



Research Article

The Double Roman Domination Number of Sunlet Graphs S_n

Wensheng Li*

School of Science, Langfang Normal University, Langfang, China

Abstract

A double Roman dominating function (DRDF) f on a given graph G is a mapping from $V(G)$ to $\{0, 1, 2, 3\}$ in such a way that a vertex u for which $f(u) = 0$ has at least a neighbor labeled 3 or two neighbors both labeled 2 and a vertex u for which $f(u) = 1$ has at least a neighbor labeled 2 or 3. The weight of a DRDF f is the value $w(f) = \sum_{u \in V(G)} f(u)$. The minimum weight of a DRDF on a graph G is called the double Roman domination number $\gamma_{dr}(G)$ of G . A Sunlet graph S_n is constructed from an n -cycle by attaching one pendant vertex to each vertex of the cycle. This paper investigates the double Roman domination of Sunlet graphs S_n for integer $n \geq 3$. Using three lemmas, we derive the exact formula $\gamma_{dr}(S_n) = \lceil 7n/3 \rceil$. We further analyze whether Sunlet graphs belong to the family of double Roman graphs. By comparing the domination number $\gamma(S_n) = n$ with the known inequality $2\gamma(G) \leq \gamma_{dr}(G) \leq 3\gamma(G)$ for double Roman graphs, we prove that S_n fails to satisfy the required inequality, which implies no Sunlet graph is a double Roman graph. The obtained result gives an explicit expression for the double Roman domination number of this class of unicyclic graphs and enriches the research achievements of double Roman domination theory for unicyclic graphs.

Keywords

Double Roman Domination, Double Roman Domination Number, Sunlet Graphs

1. Introduction

Let $G = (V, E)$ be a graph without loops and multiple edges, where $V(G)$ and $E(G)$ are the vertex set and edge set of G , respectively. If $uv \in E(G)$, we say that vertices u and v are adjacent, and u is a neighbor of v . The open neighborhood of v , denoted by $N(v)$, is a set $\{u \mid uv \in E\}$. And the close neighborhood of v , denoted by $N[v]$, is a set $N(v) \cup \{v\}$. A function $f: V \rightarrow \{0, 1, 2, 3\}$ is a double Roman dominating function, shorted by DRDF, if a vertex v for which $f(v) = 0$ has at least a neighbor labeled 3 or two neighbors both labeled 2 and a vertex v for which $f(v) = 1$ has at least a neighbor labeled 2 or 3. The sum of

the function values of all vertices in $N[v]$ is $f(N[v]) = \sum_{u \in N[v]} f(u)$, which is shortly denoted by $f[v]$.

The weight of f , denoted by $\omega(f)$, is the sum of the function values of all vertices in G . The double Roman domination number, denoted by $\gamma_{dr}(G)$, equals the minimum weight of a DRDF on G , and a DRDF of G with weight $\gamma_{dr}(G)$ is called a γ_{dr} -function of G .

The double Roman domination was introduced in [1] and has been studied in [2-9]. Double Roman domination for cardinal products of graphs was studied in [2], and double Roman trees were characterized in [3]. It is known that the decision

*Correspondence: Wensheng Li (liwensheng@lfnu.edu.cn)



problem associated with $\gamma_{dR}(G)$ is NP-complete for bipartite and chordal graphs, undirected path graphs, chordal bipartite graphs, and circle graphs [4-6]. Closely related problems to double Roman domination were studied in [7-13]. As observed in [10], the double Roman domination problem can be solved in linear time for unicyclic graphs, but explicit formulas for the double Roman domination number of general unicyclic graphs have not been established. Many results concern variants of double Roman domination on unicyclic graphs [11, 12]. However, most existing studies mainly focus on algorithmic complexity and bound analysis, while few studies provide closed-form exact values for specific unicyclic graph subclasses. In particular, the precise double Roman domination number of Sunlet graphs has not been formally characterized in previous literature. To fill this research gap, in this paper, we focus on a special family of unicyclic graphs—the Sunlet graphs, and determine the exact value of the double Roman domination number of Sunlet graphs.

The Sunlet graph S_n is obtained by taking a simple n -gon, that is, a cycle with vertices $v_1, \dots, v_n$, and attaching to each vertex v_i a new pendant vertex u_i , where $1 \leq i \leq n$. Figure 1 shows a Sunlet graph with $2n$ vertices.

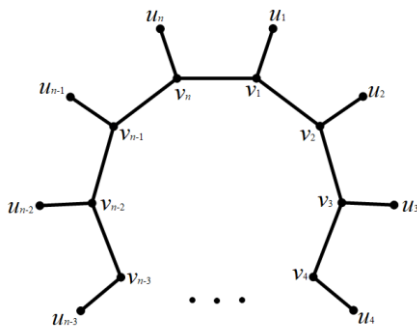


Figure 1. Sunlet graph S_n with $2n$ vertices.

In [1], Beeler et al. established the following result.

Proposition A [1] In a double Roman dominating function of weight $\gamma_{dR}(G)$, no vertex needs to be assigned the value 1.

In view of Proposition A, we only consider double Roman dominating functions of G with no vertex labeled 1 in the subsequent discussion. The following observation will be frequently utilized throughout the paper.

Proposition B [13] Let f be a DRDF on G and v is a leaf of G . Then $f[v] \geq 2$.

2. Double Roman Domination Number of S_n

Let $Q_i = Q(V_i, E_i)$ be a continuous local subgraph of S_n .

$V_i = \{v_i, v_{i+1}, v_{i+2}, u_i, u_{i+1}, u_{i+2}\}$ and $E_i = \{v_i v_{i+1}, v_{i+1} v_{i+2}, v_i u_i, v_{i+1} u_{i+1}, v_{i+2} u_{i+2}\}$ for $1 \leq i \leq n$, where subscripts are reduced modulo n as shown in Figure 2.

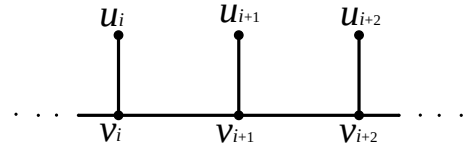


Figure 2. Q_i , a continuous local component of S_n .

Proposition C Let g be a γ_{dR} -function of S_n with no vertex assigned value 1. Then

$$g(V_i) = g(v_i) + g(v_{i+1}) + g(v_{i+2}) + g(u_i) + g(u_{i+1}) + g(u_{i+2}) \geq 7$$

where $1 \leq i \leq n$.

Proof. By Proposition B, we have

$$g(V_i) = g[u_i] + g[u_{i+1}] + g[u_{i+2}] \geq 6.$$

We now prove that the case $g(V_i) = 6$ is impossible via proof by contradiction. Suppose for contradiction that $g(V_i) = 6$. Since no vertex in V_i is labeled with value 1, the non-zero labels assigned to vertices in V_i can only take the form $\{2, 2, 2\}$ or $\{3, 3\}$.

Case 1: The non-zero labels are $\{2, 2, 2\}$.

By the definition of the labeling function, if a leaf vertex carries label 0, its supporting vertex must be assigned label 3. It immediately follows that no leaf vertex can take the label 0. Thus, the three leaf vertices u_i , u_{i+1} and u_{i+2} must each receive label 2 exclusively. Under this configuration, vertex v_{i+1} bears label 0, and its open neighborhood contains exactly one vertex with a non-zero label (value 2), which directly violates the labeling axiom.

Case 2: The non-zero labels are $\{3, 3\}$.

Among the three pairs consisting of supporting vertices and their corresponding leaf vertices $\{v_i, u_i\}$, $\{v_{i+1}, u_{i+1}\}$ and $\{v_{i+2}, u_{i+2}\}$, at least one pair is entirely assigned label 0. This configuration contradicts Proposition B.

Combining the above two contradictory cases, our initial hypothesis $g(V_i) = 6$ is invalid. Consequently, we derive $g(V_i) \neq 6$, i.e., $g(V_i) = g[u_i] + g[u_{i+1}] + g[u_{i+2}] \geq 7$.

Lemma 1 Let $n = 3k$, where $k \geq 1$. Then $\gamma_{dR}(S_n) = 7k$.

Proof. Consider the mapping $f: V(S_n) \rightarrow \{0, 2, 3\}$ specified in Table 1, where $1 \leq i \leq n$ for indices and all subscripts are taken modulo n . Then, for every vertex labeled 0,

there exists a vertex labeled 3 adjacent to it. Therefore, f is a DRDF on S_n and $\omega(f) = 7k$, which implies $\gamma_{dR}(S_n) \leq 7k$.

Table 1. The mapping $f: V(S_n) \rightarrow \{0, 2, 3\}$ with $n = 3k$.

Vertex	Label	Neighbors	Neighbors with Label 3
$v_i, i \equiv 1 \pmod{3}$	0	u_i, v_{i-1}, v_{i+1}	v_{i+1}
$v_i, i \equiv 2 \pmod{3}$	3		
$v_i, i \equiv 0 \pmod{3}$	0	u_i, v_{i-1}, v_{i+1}	v_{i-1}
$u_i, i \equiv 1 \pmod{3}$	2		
$u_i, i \equiv 2 \pmod{3}$	0	v_i	v_i
$u_i, i \equiv 0 \pmod{3}$	2		

The mapping is illustrated in Figure 3 with $n = 6$.

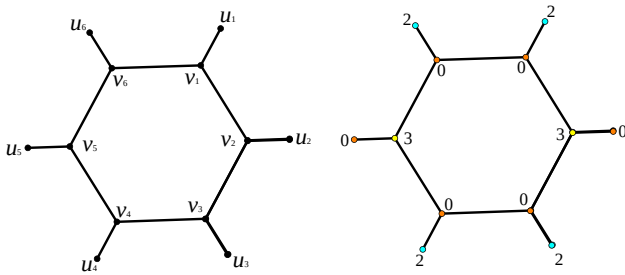


Figure 3. The mapping $f: V(S_n) \rightarrow \{0, 2, 3\}$ with $n = 6$

On the other hand, let g be a γ_{dR} -function of S_n with no

vertex assigned value 1. By Proposition C, we have

$$\omega(g) = g(Q_1) + g(Q_4) + \dots + g(Q_{n-2}) \geq 7k.$$

From the above arguments, we obtain $\gamma_{dR}(S_n) = 7k$ for $n = 3k$.

Lemma 2 Let $n = 3k + 1$, where $k \geq 1$. Then $\gamma_{dR}(S_n) = 7k + 3$.

Proof. Consider the mapping $f: V(S_n) \rightarrow \{0, 2, 3\}$ specified in Table 2, where $1 \leq i \leq n$ for indices and all subscripts are taken modulo n . Then, for every vertex labeled 0, there exists a vertex labeled 3 adjacent to it. Therefore, f is a DRDF on S_n and $\omega(f) = 7k + 3$, which implies $\gamma_{dR}(S_n) \leq 7k + 3$.

Table 2. The mapping $f: V(S_n) \rightarrow \{0, 2, 3\}$ with $n = 3k + 1$.

Vertex	Label	Neighbors	Neighbors with Label 3
$v_i, i \equiv 1 \pmod{3}$ and $i \neq n$	0	u_i, v_{i-1}, v_{i+1}	v_{i+1}
$v_i, i \equiv 2 \pmod{3}$	3		
$v_i, i \equiv 0 \pmod{3}$	0	u_i, v_{i-1}, v_{i+1}	v_{i-1}
v_n	3		
$u_i, i \equiv 1 \pmod{3}$ and $i \neq n$	2		
$u_i, i \equiv 2 \pmod{3}$	0	v_i	v_i
$u_i, i \equiv 0 \pmod{3}$	2		
u_n	0	v_n	v_n

The mapping is illustrated in Figure 4 with $n = 7$.

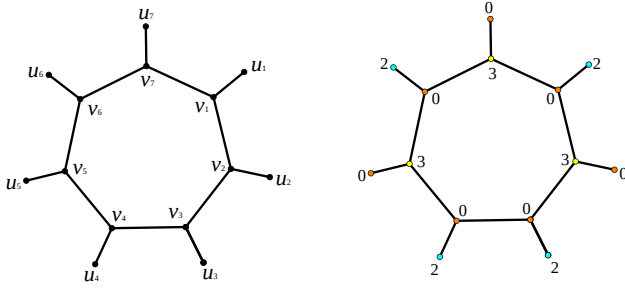


Figure 4. The mapping $f : V(S_n) \rightarrow \{0, 2, 3\}$ with $n = 7$.

On the other hand, let g be a γ_{dR} -function of S_n with no vertex assigned value 1. By Proposition B and Proposition C we have

$$\omega(g) = g(Q_1) + g(Q_4) + \cdots + g(Q_{n-3}) + g[u_n] \geq 7k + 2.$$

We next demonstrate that the case $\omega(g) = 7k + 2$ is unattainable via proof by contradiction. Assume for contradiction that $\omega(g) = 7k + 2$, which implies

$$\sum_{i=1}^n g(Q_i) = 3\omega(g) = 21k + 6.$$

Assume $n = 3k + 1$. The sum of weights over the $3k + 1$ consecutive local subgraph configurations equals $21k + 6$. By the pigeonhole principle, at least one of these configurations has weight less than 7. This contradicts Property C, which states that the weight of any consecutive local subgraph configuration is at least 7. Accordingly, we obtain $\omega(g) \geq 7k + 3$.

From the above reasoning, we conclude $\gamma_{dR}(S_n) = 7k + 3$ with $n = 3k + 1$.

Lemma 3 Let $n = 3k + 2$, where $k \geq 1$. Then $\gamma_{dR}(S_n) = 7k + 5$.

Proof. Consider the mapping $f : V(S_n) \rightarrow \{0, 2, 3\}$ specified in Table 3, where $1 \leq i \leq n$ for indices and all subscripts are taken modulo n . Then, for every vertex labeled 0, there exists a vertex labeled 3 adjacent to it. Therefore, f is a DRDF on S_n and $\omega(f) = 7k + 5$, which implies $\gamma_{dR}(S_n) \leq 7k + 5$.

Table 3. The mapping $f : V(S_n) \rightarrow \{0, 2, 3\}$ with $n = 3k + 2$.

Vertex	Label	Neighbors	Neighbor with Label 3
$v_i, i \equiv 1 \pmod{3}$ and $i \neq n$	0	u_i, v_{i-1}, v_{i+1}	v_{i+1}
$v_i, i \equiv 2 \pmod{3}$	3		
$v_i, i \equiv 0 \pmod{3}$	0	u_i, v_{i-1}, v_{i+1}	v_{i-1}
v_{n-1}	3		
v_n	0	u_n, v_{n-1}, v_1	v_{n-1}
$u_i, i \equiv 1 \pmod{3}$ and $i \neq n$	2		
$u_i, i \equiv 2 \pmod{3}$	0	v_i	v_i
$u_i, i \equiv 0 \pmod{3}$	2		
u_{n-1}	0	v_{n-1}	v_{n-1}
u_n	2		

The mapping is illustrated in Figure 5 with $n = 8$.

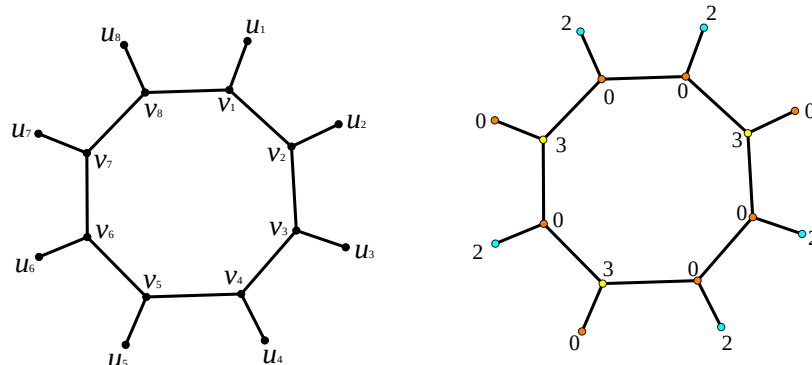


Figure 5. The mapping $f : V(S_n) \rightarrow \{0, 2, 3\}$ with $n = 8$.

On the other hand, let g be a γ_{dR} -function of S_n with no vertex assigned value 1. By Proposition B and Proposition C, we have

$$\omega(g) = g(Q_1) + g(Q_4) + \dots + g(Q_{n-4}) + g[u_{n-1}] + g[u_n] \geq 7k + 4$$

We now show that the case $\omega(g) = 7k + 4$ cannot hold. We proceed by proof by contradiction. Suppose for contradiction that $\omega(g) = 7k + 4$, which yields

$$\sum_{i=1}^n g(Q_i) = 3\omega(g) = 21k + 12$$

Assume $n = 3k + 2$. The sum of weights over the $n = 3k + 2$ consecutive local subgraph configurations equals $21k + 12$. By the pigeonhole principle, at least one of these configurations has weight less than 7. This contradicts Property C, which states that the weight of any consecutive local subgraph configuration is at least 7. Therefore, $\omega(g) \geq 7k + 5$.

Combining the above arguments, we obtain $\gamma_{dR}(S_n) = 7k + 5$ with $n = 3k + 2$.

3. Discussion

By Lemma 1, Lemma 2 and Lemma 3, we obtain the following theorem.

Theorem 1 Let S_n be a Sunlet graph where $n \geq 3$. Then

$$\gamma_{dR}(S_n) = \left\lceil \frac{7n}{3} \right\rceil.$$

In [1], Beeler et al. proposed the concept of the double Roman domination. They showed that $2\gamma(G) \leq \gamma_{dR}(G) \leq 3\gamma(G)$. Moreover, they suggested to find double Roman graphs.

In [14], It was proven that

Theorem 2 For a graph G with even order $n(n \geq 2)$

and no isolated vertices, $\gamma(G) = \frac{n}{2}$ if and only if the components of G are the cycle C_4 or the corona $H \circ K_1$ for any connected graph H .

By Theorem 2, $\gamma(S_n) = n$. Consequently, S_n is not a double Roman graph for all integers $n \geq 3$.

4. Conclusions

This paper investigates the double Roman domination number and related properties of Sunlet graphs S_n . We obtain the exact value $\gamma_{dR}(S_n) = \lceil 7n/3 \rceil$, and our results reveal that every Sunlet graph S_n with $n \geq 3$ fails to satisfy the required inequality for double Roman graphs, implying no Sunlet graph belongs to the family of double Roman graphs. Several directions deserve further exploration. First, it is worthwhile to extend the discussion to generalized Sunlet graphs with varied pendant structures. We aim to characterize the exact double Roman domination numbers of generalized Sunlet graphs and determine the parameter range under which generalized Sunlet graphs can be double Roman graphs. In deriving the double Roman domination number of Sunlet graphs, we employ local configuration decomposition based on their topological structure. For generalized Sunlet graphs, certain properties of path graphs may be required to handle their structural features. Second, one can establish tighter bounds relating the domination number and double Roman domination number for these graphs. Additionally, further research may develop universal criteria to judge whether unicyclic graphs, including generalized Sunlet graphs, are double Roman graphs.

Abbreviations

DRDF Double Roman Dominating Function

Author Contributions

Wensheng Li: Conceptualization, Methodology, Resources, Validation, Writing – original draft, Writing – review & editing

Conflicts of Interest

The author declares no conflicts of interest.

References

- [1] R. A. Beeler, T. W. Haynes, and S. T. Hedetniemi. Double Roman domination, *Discrete Appl. Math.*, 2016, vol. 211, 23–29, Oct. 2016. <https://doi.org/10.1016/j.dam.2016.03.017>
- [2] A. Klobucar; A. Klobucar. Properties of double Roman domination on cardinal products of graphs, *Ars Math. Contemp.*, vol. 19, 337–349, 2020. <https://doi.org/10.26493/1855-3974.2022.44a>
- [3] H. A. Ahangar, J. Amjadi, M. Atapour, M. Chellali, S. M. Sheikholeslami. Double Roman trees, *Ars Combin.*, vol. 145, 173–183, 2019. <https://combinatorialpress.com/ars/vol145/>
- [4] H. A. Ahangar, M. Chellali, S. M. Sheikholeslam. On the double Roman domination in graphs, *Discrete Appl. Math.*, vol. 232, 1–7, 2017. <https://doi.org/10.1016/j.dam.2017.06.014>
- [5] S. Banerjee, M. A. Henning, D. Pradhan. Algorithmic results on double Roman domination in graphs, *J. Comb. Opt.*, vol. 39, 90–114, 2020. <https://doi.org/10.1007/s10878-019-00457-3>
- [6] N. J. Rad, H. Rahbani. Some progress on the double Roman domination in graphs, *Discuss. Math. Graph Theory*, vol. 39, 41–53, 2019. <https://doi.org/10.7151/dmgt.2069>
- [7] J. Meena, T. N. M. Malini Mai, M. L. Suresh, L. Rathour, L. N. Mishra. Double Roman domination in some graphs, *Discrete Mathematics, Algorithms and Applications*, vol. 18(1), 2550016, 2026. <https://doi.org/10.1142/s1793830925500168>
- [8] S. M. Sheikholeslami, M. Esmaeili, J. J. Hamja., C. L. Armada, I. S. Aniversario. Independent double Roman domination stability in graphs, *European Journal of Pure & Applied Mathematics*, vol. 18, 1-16, 2025. <https://doi.org/10.29020/nybg.ejpam.v18i2.5984>
- [9] T. Zec, D. Matić, M. Djukanović. On double Roman domination problem for several graph classes, *Aequationes mathematicae*, vol. 99, 439-463, 2025. <https://doi.org/10.1007/s00010-024-01071-3>
- [10] D. P. Rupnik, J. Žerovnik. Double Roman domination: A Survey, *Mathematics*, vol. 11, 351, 2023. <https://doi.org/10.3390/math11020351>
- [11] S. Nazari-Moghaddam, M. Chellali, S. M. Sheikholeslami. Outer independent double Roman domination in unicyclic and bicyclic graphs. *Ars Combinatoria*, vol. 162, 51–70, 2025. <https://doi.org/10.61091/ars162-05>
- [12] S. M. Sheikholeslami, M. Esmaeili, M. Chellali, Y. Shang. Lower bounds for signed double Roman k-domination in unicyclic graphs, *Research in Mathematics*, vol. 13(1), 2690319, 2026. <https://doi.org/10.1080/27684830.2026.2690319>
- [13] W. S. Li, Z. S. Huang, Z. F. Feng. On the double Roman domination in Spider graphs, *Open Journal of Discrete Mathematics*, vol. 16(2), 13-18, 2026. <https://doi.org/10.4236/ojdm.2026.162002>
- [14] T. W. Haynes, S. T. Hedetniemi, P. J. Slater. *Fundamentals of Domination in Graphs*, 1998.

Biography



Wensheng Li received the B. S. degree in Information and Computing Science from the Hebei University of Technology, Hebei, China, in 2003. He obtained the M. S. degree in Computational Mathematics from the Dalian University of Technology, Dalian, China, in 2006. He worked as a lecturer in Department of Mathematics and Information Science, Langfang Teachers College. His research interest is in the area of graph theory.

Research Field

Wensheng Li: Optimization, Computational mathematics, Mathematical Modelling, Graph Theory and Applied mathematics