prime degree, and so by Theorem 2.4, $D$ is nilpotent. The statement also follows from Theorem 2.2, as such $D$ is cyclic which by implication $D$ is abelian. Further for any $u, v \in D$ there is $\alpha,\beta\epsilon\Omega$ such that $\alpha^u = \beta^v$=1 and so $u = v$. Hence the theorem.

Theorem 2.6 [10]: Let $G$ be a finite primitive group with abelain point stabilizer, then $G$ is either, regular or of prime degree or a Frobenius group.

Proof: Since $G$ is a primitive group and not regular, then $G$ is a Frobenus group and has a subgroup which is a Sylow $p$ −subgroup, which is regular and abelian. Therefore $G_\alpha$ is of the type $D$, and so $D$ is nilpotent which also by Theorem 2.2, it shows condition 1 imply condition 3 thus $G$ is Frobenius.

Conversely, suppose $G$ is Frobenius, it follows from Theorem 2.1 that, $G$ has a subgroup which is regular and abelian of prime degree, therefore the subgroup may be of type $D$. Hence $G$ has $G_\alpha$ asc the only proper subgroup and so $G$ is primitive.

Next we define nilpotency of a finite primitive group as it is a requirement in the attainment of a Frobenius structure.

Definition 2.7: A finite primitive group $G$ is nilpotent if and only if $G$ has an upper central and a lower central series. That is there is an integer $k \geq 0$ such that $\gamma_k(G) = 1$.

Therefore we state categorically that if $G$ is nilpotent and $N$ is normal then $G/N$ is nilpotent. Also if $G$ is nilpotent then it imply $G$ is soluble. The next result clearly gives condition for the existencet of nilpotency classes.

Theorem 2.8: Let $G$ be a finite primate group with a nilpotent point stabilizer. Then $G$ is soluble if the Sylow 2 −subgroup of the point stabilizer is also nilpotent of class at most 2.

Proof: Suppose $G$ has a nilpotent point stabilizer, then $G_\alpha$ is cyclic and so the point stabilizer is an elementary $p$ abelain group. Therefore by Definition 2.6 and the condition that $G_\alpha$ is a normal subgroup of $G$ it imply that $G$ is soluble which follows immediately from Theorem 2.2.

Conversely if $G$ is of order say, $np^a$ for $a \geq 1$ then $G$ contain a normal subgroup which a Sylow $p$ −subgroup. Therefore if can be deduce from Theorem 2.1 that $G_\alpha$ is nilpotent, and so by Theorem 2.2(2) $G_\alpha$ is nilpotent of class at most 2.

The next result followed from the work of [4].

Theorem 2.9: Let $G$ be a finite primitive group with a maximal subgroup $M$ which is abelian, then $G$ is solvable.

Proof: If $G$ is primitive then for any normal subgroup of $G$, say,$D$ is maximal implying $M \leq D$, so we assume $D = M$ Therefore the composition series for $G$ is of the form $G \rhd D \rhd 1$ and so $D$ is a normal subgroup of $G$. Moreover since $G$ is primitive it imply that $D$ is abelian and so it is of order a power of $p$.

Remark 2.10: the necessary and sufficient condition that a finite primitive group is nilpotent is $G$ is a finite $p$ −group and abelian.

Theorem 2.11: Let $G$ is $p$ −group then $G$ is nilpotent.

Proof: Suppose $|G| = 1$, the result is trivialy true. We assume that $|G| \geq 2$. Also assume inductively that the theorem holds for all $p$ −groups of order less than $|G|$ hence it iply $G/Z(G)$ is nilpotent. Thus $G$ is nilpotent if and only if $G/Z(G)$ is cyclic.

Therefore we take the statement of the following theorem whose proof will not be given here, it is based on the work carried out in [1].

Theorem 2.12 [1]: Let $G$ be a primitive group. The following are equivalent.

i. $G$ is nilpotent.
ii. Every subgroup of $G$ is subnormal in $G$
iii. Whenever $H$ a proper subgroup of is $G$ then $H$ is a proper subgroup of its normalizer in $G$.
iv. Every maximal subgroup of $G$ is normal in $G$
v. $G' \leq \vartheta(G)$
vi. Every Sylow $p$ −subgroup of $G$ is normal in $G$
vii. $G$ is a direct product of groups of prime power order.

The next theorem is as a result of the work of [17] in relation to the structure of $D$which ill help in determining the socle for $G$ having a subgroup in the form of $D$.

Theorem 2.13 {10]: Let G be a group which acts primitively and on $\Omega$ with $|\Omega| = n$. Let $H = soc(G)$ and $\alpha \in \Omega$. Then H is of type T then $G$ is affine and T is abelian of order p and $n = p^n$ $and$ $G_\alpha$ is a complement to which acts on H and is simple.

We can say therefore, if $G$ has a minimal normal subgroup $K$ say, then for some prime $p$ and some integer $(d, k)$, $G$ is a regular elementary abelian group of order $p^d$ and $soc(G) = K = C_G(K)$. We observe further, if $K$ has the structure of $D$ the $soc(G) = D = C_G(D)$ which is also an elementary abelian group of prime power $p^d$ and also isomorphic to an affine group. This suffices to say $D$ is a subgroup and is nilpotent. Thus we can clearly state that socle are subgroups of normalizers subgroups.

Theorem 2.1 [10] 4: Let G be a finite group, then $soc(G) \leq N_G(H)$ for each subnormal group $H$ of $G$.

Proof:

The result is certainly true if $G = H$. Therefore induction on $|G:H|$ with $H < G$ show that each minimal normal subgroup K of G is contained in the normalizer of $N_G(H)$.

Since it is subnormal, there exist $L \lhd G$ $with$ $H \leq L \lhd G$. Then either $K \leq L, (K, L) = KL$ . Hence $K \leq C_G(H) \leq N_G(H)$ and the result is true. So suppose that $K \leq L$. Then there exists a minimal normal subgroup of $x^{-1}L = L$, and induction shows that

$$K= \{ x^{-1}Hx\} \leq N_L(H) \leq N_G(H).$$

Since K is a minimal normal subgroup of G we have that $K = \{x^{-1}Hx | \ x \in G\}$. Implying $K \leq N_G(H)$.

## 3. Main Results

Theorem 3.1: Let $G$ be a finite group of order n and $H$ a subgroup of order $p^a$ for $a \geq 1$ then $soc(G)$ is a regular abelian group of order a power of $p$.

Proof:

For $G$ to regular imply $G$ then by Theorem 2.6 show that $G$ is either a Frobenius group or reguar if and only if $G$ has an abelian stabilizer so let $H$ be a subgroup of $G$ with the structure as defined in $D$. If $H = G_\alpha$ then $H$ is regular and abelian, also suppose $H$ and maximal with $H = soc(G)$ imply $H$ is nilpotent.

Conversely suppose $G$ is nilpotent and $H$ is normal in $G$ then every Sylow $2-$subgroup of $G$ is an elementary $p-$abelian group of order a power of $p$. Therefore if $H$ is maximal Theorem 2.13 imply $soc(G)$ is regular and abelian of order $p$.

Theorem 3.2: Let $G$ be a finite nilpotent group with a regular normal subgroup $H$ of prime order. Then $soc(G)$ is nilpotent.

Proof: Since $G$ is niplotent it shows that the normal subgroups of $G$ is abelian and there let $H$ be a normal of $G$ with the structure as defined in $D$ then $H$ is regular and abelian and is also maximal, otherwise there may be a chain of subgroup with another proper subgroup say, $M$ such that $G \supset H \supset M \supset \{e\}$ for $M \leq H$, this shows that $G$ is nilpotent of class 3. But $H$ is a maximal subgroup of $G$ therefore it is nilpotent of class at most 2 by Theorem 2.7, and so, suppose $H = G_\alpha$ and that $soc(G) = H$ then by theorem 2.12 $G$ is an elementary $p-$abelian group swhich imply $soc(G)$ is nilpotent.

Theorem 3.3: Let $G$ be a finite nilpotent group and $H$ a subnormal group of $G$, therefore $soc(G)$ is an elementary abelian $p-$. group.

Proof: suppose $G$ is primitive of degree $n$, imply that $|G| = n = p^a m$, and $G$ has Sylow subgroup by first sylows theorem. Let $H$ be the subgroup of $G$, since $G$ is primitive show that $H$ is maximal and so by Theorem 3.3,$H$ is of type $D$ above consequently $H = soc(G)$ which is also elementary $p-$ abelian.

Conversely if $G$ is an abelian $p-$group, it imply that any chain of subgroup of $G$ has a maximal subgroup $H$ say, therefore by Theorem 2.7 it imply that $H = soc(G)$ and so $H$ is nilpotent of class 2 and so by Theorem 3.3, the chain $G \supset H \supset M \supset \{e\}$ has a subnormal group $M$ in line with Theorem 2.14 and so $M = H$, hence $G$ is nilpotent.

## 4. Conclusion

The regular subgroup of the type as defined in $D$ and their nilpotency classes were obtained for Frobenius groups with regular normal abelian subgroups. These groups were of order a power of $p$ in which most of the subgroups had the structure of $D$. The socle of $G$ had a direct relation with the classification scheme foe finite simple groups based on the socle type as is in the work of [17]. The case for which $p$ was of order 2 was determined for groups of even order. It showed that the socle of the groups having the structure of $D$ were transitive and nilpotent.

## Abbreviations

| | |
|---|---|
| soc(G) | Socle of the Group G |
| fix(x) | Fix of an Element x in G |
| Supp(x) | Support of the Element x in G |

## Conflicts of Interest

The authors declare there is no conflict of interest.

## References

[1] Audu, M. S., & Momoh, S. U (1990): On transitive Permutation groups. *Abacus*, 19, (2). 17-23.

[2] Burnside, W (1911). *Theory of groups of finite order*. Cambridge University press. London.

[3] Cameron, P. J. (1981). Finite permutation groups and finite simple groups*; London Math. Soc:* https://doi.org/101112/BLMS/13.1.1

[4] Cameron, P. J. (1999). Permutation groups. Math. Soc, Student Text 45, Cambridge University Press.

[5] Cameron, P. J. (2000). Aspect of infinite permutation groups: School of Mathematical Sciences, Queen Mary, *University of London.*

[6] Cameron, P. J. (2000). t-orbit homogenous permutations. London Math Soc. Subject Classification 20B10, https://doi.org/10.1112/Sooooooooooooooo

[7] Cameron, P. J. (2004). The encyclopaedia of design theory on primitive permutation groups. www.maths.qmul.ac.uk Retrieved on 12/07/2015.

[8] Cameron, P. J (2012). Permutation groups and regular semi groups. London Math Subject classification. 1273-1285.

[9] Cameron, P. J. (2013) Permutation group and transformation semi-group, Novi sad Algebraic Conference. *University of St. Andrews.*

[10] Dixion, J., & Mortimer, B (1996). *Permutation groups Graduate texts in mathematics.* Springer New York`.

[11] Frobenius, F. G. (1904). Uber die charaktere der mehch trasitiven gruppen. mouton de grupter 558-571. *Berlin.*

[12] Kantor, W. M. (1972). *k*-Homogeneous permutation groups. Math Z, 124, 261-265. *Springer Verlag*.

[13] Khukhr, O (2013). Finite *p*-groups with a Frobenius group of automorphism whose kernel is cyclic *p*-group. *Manchester institute for Mathematical Sciences*: *School of Mathematics*, *University of Manchester*.

[14] Liebeck, W., & Praeger C. E, & Sax, I. J. (2000). Primitive permutation groups with a regular subgroup. *Journal of Mathematics subject classification.* 20B15, 05C25.

[15] Livingstone, D., & Wagner, A. (1965). Transitivity of finite permutation groups on unordered sets. *Math Z.* 90 393-403.

[16] Martin, W. J., & Segan, B (2000). New notion of transitivity for groups and set of permutations. *Journal of Math Soc. Math Subject Classification 20B20*.

[17] O′Nan, M. E., & Scott, L (1979). Finite groups. Santa Cruz conference. London *Math Soc* (2): 32.

[18] Wielandt, H. (1969). Permutation groups through the invariant relation and invariant functions. Columbus. 399514.