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Fixed Point Theorem in Fuzzy b-Metric Space Using Compatible Mapping of Type (A)

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Abstract: One of the most active and developing fields in both pure and applied mathematics is the theory of fixed points. It is possible to formulate a large number of nonlinear issues that arise in many scientific domains as fixed point problems. Since Zadeh first introduced the concept of fuzzy mathematics in 1965, the interest in fuzzy metrics has grown to the point that several studies have concentrated on examining their topological characteristics and applying them to mathematical issues. This was primarily because, in certain situations, fuzziness rather than randomization was the cause of uncertainty in the distance between two spots. Many mathematicians have examined and developed the concept of distance in relation to fuzzy frameworks because it is a naturalist concept. Generally speaking, it is impossible to determine the precise distance between any two locations. Thus, we deduce that if we measure the same distance between two locations at different times, the results will differ. There are two approaches that can be used to manage this situation: statistical and probabilistic. But instead of employing non-negative real numbers, the probabilistic approach makes use of the concept of a distribution function. Since fuzziness, rather than randomness, is the cause of the uncertainty in the distance between two places. Because of the positive real number $b \geq 1$, the area of fuzzy b-metric space is larger than fuzzy metric space. Thus, this field is the source of our concern. This study aims to use the notion of compatible mappings and semicompatible mappings of type (A) to develop some common fixed point theorems in fuzzy b-metric space. A few ramifications of our primary discovery are also provided. Included are pertinent examples to highlight the importance of these key findings. Our results add to a number of previously published findings in the literature.

Keywords: Fuzzy b-Metric Space, Compatible Mappings, Common Fixed Point

1. Introduction

The theory of fixed points is one of the most dynamic and expanding area in applied and pure mathematics as well. A wide range of nonlinear problems encountered in various scientific fields can be formulated as fixed point problems. The Banach contraction principle plays a key role in addressing such problems. Over time, fixed point theory (FPT) has proven effective in solving a diverse array of problems, significantly contributing to real-world applications. Many strong fixed point theorems have been established, though often based on strong assumptions. Recent research has focused on understanding the core principles of fixed point problems while relaxing these stringent conditions by using modified assumptions. In general it is not possible to measure the exact distance between any two places precisely. Thus we conclude that while measuring the same distance between two places in different times, we will get the different results. This situation can be handled by two ways probabilistic and statistical approach. But by using the probabilistic approach, it uses the idea of distribution function instead non-negative real numbers. As the uncertainty in the distance between two points is due to fuzziness instead of randomness.

In 1965, L. A. Zadeh [17] introduced the concept that addresses ambiguity, imprecision, and manipulation, which stands in contrast to classical set theory, offering a more intriguing and practical approach. These methods have been applied across various scientific and technical fields, including navigation, image processing, and fractals. Since then, the theory of fuzzy sets has been widely expanded, with numerous authors applying it to the areas such as topology and analysis. In general it is not possible to measure the exact distance between any two places precisely. Thus we conclude that while measuring the same distance between two places in different times, we will get the different results. This situation can be handled by two ways probabilistic and statistical approach. But by using the probabilistic approach it uses the idea of distribution function instead non-negative real numbers. As the uncertainty in the distance between two points is due to fuzziness instead of randomness. Motivated from the this principle, In 1975, I. Karmosil and J. Michalek [10] introduced the concept of fuzzy metric space (FMS), as an extension of the metric space, incorporating fuzzy scenarios. A. George and P. Veeramani [4] further refined the idea of fuzzy metric spaces, which has important implications in quantum particle physics as well. M. Grabiec [5], in 1983, explored the completeness property of fuzzy metrics and extended Banach's contraction theorem to these spaces. Since then, numerous researchers have contributed to further generalizations and extensions.

In 1889, Bourbaki, I. Bakhtin [1] with S. Czerwik [2] initially proposed the idea of b-metric space. Later formalizing the definition of b-metric spaces. Various researchers have explored examples and fixed-point results within these spaces. S. Sedghi and N.Shobe [14] extended this work by introducing fuzzy b-metric spaces, which are broader than fuzzy metric spaces, using a weaker form of the triangle inequality. S. Nadaban [11] introduced some of the topological aspects of a fuzzy b-metric space and presented the concept of a fuzzy b-metric space and also introduced some of its basic properties in terms of topology.

G. Jungck [6] made the first significant contribution to the concept of fixed point theory for the mapping of compatible by generalizing previous results. Later, Y. J. Cho [7] and other researchers explored a more modified form called type (A) compatible mappings, that under certain conditions is similar to the idea of compatible mappings. Also they gave the proof of a common type FPT to these mappings in fuzzy b-metric spaces.

Here we would like to introduce the properties of type (A) compatible mappings of fixed point theorems in fuzzy b-metric spaces and give the proof of the related theorems by modifying the results of R. Yumnam [16], K.Jha and K.B. Manandhar [9] and Jungck , Murthy and Y.J. Cho [7]. Also generalize many previous results existing in the literature.

2. Preliminaries

In 1960, Schweizer and Sklar introduced the operation of t-norm and using the concept of continuity.

Definition 2.1 [13] If we define a map $*$ from $[0,1]\times[0,1]$ to $[0,1]$ is known as a continuous triangular norm (t-norm) with the following properties.

(i) Symmetry: $m * n = n * m$, for $m, n \in [0,1]$;

(ii) Monotonocity: $m * n, \leq o * p$ whenever $m \leq n$ and $o \leq p$;

(iii) Associativity: $(m * n) * o) = (m * (n * r))$, where $m, n, o, p \in [0,1]$

(iv) Boundary condition: $1 * p = p$, for all $p \in [0,1]$.

Definition 2.2 [10] A triplet $(U, F, *)$ is known as a fuzzy metric space if U is any set, $*$ is a continuous t-norm and F is a fuzzy set defined on $U \times U \times \mathbb{R}$ to the unit interval $[0,1]$ if it satisfies the properties given as below, for all $u, v, w \in U$ and $t, s > 0$

(FK-1) $F(u, v, 0) = 0$,

(FK-2) $F(u, v, t) = 1$ for all $t > 0 \iff u = v$

(FK-3) $F(u, v, t) = F(v, u, t)$,

(FK-4) $F(u, v, t) * F(v, w, s) \leq F(u, v, t + s)$ for all $t, s > 0$,

(FK-5) $F(u, v, .) : (0, \infty) \to [0,1]$ is continuous from left

The degree of nearness between u and v with respect to $t > 0$ is denoted by $F(u, v, t)$.

Motivated from the work of Karmosil and Michalek, especially the notation of convergence, M. Grabic [3] studied the concept of convergence sequence in fuzzy metric space. But it was observed that the notation of completeness of fuzzy metric space given by Grabiec was incomplete. Therefore, George and Veeramani modified some properties established by Karmosil and Michalek and introduced the new idea of fuzzy metric space.

Definition 2.3 [4] The triplet $(U, F, *)$ is known as a fuzzy metric space if U is any set, $*$ is a continuous t-norm and F is a fuzzy set defined on $U \times U \times (0, \infty) \to [0,1]$ if it satisfies the properties given as below, for all $u, v, w \in U$ and $t, s > 0$

(FG-1) $F(u, v, t) > 0$,

(FG-2) $F(u, v, t) = 1 \iff u = v$

(FG-3) $F(u, v, t) = F(v, u, t)$,

(FG-4) $F(u, v, t) * F(v, w, s) \leq F(u, v, t + s)$ for all $t, s > 0$,

(FG-5) $F(u, v, .) : (0, \infty) \to [0,1]$ is continuous

The property (FG-2) indicates that

$F(u, u, t) = 1$ for all $u \in U, t > 0$ and $F(u, v, t) < 1$ for all $u \neq v, t > 0$

Example 2.4 [4] Consider (S, d) be a metric space and define a mapping $g : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing and continuous. We define $F : S \times S \times (0, \infty) \to [0,1]$ given as

$$F(u, v, t) = \frac{g(t)}{g(t) + \lambda d(u, v)}$$

for all $u, v \in S$, $\lambda \in \mathbb{R}^+$. Then $(U, F, *)$ satisfies the properties of FMS, where the t-norm $*$ is the product norm.

Particularly, consider $g(t) = t^n$ for $n \in \mathbb{N}$ and $\lambda = 1$ in (2.3), then we obtain

$$F(u, v, t) = \frac{t^n}{t^n + d(u, v)}.$$

In this case, $(S, F, *)$ is a fuzzy metric space.

By taking $n = 1$ for above equation, then we will get result for standard fuzzy metric space as stated:

$$F(u,v,t) = \frac{t}{t + d(u,v)}.$$

A fuzzy metric space is an extension of traditional metric space. While measuring the distance between two places is not exact, in such situation we use the concept of fuzzy metric.

In 2016, Nadaban [11] explored the concept of fuzzy b-metric space as the generalized notation for fuzzy metric spaces explored by Kramosil and Michalek.

Definition 2.5 [11] Assume U be a nonempty set, and $k \geq 1$ as a real number and $*$ is a continuous t-norm a fuzzy set F on $U \times U \times \mathbb{R}^+$.

If the following axioms are full filled so the order tuple $(U, F, *)$ is known as the fuzzy b-metric space for all $u, v, w \in U$ and $t, s > 0$

(Fb-i) $F(u,v,t) > 0$;

(Fb-ii) $F(u,v,t) = 1$ for all $t > 0$ if and only if $u = v$;

(Fb-iii) $F(u,v,t) = F(v,u,t)$;

(Fb-iv) $F(u,v,t) * F(v,w,s) \leq F(u,w,k(t+s))$ for all $t, s > 0$;

(Fb-v) $F(u,v,.) : (0,\infty) \to [0,1]$ is left continuous and

(Fb-vi) $\lim_{t\to\infty} F(u,v,t) = 1$

In fuzzy b-metric space if we put $k = 1$ then it becomes a fuzzy metric space.

Example 2.6 [11]

Let $F(u,v,t) = e^{\frac{-d(u,v)}{t^2}}$, as d is considered as a b-metric on U, and let $m * n = m.n$ for $m, n \in [0,1]$.

Now we will show $F(u,v,t)$ is a fuzzy b-metric space.

Let $U = \mathbb{R}$. For a real number $k \geq 1$, we have

(i) For all $t > 0$ and $d(u,v) > 0$ so we have $F(u,v,t) = e^{\frac{-d(u,v)}{t^2}} > 0$, so $F(u,v,t) > 0$.

(ii) If $u = v$ then $d(u,v) = 0$, and hence we easily conclude that $F(u,v,t) = e^{-\frac{d(u,v)}{t^2}} = e^0 = 1$.

conversely, let us consider $F(u,v,t) = 1$ and then $e^{-\frac{d(u,v)}{t^2}} = e^0 = 1$. Which gives $u = v$.

So $F(u,v,t) = 1$ iff $u = v$.

(iii) Now we will show that $F(u,v,t) = F(u,v,t)$

Since $e^{-\frac{d(u,v)}{t^2}} = e^{-\frac{d(v,u)}{t^2}}$ for all $u, v \in \mathbb{R}$.

It follows that for all $u, v \in U$ and for all $t > 0$

$F(u,v,t) = e^{-\frac{d(u,v)}{t^2}} = e^{-\frac{d(v,u)}{t^2}} = F(v,u,t)$.

(iv) for $u, v, w \in U$ and $t_1, t_2 > 0$, we have

$$F(u,w,t_1,t_2) = e^{-\frac{d(u,w)}{t_1^2 + t_2^2}}$$
$$\geq e^{-}k(\frac{d(u,v) + d(v,w)}{t_1^2 + t_2^2}), \text{ for } k \geq 1.$$
$$= e^{-}k(\frac{d(u,v)}{t_1^2 + t_2^2}).e^{-}k(\frac{d(v,w)}{t_1^2 + t_2^2})$$
$$\geq e^{-}k(\frac{d(u,v)}{t_1^2}).e^{-}k(\frac{d(v,w)}{t_2^2})$$
$$= e^{-\left(\frac{d(u,v)}{\frac{t_1^2}{k}}\right)}.e^{-\left(\frac{d(v,z)}{\frac{t_2^2}{k}}\right)}$$
$$= F(u,v,\frac{t_1}{k}) * F(v,w,\frac{t_2}{k}).$$

Thus

$F(u,w,t_1+t_2) \geq F(u,v,\frac{t_1}{k}) * F(v,w,\frac{t_2}{k})$

(v) Let us consider a sequence $\{t_n\}$ in $(0,\infty)$ such that the sequence $\{t_n\}$ converges to t in $(0,\infty)$ i.e.

$\lim_{n\to\infty} |t_n - t| = 0$

We have $e^{-\frac{d(u,v)}{t_n}}$ converges to $e^{-\frac{d(u,v)}{t}}$ as $\{t_n\}$ converges to t with respect to usual metric.

Hence $F(u,v,.) : (0,\infty) \to [0,1]$ is left continuous and $\lim_{t\to\infty} F(u,v,t) = 1$

Hence all the properties of fuzzy b-metric (Fb-MS) spaces are satisfied so $F(u,v,t)$ is a fuzzy b-metric space.

Definition 2.7 [4] Consider $(U, F, *)$ as a fuzzy b-metric space. Then a sequence $\{u_n\}$ in U is called the convergent in U if

$\lim_{n\to\infty} F(u_n, u, t) = 1$ for each $t > 0$.

Since $\lim_{n\to\infty} u_n = u$

Definition 2.8 [4] A sequence $\{u_n\}$ in U is known as the Cauchy sequence in U if

$\lim_{n\to\infty} F(u_n, u_{m+n}, t) = 1$ where $t > 0$ and $m, n > 0$.

When every Cauchy sequence is the convergent in a fuzzy b-metric space (Fb-MS) then the space is known as the complete fuzzy b-metric space.

Definition 2.9 [15] The mappings f and g defined in a fuzzy b-metric space $(U, F, *)$ to itself are known as the compatible if $\lim_{n\to\infty} F(fgu_n, gfu_n, t) = 1$ for all $t > 0$, as $\{u_n\}$ be a sequence in U where $\lim_{n\to\infty} fu_n = \lim_{n\to\infty} gu_n = w$, for some $w \in U$.

Definition 2.10 [7] The self mappings f and g in a fuzzy b-metric space $(U, F, *)$ are known as the compatible of type (A) if $\lim_{n\to\infty} F(fgu_n, ggu_n, t) = \lim_{n\to\infty} F(gfu_n, ffu_n, t) = 1 \ \forall \ t > 0$, whenever $\{u_n\}$ is a sequence in U such that $\lim_{n\to\infty} fu_n = \lim_{n\to\infty} gu_n = w$, for some $w \in U$.

Definition 2.11 [12] The self mappings f and g of a FBM $(U, F, *)$are said to be compatible of type (P) if $\lim_{n\to\infty} F(ggu_n, ffu_n, t) = 1$ whenever $\{u_n\}$ is a sequence in U such that $\lim_{n\to\infty} fu_n = \lim_{n\to\infty} gu_n = u$ for some $u \in U$ and $t > 0$.

Definition 2.12 [9] The self mappings f and g of a metric space (U, d) are called compatible of type (K) if $\lim_{n\to\infty} ffu_n = gu$ and $\lim_{n\to\infty} ggu_n = fu$, whenever $\{u_n\}$ is a sequence in U such that $\lim_{n\to\infty} fu_n = \lim_{n\to\infty} gu_n = u$ for some $u \in U$.

Definition 2.13 [3] Let f and g be a pair of mappings of a fuzzy metric space $F(U, F*)$. Then the mappings are said to be semi-compatible if

$$\lim_{n\to\infty} F(fgu_n, hu_n, t) = 1, \quad \text{for all } t > 0$$

whenever $\{u_n\}$ be a sequence in U which gives

$$\lim_{n\to\infty} fu_n = \lim_{n\to\infty} gu_n = u$$

for some $u \in U$

Definition 2.14 [8] Let f and g be a pair of mappings of a FMS $(F, U, *)$. Then these mappings are known as weakly compatible if they commute at the coincidence points for $u \in U$

$$fu = gu \implies fgu = ghu.$$

Clearly, if (f, g) is semi-compatible and $fu = gu$, then $fgu = gfu$. Thus we can say, semicompatibility implies weak compatibility, but the converse is not true.

Lemma 2.15 Let $(U, F, *)$ be a fuzzy b-metric space, for a given real number $b \geq 1$ If there is a real number$k \in \left(0, \frac{1}{b}\right)$ which gives $F(u, v, kt) \geq F(u, v, t)$ then $u = v$.

Corollary 2.16: Let f and g be the compatible self mappings of a fuzzy b-metric space $(U, F, *)$ and there is real number $b \geq 1$ and

(a) If $fv = gv$ then $fgv = gfv$.

(b) If $fu_n, gu_n \to v$, for some $v \in U$ then

(c) $gfu_n \to fv$ if f is continuous.

(d) Consider the the continuous mapping If f and g at a point v then $fv = gv$ and $fgv = gfv$

Proof: (a) Assume $fv = gv$ and $\{u_n\}$ be a sequence in U with $u_n = v \; \forall \; n$. Then $fu_n, gu_n \to fv$. Now by using the compatibility of f and g, we get $F(fgv, gfv, t) = F(fgu_n, gfu_n, t) = 1$, it gives $fgv = gfv$.

(b) If $fu_n, gu_n \to v$, for some $v \in U$ then

(c) By the continuity of f, $fgu_n \to fv$ and using the compatibility of f, g

$F(fgu_n, gju_n, t) = 1$ as $n \to \infty$, which gives $gfu_n \to fv$.

(d) If f and g are continuous then from (c) we have $gfu_n \to fv$.

But by the continuity of g, $gfu_n \to gv$.

So by uniqueness of the limit $fv = gv$. Hence $fgv = gfv$ from (a).

Lemma 2.17 Let $(U, F, *)$ be a fuzzy b- metric space. There exists a real number $b \geq 1$ and $k \in \left(0, \frac{1}{b}\right)$ where $F(u, v, kt) \geq F\left(u, v, \frac{t}{k^n}\right)$ for any positive integer n. Then $\lim_{n\to\infty} F(u, v, t) \geq 1$ then $u = v$.

Corollary 2.18 Let $(U, F, *)$ is a fuzzy b-metric (Fb-MS) space also assume the continuous mappings f and g of U so the mappings f and g are are compatible iff if they are type (A) compatible.

Corollary 2.19 Assume that $(U, F, *)$ be a fuzzy b-metric space and let f and g be the mappings of compatible of types (A) also $fw = gw$ for some $w \in U$then $ffw = fgw = gfw = ggw$.

corollary 2.20 Let $(U, F, *)$ be a (Fb-MS) and also assume f and g be the compatible mappings of type (A) and suppose $fu_n, gu_n \to w$ as $n \to \infty$, for some $w \in U$ then $ffw = fgw = gfw = ggw$. Then

(a) $\lim_{n\to\infty} fgu_n = gw$ if g is continuous at w.

(b) $fgw = gfw$ and $fw = gw$ if f and g are the continuous mappings at point w.

3. Main Result

Theorem 3.1 Consider $(U, F, *)$ as a complete Fb-MS and given a real number $b \geq 1$ and assume that the self mappings f, g, h, and J of U full fill the following axioms:

(a) $f(U) \subset J(U)$, $g(U) \subset h(U)$,

(b) g and J are continuous,

(c) The pairs (f, h) and (g, J) are the type (A)compatible maps on U,

(d) There is $k \in \left(0, \frac{1}{b}\right)$ where for every $u, v \in U$ and $t > 0$,

$$\begin{aligned} F(fu, gv, kt) \geq\ & F(hu, Jv, t)* \\ & F(fu, hu, t) * F(gv, Jv, t) * F(fu, Jv, t). \end{aligned}$$

Then U contains the unique common fixed point of f, g, h, and J.

proof: As we have $f(U) \subset J(U)$ and $g(U) \subset h(U)$, for any $u_0 \in U$, $\exists \; u_1 \in U$ where $fu_0 = ju_1$, and for this $u_1 \in U$, $\exists \; u_2 \in U$
which gives $gu_1 = hu_2$.

Let us define a sequence in inductive form as $\{v_n\}$ in U such that

$$v_{2n-1} = Ju_{2n-1} = fu_{2n-2} \text{and} v_{2n} = hu_{2n} = gu_{2n-1}, \quad \text{for all } n = 0, 1, 2, \ldots$$

By using the axiom of above definition (d),

$$\begin{aligned} F(v_{2n+1}, v_{2n+2}, kt) & \\ &= F(fu_{2n}, gu_{2n+1}, kt) \\ &\geq F(hu_{2n}, Ju_{2n+1}, t) * F(fu_{2n}, hu_{2n}, t) * F(gu_{2n+1}, Ju_{2n+1}, t) * F(fu_{2n}, Ju_{2n+1}, t) \\ &= F(v_{2n}, v_{2n+1}, t) * F(v_{2n+1}, v_{2n}, t) * F(v_{2n+2}, v_{2n+1}, t) * F(v_{2n+1}, v_{2n+1}, t) \\ &\geq F(v_{2n}, v_{2n+1}, t) \\ &* F(v_{2n+1}, v_{2n+2}, t). \end{aligned}$$

From corollary 2.13, we have

$$F(v_{2n+1}, v_{2n+2}, kt) \geq F(v_{2n}, v_{2n+1}, t). \tag{1}$$

Similarly, we have

$$F(v_{2n+2}, v_{2n+3}, kt) \geq F(v_{2n+1}, v_{2n+2}, t). \tag{2}$$

Combining (1) and (2), we have

$$F(v_{n+1}, v_{n+2}, kt) \geq F(v_n, v_{n+1}, t). \tag{3}$$

Now using (3), we have

$$\begin{aligned} F(v_n, v_{n+1}, t) &\geq F\left(v_n, v_{n-1}, \frac{t}{k}\right) \\ &\geq F\left(v_{n-2}, v_{n-1}, \frac{t}{k^2}\right) \\ &\geq \dots \\ &\geq F\left(v_1, v_2, \frac{t}{k^n}\right) \to 1 \quad \text{as } n \to \infty. \end{aligned}$$

So, $F(v_n, v_{n+1}, t) \to 1$ as $n \to \infty$ and $t > 0$.
For each $\epsilon > 0$ and $t > 0$, let us take $n_0 \in \mathbb{N}$ which gives $F(v_n, v_{n+1}, t) > 1 - \epsilon \; \forall n > n_0$.
Let $m, n \in \mathbb{N}$, we assume that $m \geq n$. Now we have as

$$\begin{aligned} F(v_n, v_m, t) &\geq F\left(v_n, v_{n+1}, \frac{t}{m-n}\right) * F\left(v_{n+1}, v_{n+2}, \frac{t}{m-n}\right) * \cdots * F\left(v_{m-1}, v_m, \frac{t}{m-n}\right) \\ &\geq (1-\epsilon) * (1-\epsilon) * \dots \text{(m - n) times} \\ &\geq (1-\epsilon). \end{aligned}$$

So we can say, $\{v_n\}$ as a Cauchy sequence in U.
As $(U, F, *)$ is complete so the sequence $\{v_n\}$ convergent sequence in $w \in U$, therefore $\{fu_{2n-2}\}$, $\{hu_{2n}\}$, $\{gu_{2n-1}\}$, and $\{Ju_{2n-1}\}$ also converge to w. By using the corollary 2.13 and given (c), we have

$$fhu_{2n-2} \to hw \tag{4}$$

and

$$gJu_{2n-1} \to Jw. \tag{5}$$

Using the axiom of definition (d), we obtain

$$\begin{aligned} F(fhu_{2n-2}, gJu_{2n-1}, kt) \geq{}& F(hhu_{2n-2}, JJu_{2n-1}, t) * F(fhu_{2n-2}, hhu_{2n-2}, t) \\ & * F(gJu_{2n-1}, JJu_{2n-1}, t) * F(fhu_{2n-2}, JJu_{2n-1}, t). \end{aligned}$$

Having the limit as $n \to \infty$ and by combining (4) and (5), we get

$$\begin{aligned} F(hw, Jw, kt) &\geq F(hw, Jw, t) * F(hw, hw, t) * F(Jw, Jw, t) * F(hw, Jw, t) \\ &\geq F(hw, Jw, t) * 1 * 1 * F(hw, Jw, t) \\ &\geq F(hw, Jw, t). \\ & It\ gives\ that\ hw = Jw. \end{aligned} \tag{6}$$

Again using the condition (d) we have,

$$\begin{aligned} & F(fw, gJu_{2n-1}, kt) \\ & \geq F(hw, JJu_{2n-1}, t) * F(fw, hw, t) \\ & * F(ghu_{2n-1}, JJu_{2n-1}, t) * F(fw, JJu_{2n-1}, t). \end{aligned}$$

Again, taking the limit as $n \to \infty$ and using (5) and (6), we have

$$\begin{aligned} F(fw, Jw, kt) &\geq F(hw, hw, t) * F(fw, Jw, t) * F(Jw, Jw, t) * F(fw, Jw, t) \\ &\geq F(fw, Jw, t). \\ &and\ hence\ fw = Jw. \end{aligned} \tag{7}$$

From (d), (6), and (7),

$$\begin{aligned} F(fw, gw, kt) &\geq F(hw, Jw, t) * F(fw, hw, t) * F(gw, Jw, t) * F(fw, Jw, t) \\ &= F(fw, fw, t) * F(fw, fw, t) * F(gw, fw, t) * F(fw, fw, t) \\ &\geq F(fw, gw, t). \\ &and\ hence\ fw = gw. \end{aligned} \tag{8}$$

From (6), (7), and (8), we have

$$fw = gw = Jw = hw. \tag{9}$$

Now, we will show that $gw = w$.
From definition (iv),

$$\begin{aligned} &F(fu_{2n}, gw, kt) \\ &\geq F(hu_{2n}, Jw, t) * F(fu_{2n}, hu_{2n}, t) * F(gw, Jw, t) * F(fu_{2n}, Jw, t). \end{aligned}$$

Now, using the limit as $n \to \infty$ and appling (6) and (7), we get

$$\begin{aligned} F(w, gw, kt) &\geq F(w, Jw, t) * F(w, w, t) * F(gw, Jw, t) * F(w, Jw, t) \\ &= F(w, gw, t) * 1 * F(fw, fw, t) * F(w, gw, t) \\ &\geq F(w, gw, t). \end{aligned}$$

And hence $gz = w$.
Hence, from (9), $w = fw = gw = Jw = hw$. So w is a fixed point which is common in f, g, h, and J.
To show the uniqueness, assume z be the next fixed point as common in the mappings f, g, h, and J. So we have

$$\begin{aligned} F(w, z, kt) &= F(fw, gz, kt) \\ &\geq F(hw, Jz, t) * F(fw, hw, t) * F(gz, Jz, t) * F(fw, Jz, t) \\ &\geq F(w, z, t). \end{aligned}$$

by using the Lemma 2.12, $z = w$.

Which completes the proof. Now by using the theorem 3.1 we introduce following result.

Corollary 3.2 Assume $(U, F, *)$ is a complete Fb-MS, there exist a $b \geq 1$ and suppose f, g, h and J be the mappings defined from U to itself satisfying the axioms (a), (b) and (c) of above theorem (3.1) and $\exists\ k \in \left(0, \frac{1}{b}\right)$ which gives

$$F(fu, gu, kt) \geq F(hu, Ju, t)$$

for every $u, v \in U$ and $t > 0$. f, g, h and J have a common fixed point in U which is unique.

Theorem 3.3 Suppose $(U, F, *)$ be a Fb-MS which is complete. Then self-mappings f, g and h of U which are continuous contain a common fixed point in U iff $\exists$ a self-mapping f, g and h of U satisfying the following axioms

(a) $f(U) \subseteq h(U) \cap g(U)$,

(b) the pairs (f, g) and (f, h) are type (A) compatible on U,

(c) $\exists\ k \in \left(0, \frac{1}{b}\right)$ which gives for every $u, v \in U$ and $t > 0$,

$$\begin{aligned} F(fu, fv, kt) \geq\ &F(gu, hv, t) * F(fu, gu, t) * \\ &F(fv, hv, t) * F(fu, hv, t). \end{aligned}$$

In general, f, g, and h have a in U which is unique.

proof: Let us assume the mappings g and h contain a fixed point which is common in U, assume that it is w. So we have $gw = w = hw$. Let $fu = w\ \forall\ u \in U$.

Since $f(U) \subseteq h(U) \cap g(U)$ and we know that (f, g) and (f, h) are the type (A) compatible mappings, in fact $f \circ g = g \circ f$ and $f \circ h = h \circ f$, so the axioms (a) and (b) are verified.

By taking $k \in \left(0, \frac{1}{b}\right)$, we have

$$F(fu, fv, kt) = 1 \geq F(gu, hv, t) * M(fu, gu, t)$$

$$* F(fv, hv, t) * F(fu, hv, t)$$

for every $u, v \in U$ and $t > 0$, and hence condition (c) is verified.

Now by using the Theorem 3.1 we can say that $f = g$. Then f, g, and h have a common fixed point which is unique in U.

Example 3.4 Let $U = [0, 2]$ with the usual metric $d(u, v) = |u - v|$, define

$$F(u, v, t) = \frac{t}{t + d(u, v)}$$

for all $u, v \in U$, $t > 0$, and $p * q = p.q$ for all $p, q \in [0, 1]$. Then $(U, F, *)$ is a fuzzy b-metric space.

We define the self-mappings f and g as:

$$f(u) = g(u) = 1 \quad \text{for} \quad u \in [0, 1),$$

$$f(u) = g(u) = \frac{4}{3} \quad \text{for} \quad u = 1,$$

and

$$f(u) = 2 - u, \quad g(u) = u \quad \text{for} \quad u \in (1, 2].$$

Consider a sequence $\{u_n\}$ in U such that $u_n = 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$. Then, we have

$$f(u_n) = (2 - u_n) \to 1 = u, \quad S(u_n) = u_n \to 1 = u.$$

Since $2 - u_n < 1$ for all $n \in \mathbb{N}$, we have,

$$f(f(u_n)) = f(2 - u_n) = 1 \to 1,$$

$$f(g(u_n)) = f(u_n) = 2 - u_n \to 1,$$

$$g(g(u_n)) = g(u_n) = u_n \to 1,$$

$$g(f(u_n)) = g(2 - u_n) = 1 \to 1.$$

Also, we have,

$$f(u) = \frac{4}{3} = g(u),$$

but

$$f(g(u)) = f(1) = f\left(\frac{4}{3}\right) = 2 - \frac{4}{3} = \frac{2}{3},$$

$$g(f(u)) = g(1) = g\left(\frac{4}{3}\right) = \frac{4}{3}.$$

However, we have

$$\frac{2}{3} = f(g(u)) \neq g(f(u)) = \frac{4}{3}, \quad \text{at} \quad u = 1.$$

Therefore, (f, g) is compatible, compatible of type (A).

Theorem 3.5 Assume f and g are self mappings of a Fb-MS $(U, F, *)$ and for a given real number $b \geq 1$. If f and g are continuous, then the pair of mappings (f, g) is compatible of type (A) iff f and g are semicompatible.

Proof: Suppose $\lim_{n\to\infty} fu_n = \lim_{n\to\infty} gu_n = w$ for some $w \in U$ and assume the pair of self mappings (f, g) is compatible of type (A). But the mappings f and g are continuous, so

$$\lim_{n\to\infty} fgu_n = fw, \lim_{n\to\infty} ffu_n = fw,$$

$$\text{and} \quad \lim_{n\to\infty} gfu_n = gw.$$

Hence, as we have $k \in \left(0, \frac{1}{b}\right)$

$$\begin{aligned}
\lim_{n\to\infty} F(fgu_n, gw, t) &\geq \lim_{n\to\infty} F\left(fgu_n, gfu_n, \frac{t}{k}\right) * \lim_{n\to\infty} F\left(gfu_n, gw, \frac{t}{k}\right) \\
&\geq \lim_{n\to\infty} F\left(fgu_n, ffu_n, \frac{t}{k^2}\right) * \lim_{n\to\infty} F\left(ffu_n, gw, \frac{t}{k^2}\right) * \lim_{n\to\infty} F\left(gfu_n, gw, \frac{t}{k}\right) \\
&= 1 * 1 * 1 = 1
\end{aligned}$$

Which gives

$$\lim_{n\to\infty} F(fgu_n, gw, t) \geq 1 * 1 * 1 = 1$$

so the pair (f, g) is semicompatible.

Conversely, let us assume that f and g are semicompatible. Then we have

$$\begin{aligned}
\lim_{n\to\infty} F(fgu_n, ffu_n, t) &\geq \lim_{n\to\infty} F\left(fgu_n, gw, \frac{t}{k}\right) * \lim_{n\to\infty} F\left(gw, ffu_n, \frac{t}{k}\right) \\
&\geq F\left(fgu_n, gw, \frac{t}{k}\right) * \lim_{n\to\infty} F\left(gw, fgu_n, \frac{t}{k^2}\right) * \lim_{n\to\infty} F\left(fgu_n, ffu_n, \frac{t}{k^2}\right) \\
&= 1 * 1 * 1
\end{aligned}$$

so the pair of mappings (f, g) is type (A) compatible.

Example 3.6 Assume $U = [0, 1]$ and $(U, F, *)$ be a Fb-MS with $b \geq 1$ and

$$F(u, v, t) = \exp\left(\frac{-|u - v|}{t}\right), \quad \text{for all } u, v \in U, t > 0.$$

If we define a mapping as follows

$$f(u) = \begin{cases} u & \text{if } 0 \leq u \leq \frac{1}{3}, \\ 1 & \text{if } u \geq \frac{1}{3}. \end{cases}$$

Let us assume that g be the identity mapping on U and $\{u_n\} = \frac{1}{3} - \frac{1}{n}$. Then:

$$\lim_{n\to\infty} \{fu_n\} = \lim_{n\to\infty} \left\{\frac{1}{3} - \frac{1}{n}\right\} \to \frac{1}{3}$$

$$\lim_{n\to\infty} \{gu_n\} = \lim_{n\to\infty} \left\{\frac{1}{3} - \frac{1}{n}\right\} \to \frac{1}{3}$$

$$\begin{aligned} \lim_{n\to\infty} \{fu_n\} &= \lim_{n\to\infty} \{gu_n\} \\ &= \lim_{n\to\infty} \left\{\frac{1}{3} - \frac{1}{n}\right\} \to \frac{1}{3}, \\ \lim_{n\to\infty} \{gu_n\} &= \lim_{n\to\infty} \{fu_n\} \\ &= \lim_{n\to\infty} \left\{\frac{1}{3} - \frac{1}{n}\right\} \to \frac{1}{3} \\ \lim_{n\to\infty} \{ffu_n\} &= \lim_{n\to\infty} \{fu_n\} \\ &= \lim_{n\to\infty} \left\{\frac{1}{3} - \frac{1}{n}\right\} \to \frac{1}{3} \end{aligned}$$

So we conclude the mappings (f, g) and (g, f) are type (A) compatible.
Again,

$$\lim_{n\to\infty} \{fgu_n\} = \lim_{n\to\infty} \{fu_n\} = \lim_{n\to\infty} \left\{\frac{1}{3} - \frac{1}{n}\right\} \to \frac{1}{3} = g\left(\frac{1}{3}\right)$$

Hence (f, g) is semicompatible but

$$\lim_{n\to\infty} \{gfu_n\} = \lim_{n\to\infty} \{fu_n\} = \lim_{n\to\infty} \left\{\frac{1}{3} - \frac{1}{n}\right\} \to \frac{1}{3} \neq f\left(\frac{1}{3}\right)$$

So (g, f) is not semicompatible.

4. Conclusion

In this article by using the concept of compatible mapping and semicompatible mapping of type (A)in fuzzy b-metric space we proved the theorems. Also we introduced some properties of these result sand verify the results with some suitable examples. We have built a fertile ground to study in further different types of compatible mappings such as compatible mappings of type (P), type (E), type (K) and many mere, by using the concept of fuzzy b-metric space.

Abbreviations

CFPT	Common Fixed Point Theorem
FBM	Fuzzy Metric Space
FPT	Fixed Point Theorem
Fb-MS	Fuzzy b-Metric Space

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Conflicts of Interest

The authors declare that they have no conflict of interests.

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