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Research Article

Consequences of Generalized Ratio Tests That Lead to More Efficient Convergence and Divergence Tests

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Abstract

The Ratio Test, developed by Jean Le Rond d'Alembert, is a fundamental method for determining the convergence or divergence of an infinite series and is commonly used as a primary test for many series. However, due to its restricted range of applicability, several generalized forms of the Ratio Test have been introduced to extend its usefulness. In this paper, we combine two of the most effective and reliable generalized Ratio Tests to create more efficient convergence tests. To this end, we show that if a positive valued function, *f*, is defined for all numbers greater than or equal to one, and if the improper integral of the reciprocal of *f*, over the interval from one to infinity, diverges, then f has a close relationship with a sister function φ. We then show that these paired functions satisfy a remarkable relationship that completely characterizes all monotonically decreasing sequence of terms whose sum diverge. We demonstrate through several examples the ease with which φ can be found if *f* is known and vice-versa. Next, we combine the generalized Ratio Tests of Dini and Ermakoff by focusing on a 'thin' subsequence of the terms of a large category of infinite series to develop other convergence and divergence tests. Furthermore, we refine these tests to produce practical and easier to apply convergence and divergence tests. Lastly, we demonstrate that for many infinite series, one can factor their terms into the product of the reciprocal of f and L. We then show that the limit superior and limit inferior of an expression based on L determines the convergence or divergence of the original series.

Keywords

Series, Convergence, Divergence

1. Introduction

In this paper, we shall show how one can combine the ratio tests of Dini and Ermakoff (see [1], pages 37-45) to develop other tests which can lead to a rapid determination of convergence and divergence for certain infinite series.

Dini's test for convergence and divergence states that:

If $\sum_{n=1}^{\infty} D_n{}^{-1}$ is a divergent series and if $T_n = D_n \frac{a_n}{a_{n+1}} - D_{n+1}$, then $\sum_{n=1}^{\infty} a_n$ is convergent if $\underline{lim} T_n > 0$ and divergent if $\overline{lim} T_n < 0$ (see [1], page 37).

Ermakoff's tests state that: If $f(x)$ and $\varphi(x)$ satisfy the following conditions:

$f(x)$ is strictly increasing on $[1, \infty)$,

$f(n) = a_n > 0$, where $n \epsilon Z^+$, and

$\varphi(x)$ is an increasing function that satisfies $\varphi(x) > x$, then:

$\sum_{n=1}^{\infty} a_n$ converges if:

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$\overline{lim}\left(\frac{\varphi'(x)f(\varphi(x))}{f(x)} < 1\right)$ and diverges if $\underline{lim}\left(\frac{\varphi'(x)f(\varphi(x))}{f(x)} > 1\right)$ (see [2], page 44).

If one rephrases Dini's tests as a function of the limit infimum and limit infimum (see [3]) of $\left(\frac{(\delta+D_{n+1})a_{n+1}}{D_n a_n}\right)$, where $\delta \in R$, it becomes apparent that it bears some resemblance to Ermakoff's tests.

The speculation that some advantage may be gained by combining Ermakoff and Dini's tests forms the basis of this paper.

In the first section (Preliminaries) we give a characterization of all diverging series via the equation $\varphi'(x)f(x) = f(\varphi(x))$ (see [2, 4]).

In the third section (Generalized Ratio Tests), we prove two convergence and divergence theorems. In the next section (Main Results), we provide refinements of these two key theorems, by focusing our attention on infinite series that are expressible in the form $\sum_{n=1}^{\infty}\frac{1}{f(n)(L(n))^{\alpha}}$, where α is a non-zero real number.

A key result from the first section (Preliminaries), is used to devise this refinement scheme. In fact, we show that the problem of finding whether a series converges or diverges, for many infinite series, can be reduced to one of knowing a few basic divergent series and the $\underline{lim}$ or the $\overline{lim}$ of quantities of the form $L(\varphi(n)) - L(n)$.

We conclude our paper in the last section with a couple of examples illustrating the use of some of our results.

Throughout this paper, f will denote a continuous, positive valued, and a non-decreasing function on $[1,\infty)$.

2. Preliminaries

Definition 2.1

We shall denote by D_τ the set of all functions $\varphi: [1,\infty) \to R$, that satisfy:

$$\varphi(x) > x \text{ and} \tag{1}$$

$$\varphi'(x) \geq 1. \tag{2}$$

Definition 2.2

We shall denote by D_ϕ the set of all functions $f: [1,\infty) \to R^+$ satisfying:

$f(x)$ is strictly increasing on $[1,\infty)$, and

$$\int_1^\infty \frac{1}{f(x)}dx = \infty. \tag{3}$$

Next, we shall continue with a couple of preliminary but important results which would lead us, ultimately, to the fact that: if φ' is an increasing function on $[1,\infty)$, then a function from either D_ϕ or D_τ has at least one corresponding function in the other set. Furthermore, these paired functions satisfy a remarkable relationship that completely characterizes all monotonically decreasing sequence of terms whose sums diverge.

Lemma 2.1

Let $f \in D_\phi$ and $\varphi \in D_\tau$. Then the following conditions are equivalent:

$$\varphi'(x)f(x) = f(\varphi(x)), \tag{4}$$

$$\int_x^{\varphi(x)}\frac{1}{f(t)}dt = \int_1^{\varphi(1)}\frac{1}{f(t)}dt, \text{ for all } x \geq 1. \tag{5}$$

Proof:

To prove that (5) implies (4), differentiate both sides of (5) with respect to x (see [5]).

Next, suppose that (4) is true. Then,

$$\int_1^x \frac{1}{f(t)}dt = \int_1^x \frac{\varphi'(t)}{f(\varphi(t))}dt = \int_{\varphi(1)}^{\varphi(x)}\frac{1}{f(u)}du,$$

where $u = \varphi(t)$ (see [6]).

Hence,

$\int_1^{\varphi(1)}\frac{1}{f(t)}dt = \int_x^{\varphi(x)}\frac{1}{f(t)}dt$, for all $x > 1$.

Theorem 2.1

Let $f \in D_\phi$, then there exists at least one $\varphi \in D_\tau$ such that:

$$\varphi'(x)f(x) = f(\varphi(x)). \tag{6}$$

Proof:

Let $F(x) = \int_1^x \frac{1}{f(t)}dt$. Then $F'(x)$ exists, is non-zero, and $F^{-1}(x)$ exists (see [7, 8]).

Let $\varepsilon > 0$ and define $\varphi(x)$ as:

$$\varphi(x) = F^{-1}(F(x) + \varepsilon). \tag{7}$$

The rest of the proof is routine.

Theorem 2.2

Let $\varphi \in D_\tau$. If in addition φ' is an increasing function on $[1,\infty)$, then there exists at least one $f \in D_\phi$ such that the following conditions hold:

$$\varphi'(x)f(x) = f(\varphi(x)). \tag{8}$$

$$\int_1^\infty \frac{1}{f(x)}dx = \infty. \tag{9}$$

Proof:

Let f be continuous, positive, and a non-decreasing function on $[1, \varphi(1)]$ that satisfies:

$$f(\varphi(1)) = \varphi'(1)f(1).$$

We shall then extend f to $[1,\infty)$ via the equation

$$f(\varphi(x)) = \varphi'(x)f(x).$$

Clearly, f satisfies condition (6). Therefore, by Lemma

1.1.1, condition (7) also holds and $\int_1^\infty \frac{1}{f(x)}dx = \infty$, since

$$\int_1^{\varphi(1)} \frac{1}{f(t)}dt = \int_x^{\varphi(x)} \frac{1}{f(t)}dt.$$

Our next result, which is an immediate consequence of Theorems 2.1 and 2.2, can be viewed as a characterization of all monotonically decreasing sequence of positive terms whose sums diverge. Note that the condition $\int_1^\infty \frac{1}{f(t)}dt = \infty$, is crucial in the proof of our next result.

Theorem 2.3

Suppose that $\{a_n\}_{n=1}^\infty$ is a sequence of monotonically decreasing sequence of positive terms. Then, there exists a continuous, positive valued, and a non-decreasing function f on $[1,\infty)$ that satisfies $(n) = \frac{1}{a_n}$, for all $n \epsilon Z^+$, and at least one function $\varphi \in D_\tau$ such that $\varphi'(x)f(x) = f(\varphi(x))$.

Theorem 1.1.3 is a direct consequence of Theorem 1.1.

In the next section, we shall prove a convergence and a divergence result.

For the remainder of this paper, $f \in D_\phi$, $\varphi \in D_\tau$, and g will denote a non-increasing, almost everywhere continuous function on $[1,\infty)$. For such functions g, it follows from [9] (see page 138, problem number 8 and page 323, Theorem 11.33) that $\int_1^\infty g(t)dt$ is well defined.

3. Generalized Ratio Tests

Theorem 3.1

Let $D(x)$ be a non-decreasing, positive function on $[1,\infty)$, such that $\sum_{n=1}^\infty \frac{1}{D(\varphi^n(1))}$ diverges. If $\frac{\varphi'(x)D(\varphi(x))g(\varphi(x))}{D(x)g(x)} \geq 1$, for all $x > \xi > 1$, then $\sum_{n=1}^\infty g(n)$ diverges.

Proof:

If $\frac{\varphi'(x)D(\varphi(x))g(\varphi(x))}{D(x)g(x)} \geq 1$, for all $x > \xi$ then

$$\int_\xi^x \varphi'(t)D(\varphi(t))g(\varphi(t))dt \geq \int_\xi^x D(t)g(t)dt.$$

That is, $\int_{\varphi(\xi)}^{\varphi(x)} D(u)g(u)du \geq \int_\xi^x D(t)g(t)dt$.

Therefore,

$$\int_x^{\varphi(x)} D(u)g(u)du \geq \int_\xi^{\varphi(\xi)} D(t)g(t)dt = \beta > 0.$$

Since $D(x)$ is a non-decreasing, positive function on $[1,\infty)$, it follows that

$$\int_x^{\varphi(x)} g(u)du \geq \frac{1}{D(\varphi(x))}\int_x^{\varphi(x)} D(u)g(u)du \geq \frac{\beta}{D(\varphi(x))}.$$

Thus,

$\int_1^\infty g(x)dx \geq \beta \sum_{n=1}^\infty \frac{1}{D(\varphi^n(1))} = \infty$ and $\sum_{n=1}^\infty g(n)$ diverges by the Integral Test (see [10]).

Theorem 3.2

Suppose that $C(x)$ is a real-valued function on $[1,\infty)$ such that $\underline{lim} C(x) > \lambda > 0$.

If $\frac{\varphi'(x)C(\varphi(x))g(\varphi(x))}{C(x)g(x)} \leq 1 - \frac{\lambda}{C(x)}$, for all $x > \xi$, then $\sum_{n=1}^\infty g(n)$ converges.

Proof

We can assume without loss of generality that for all $x > \xi$, $C(x) > \lambda$. Thus,

$$\int_\xi^x \varphi'(t)C(\varphi(t))g(\varphi(t))dt \leq \int_\xi^x C(t)g(t)dt - \lambda\int_\xi^x g(t)dt\ .$$

Hence,

$$\lambda\int_\xi^x g(t)dt \leq \int_\xi^x C(t)g(t)dt - \int_{\varphi(\xi)}^{\varphi(x)} C(u)g(u)du$$

Therefore,

$$\lambda\int_\xi^x g(t)dt \leq \int_\xi^{\varphi(\xi)} g(t)C(t)dt - \int_x^{\varphi(x)} C(u)g(u)du$$

$$\leq \int_\xi^{\varphi(\xi)} C(t)g(t)dt = \beta < \infty\,.$$

Since x is arbitrary, it follows that $\int_1^\infty g(x)dx$ converges. Consequently, $\sum_{n=1}^\infty g(n)$ converges (see [10, 11]).

4. Main Results

In this Section, we shall show how the results from Section 1.1 can be used, in conjunction with Theorems 2.1 and 2.5, to attain practical tests to check for the convergence or divergence of infinite series. We shall focus our attention on infinite series that can be written in the form $\sum_{n=1}^\infty \frac{1}{f(n)(L(n))^\alpha}$.

Note that if $g(x)$ is positive and decreasing on $[1,\infty)$, $\lim_{x\to 0} g(x) = 0$, then $g(x)$ is expressible in the form $g(x) = \frac{1}{f(x)(L(x))^\alpha}$ where $f \in D_\phi$, L is non-decreasing on $[1,\infty)$, $\lim_{n\to\infty} L(x) = \infty$, and $\alpha \in R$.

Lemma 3.1

If $\beta > 0$ and $0 < \alpha < 1$, then there exists a $k > 0$ such that $(1-\alpha)^\beta \leq 1 - k\alpha$.

Proof

If $\beta \geq 1$, then since $0 < \alpha < 1$, it follows that $(1-\alpha)^\beta \leq (1-\alpha)$.

So, assume that $0 < \beta < 1$. Then there exists positive integers m, n such that $\frac{m}{n} < \beta$.

Therefore, $(1-\alpha)^\beta \leq (1-\alpha)^{\frac{m}{n}} \leq (1-\alpha)^{\frac{1}{n}}$.

Now, let $h(\alpha) = (1-k\alpha)^n - (1-\alpha)$, where $k = \frac{1}{n+1}$.

Then, $h(0) = 0$ and $h'(\alpha) = 1 - nk(1-k\alpha)^{n-1}$.

Thus, $h'(\alpha) > 0$ and $h(\alpha) > 0$ for all $\alpha\epsilon(0,1)$ (see

[12]).

Consequently, if $\beta > 0$ and $0 < \alpha < 1$, then there exists $k > 0$ such that:

$$(1-\alpha)^{\beta} \le (1-\alpha)^{\frac{m}{n}} \le (1-\alpha)^{\frac{1}{n}} \le 1 - k\alpha.$$

Corollary 3.1

Suppose that $f \in D_\phi$ and $\varphi \in D_\tau$ satisfy: $\varphi'(x)f(x) = f(\varphi(x))$.

If $(x) = \frac{1}{f(x)(L(x))^{\alpha}}$, $\alpha > 1$, and $\underline{lim}(L(\varphi(n) - L(n)) \ge \omega > 0$, then $\sum_{n=1}^{\infty} \frac{1}{f(x)(L(x))^{\alpha}}$ converges.

Proof

Now, let $C(x) = L(x)$.

Then,

$$\frac{\varphi'(x)C(\varphi(x))g(\varphi(x))}{C(x)g(x)} = \left(\frac{L(x)}{L(\varphi(x))}\right)^{\alpha-1}.$$

Since $\underline{lim}(L(\varphi(n) - L(n)) \ge \omega > 0$, it follows that $\lim_{n\to\infty} L(x) = \infty$.

Thus,

$\frac{L(x)}{L(\varphi(x))} \le 1 - \frac{\omega}{L(x)+\omega} \le 1 - \frac{\omega}{2L(x)}$ for sufficiently large x.

If $\beta \ge 2$, then for sufficiently large x,

$$\left(\frac{L(x)}{L(\varphi(x))}\right)^{\beta-1} \le \left(1 - \frac{\omega}{2L(x)}\right)^{\beta-1} \le 1 - \frac{\omega}{2L(x)}$$

If $1 < \beta < 2$, then $0 < \beta - 1 < 1$, and for sufficiently large x, it follows from Lemma 3.1 and above that there exists a $k > 0$ such that:

$$\left(\frac{L(x)}{L(\varphi(x))}\right)^{\beta-1} \le 1 - \frac{k\omega}{2L(x)} = 1 - \frac{\lambda}{C(x)},$$

where $L(x) = C(x)$ and $\lambda = \frac{kw}{2}$.

It follows from Theorem 3.1 that $\sum_{n=1}^{\infty} \frac{1}{f(x)(L(x))^{\alpha}}$ converges and hence, Corollary 3.1 holds.

Corollary 3.2

Let $f \in D_\phi$ and $\varphi \in D_\tau$ satisfy: $\varphi'(x)f(x) = f(\varphi(x))$.

If $g(x) = \frac{1}{f(x)(L(x))^{\alpha}}$, $\alpha \le 1$, and $\overline{lim}(L(\varphi(n) - L(n)) < \infty$, then $\sum_{n=1}^{\infty} \frac{1}{f(x)(L(x))^{\alpha}}$ diverges.

Proof:

Now, $D(x) = L(x)$.

Then,

$$\frac{\varphi'(x)D(\varphi(x))g(\varphi(x))}{D(x)g(x)} = \left(\frac{L(\varphi(x))}{L(x)}\right)^{1-\alpha} \ge 1.$$

Since $\overline{lim}(L(\varphi(n) - L(n)) < \infty$, it follows that for sufficiently large x,

$$0 < L(\varphi(x)) - L(x) \le \beta.$$

Thus,

$$\sum_{j=0}^{\infty} \frac{1}{D(\varphi^{j}(1))} \ge \frac{1}{\beta} \sum_{j=0}^{\infty} \frac{1}{L(1)+j\beta} = \infty.$$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{f(x)(L(x))^{\alpha}}$ diverges by Theorem 3.2.

5. Examples

Next, we shall include a couple of examples of functions $f \in D_\phi$ and their corresponding functions $\varphi \in D_\tau$.

Example 4.1

Let $f(x) = 1$. Then, $F(x) = \int_1^x \frac{1}{f(t)} dt = x - 1$ and $F^{-1}(x) = x + 1$. Thus,

$if\ \varepsilon = 1,\ \varphi(x) = F^{-1}(F(x) + 1) = x + 1.$

Example 4.2

$Let\ f(x) = x$. Then, $F(x) = \int_1^x \frac{1}{t} dt = \ln(x)$ Therefore, $F^{-1}(x) = e^x$. Hence, for $\varepsilon = In2$,

$\varphi(x) = F^{-1}(F(x) + \ln(2)) = e^{\ln(2x)} = 2x.$

Example 4.3

Let $f(x) = xInx$. Then,

$F(x) = \int_e^x \frac{1}{tlnt} dt = \ln(\ln(x)$ and $F^{-1}(x) = e^{e^x}$.

Hence, $for\ \varepsilon = In2, \varphi(x) = x^2$.

We shall conclude this paper by demonstrating the ease with which Corollaries 3.2 and 3.3 can be used to determine the convergence and divergence of some well-known series.

Example 4.4

$\sum_{n=1}^{\infty} \frac{1}{n^p}$.

Let $f(x) = 1$. Hence, $\varphi(x) = x + 1$. Since $L(x) = x$, $L(\varphi(x)) - L(x) = 1$.

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $\alpha = p > 1$ and diverges if $\alpha = p \le 11$ (see [13]).

Example 4.5

$\sum_{n=1}^{\infty} \frac{1}{n(\ln(n))^p}$. Let $f(x) = x$. Then, $L(x) = Inx$ and $\alpha = p$. Since

$f(x) = x,\quad \varphi(x) = 2x$. Now, $L(\varphi(x)) - L(x) = In(2x) - In(x) = In2$. Hence,

$\sum_{n=1}^{\infty} \frac{1}{n(\ln(n))^p}$ converges if $\alpha = p > 1$, and diverges if $\alpha = p \le 1$ (see [14] and [15]) for the interested reader.

6. Conclusion

In this paper, we proved that all monotonically decreasing sequence of positive terms whose sum diverges can be characterized by an equation satisfying $\varphi'(x)f(x) = f(\varphi(x))$, where φ' is a non-decreasing function on $[1, \infty)$.

We also showed how the tests of Dini and Ermakoff can be combined to establish other convergence and divergence theorems for certain infinite series. These new tests proved to be more malleable and easier to refine by focusing attention on series that are expressible in the form $\sum_{n=1}^{\infty}\frac{1}{f(n)(L(n))^{\alpha}}$, where α is a real number.

In fact, we proved that the problem of finding whether a series converges or diverges can, in many cases, be reduced to one of knowing a few basic divergent series and the limit superior or the limit inferior of quantities of the form $L\big(\varphi(n)\big)-L(n)$.

Abbreviations

pp Pages
$\underline{lim}$ Limit Inferior
$\overline{lim}$ Limit Superior

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Author Contributions

Joseph Granville Gaskin is the sole author. The author read and approved the final manuscript.

Conflicts of Interest

The author declares no conflict of interest.

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Research Field

Joseph Granville Gaskin: Real Analysis, Sequences and series, Number Theory, Differential Equations, Point Set Topology.