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Research Article

Properties of the Caputo Fractional-order Ferivative and Its Composition with Riemann-Liouville Fractional-order Derivative

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Abstract

Fractional calculus provides a powerful mathematical framework for modeling systems and phenomena characterized by memory, non-locality, and complex dynamic behavior. Among the different fractional differential operators, the Caputo and Riemann-Liouville fractional derivatives play important roles in theoretical and applied fractional calculus. This paper investigates and establishes several properties involving the Caputo fractional derivative by exploiting its fundamental relationship with Riemann-Liouville fractional derivative. In particular, four principal results are developed. First, a relationship between the ordinary integer-order derivative of a Caputo fractional derivative and a high-order Caputo fractional derivative is established. Second, explicit expression is derived for the composition of a Caputo fractional derivative with a Riemann-Liouville fractional derivative. Third, a formula is obtained for the composition of two Caputo fractional derivatives, including the associated initial-value terms. Finally, a generalized Leibniz rule for the Caputo fractional derivative of a product of two functions is established under appropriate continuity conditions. The proofs are developed using fundamental properties of fractional integrals and derivatives, the relationship between the Caputo and Riemann-Liouville operators, mathematical induction, and integral identities involving the Gamma and Beta functions. These results extend several established properties of the Riemann-Liouville fractional derivative to the Caputo framework and provide additional analytical tools for the study of fractional differential operators. The findings may contribute to the theoretical development of fractional calculus and its applications in areas involving fractional-order mathematical models.

Keywords

Caputo Fractional Derivative, Riemann-Liouville Fractional Derivative, Riemann-Liouville Fractional Integral, Fractional Calculus

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1. Introduction

A fractional-order derivative is just an operator which generalizes the ordinary derivative, such that if the fractional derivative is represented by the operator symbol D^{α} then, when $\alpha = 1$, it coincides with the ordinary differential operator D [20]. Their origin dates back to 1695, when Gottfried Wilhelm Leibniz, responding to a letter, speculated on the meaning of a derivative of order $\frac{1}{2}$ [20]. Fractional calculus is applied in diverse areas of engineering such as control of dynamic systems [3–5], electrical circuits [6–8], natural sciences [9–11], and signal and image processing [12–19]. The main utility of fractional calculus lies in modeling complex and non-linear systems with memory or inherited properties, where traditional integer-order calculus is not sufficient. In this context, fractional calculus has emerged as a promising approach in the design of mathematical models capable of describing heterogeneity and material structures at different scales, which cannot be adequately described by classical local theories. Therefore, fractional derivatives can play an important role in studying the macroscopic behavior of some materials, related to non-local interactions, which are common in determining their properties [21]. Today, there is virtually no area of science in which models involving fractional calculus have not been applied. However, despite the advantages of fractional-order operators over integer-order derivatives, a gap remains in our understanding of certain problems in fractional calculus. One of these problems is the lack of a universal geometric interpretation for non-integer-order derivatives. This is due to the existence of multiple definitions with different properties and limitations. In this paper, we develop some interesting properties concerning Caputo fractional order derivative. Before it, we recall some properties of fractional calculus:

Let $[a, b]$ $(-\infty < a < b < \infty)$ be a finite integral and let $AC[a, b]$ be the space of functions f which are absolutely continuous on $[a, b]$. It is known (see [2], p. 338) that $AC[a, b]$ coincides with the space of primitives of Lebesgue summable functions:

$$f(t) \in AC[a,b] \Leftrightarrow f(t) = c + \int_a^x \phi(t)dt, \quad \phi(t) \in L(a,b), \quad (1)$$

and therefore an absolutely continuous function $f(x)$ has a summable derivative $f'(x) = \phi(x)$ almost everywhere on $[a, b]$. Thus (1) yields $\phi(t) = f'(t)$ and $c = f(a)$. For $n \in \mathbb{N} := \{1, 2, 3, \cdots\}$ we denote by $AC^n[a, b]$ the space of complex-valued functions $f(x)$ which have continuous derivatives up to order $n-1$ on $[a, b]$ such that $f(x) \in AC[a, b]$:

$$AC^n[a,b] = \{f : [a,b] \to \mathbb{C} \quad and \quad \frac{d^{n-1}}{dx^{n-1}} f(x) \in AC[a,b]\}$$

Definition 1.1. Let $\alpha > 0$, $a, b \in \mathbb{R}$ with $a < b$, and f an element of the space $AC[a, b]$. The Riemann-Liouville fractional integral operator of order α, is defined by [2]:

$$J_{at}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds.$$

Definition 1.2. Let $a, b \in \mathbb{R}$, $a < b$, $n-1 < \alpha \leq n \in \mathbb{N}$ and f an element of the space $AC^n[a, b]$. The Riemann-Liouville fractional differential operator of order α, is defined by [20]:

$$\begin{aligned} {}^{RL}D_{at}^{\alpha} f(t) &= \frac{d^n}{dt^n}\left(J_{at}^{n-\alpha} f(t)\right) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-(\alpha+1)} f(s)\, ds. \end{aligned} \quad (2)$$

Definition 1.3. Let $a, b \in \mathbb{R}$, $a < b$, $n-1 < \alpha \leq n \in \mathbb{N}$ and f an element of the space $AC^n[a, b]$. The Caputo fractional differential operator of order α, is defined by [20]:

$$\begin{aligned} {}^{C}D_{at}^{\alpha} f(t) &= J_{at}^{n-\alpha}\left(\frac{d^n}{dt^n} f(t)\right) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-(\alpha+1)} f^{(n)}(s)\, ds. \end{aligned} \quad (3)$$

Now, calculating Caputo's derivative of t^p with $p \in \mathbb{R}$, we arrive at:

$${}^{C}D_{at}^{\alpha} t^p = \frac{\Gamma(p+1) t^{p-\alpha}}{\Gamma(p-\alpha+1)}. \quad (4)$$

If $f(t)$ is a function for which the Caputo fractional derivative, ${}^{C}D_{at}^{\alpha} f(t)$, of order $\alpha > 0$ exists together with the Riemann-Liouville fractional derivative, ${}^{RL}D_{at}^{\alpha} f(t)$, then, they are connected with each other by the following relation:

$${}^{C}D_{at}^{\alpha} f(t) = {}^{RL}D_{at}^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(a). \quad (5)$$

The Leibniz rule for fractional differentiation is the following. If $f(\tau)$ is continuous in $[a, t]$ and Φ has $n+1$ continuous derivatives in $[a, t]$, then the Riemann-Liouville fractional derivative of the product $\Phi(t) f(t)$ is given by

$${}^{RL}D_{at}^{\alpha}(\Phi(t) f(t)) = \sum_{k=0}^{n} \binom{\alpha}{k} \Phi^{(k)}(t) {}^{RL}D_{at}^{\alpha-k} f(t) - R_n^{\alpha}(t), \quad (6)$$

where $n \geq p+1$ and $R_n^{\alpha}(t) =$

$$\frac{1}{n!\Gamma(-\alpha)} \int_a^t (t-\tau)^{-\alpha-1} f(\tau)\, d\tau \int_{\tau}^t \Phi^{(n+1)}(\psi)(\tau-\psi)^n\, d\psi. \quad (7)$$

It is demonstrated that

$$\lim_{n\to\infty} R_n^{\alpha}(t) = 0,$$

if $f(\tau)$ and $\Phi(\tau)$ along with all its derivatives are continuous in $[a,t]$. Under this condition the Leibniz rule for Riemann-Liouville fractional (6) takes the form:

$$ {}^{RL}D_{at}^{\alpha}\left(\Phi(t)f(t)\right)=\sum_{k=0}^{\infty}\binom{\alpha}{k}\Phi^{(k)}(t)\,{}^{RL}D_{at}^{\alpha-k}f(t). \quad (8)$$

The formula

$$\int_{\Psi}^{t}(t-\tau)^{m-\beta-1}(\tau-\Psi)^{-\alpha-1-m}\,d\tau=$$

$$\frac{\Gamma(-\alpha-m)\Gamma(m-\beta)}{\Gamma(-\alpha-\beta)}(t-\Psi)^{-\beta-\alpha-1} \quad (9)$$

which will be used later, is obtained by using the substitution $\tau=\Psi+z(t-\Psi)$, $d\tau=(t-\Psi)dz$. Effectively:

$$\int_{\Psi}^{t}(t-\tau)^{m-\beta-1}(\tau-\Psi)^{-\alpha-1-m}d\tau$$

$$=\int_{0}^{1}\left[t-\Psi-z(t-\Psi)\right]^{m-\beta-1}$$

$$z^{-\alpha-1-m}(t-\Psi)^{-\alpha-1-m}(t-\Psi)dz$$

$$=\int_{0}^{1}(t-\Psi)^{m-\beta-1}(1-z)^{m-\beta-1}(t-\Psi)^{-\alpha-1-m}$$

$$(t-\Psi)z^{-\alpha-1-m}dz$$

$$=\int_{0}^{1}(t-\Psi)^{-\beta-\alpha-1}z^{-\alpha-1-m}(1-z)^{m-\beta-1}dz$$

$$=(t-\Psi)^{-\beta-\alpha-1}\int_{0}^{1}z^{-\alpha-m-1}(1-z)^{m-\beta-1}dz$$

$$=(t-\Psi)^{-\beta-\alpha-1}B(-\alpha-m,m-\beta)=$$

$$\frac{\Gamma(-\alpha-m)\Gamma(m-\beta)}{\Gamma(-\alpha-\beta)}(t-\Psi)^{-\beta-\alpha-1}. \quad (10)$$

The composition of Riemann-Liouville fractional differential operator ${}^{RL}D_{at}^{\alpha}$ with the usual derivative $\frac{d^n}{dt^n}$ is given by the following theorem [1]:

Theorem 1.1. Let $a\in\mathbb{R}$, $m-1<\alpha\le m\in\mathbb{N}$. Then:

$$\frac{d^m}{dt^m}\left({}^{RL}D_{at}^{\alpha}f(t)\right)={}^{RL}D_{at}^{\alpha+m}f(t), \quad (11)$$

$${}^{RL}D_{at}^{\alpha}\left({}^{RL}D_{at}^{m}f(t)\right)={}^{RL}D_{at}^{\alpha+m}f(t). \quad (12)$$

The purpose of this paper is to prove the following formulas for Caputo fractional derivative:

$$\frac{d^m}{dt^m}\left[{}^{C}D_{at}^{\alpha}f(t)\right]={}^{C}D_{at}^{\alpha+m}f(t), \quad (13)$$

$${}^{C}D_{at}^{\beta}\left[{}^{RL}D_{at}^{\alpha}f(t)\right]=$$

$$(-1)^m\Gamma(\alpha+m+1)\Gamma(-\alpha-m)J_{at}^{-\alpha-\beta}f(t), \quad (14)$$

$${}^{C}D_{at}^{\alpha}\left({}^{C}D_{at}^{\beta}f(t)\right)=$$

$$(-1)^n\Gamma(\beta+n+1)\Gamma(-\beta-n)\cdot$$

$$\cdot J_{at}^{-\beta-\alpha}f(t)-\sum_{k=0}^{m-1}\frac{f^{(k)}(a)(t-a)^{k-\beta-\alpha}}{\Gamma(k-\beta-\alpha+1)}, \quad (15)$$

$${}^{C}D_{at}^{\alpha}\left(\Phi(t)f(t)\right)=\sum_{k=0}^{\infty}\binom{\alpha}{k}\Phi^{(k)}(t)\,{}^{RL}D_{at}^{\alpha-k}f(t)-$$

$$\sum_{k=0}^{n-1}\frac{(t-a)^{k-\alpha}}{\Gamma(k+1-\alpha)}\left(\Phi(t)f(t)\right)^{(k)}(a). \quad (16)$$

The main mathematical instrument to use for the proof of the above results is the relationship between Caputo and Riemann-Liouville fractional derivatives, given by the formula (5).

2. Main Results

First, we prove the formula (13).

Theorem 2.1. Let $a\in\mathbb{R}$, $m-1<\alpha\le m\in\mathbb{N}$. Then:

$$\frac{d^m}{dt^m}\left[{}^{C}D_{at}^{\alpha}f(t)\right]={}^{C}D_{at}^{\alpha+m}f(t).$$

Proof. First, we show the formula:

$$\frac{d^m}{dt^m}(t-a)^{k-\alpha}=\frac{\Gamma(k-\alpha+1)}{\Gamma(k-\alpha-m+1)}(t-a)^{k-\alpha-m}, \quad (17)$$

where k and m are both positive entire numbers, and α is a real positive number. To this end, we use the method of mathematical induction:

For $m=1$, we have

$$\frac{d}{dt}(t-a)^{k-\alpha}=(k-\alpha)(t-a)^{k-\alpha-1}$$

$$=\frac{(k-\alpha)!}{(k-\alpha-1)!}(t-a)^{k-\alpha-1}$$

$$=\frac{\Gamma(k-\alpha+1)}{\Gamma(k-\alpha)}(t-a)^{k-\alpha-1}. \quad (18)$$

Let suppose that formula (17) holds true for $m=n$, that is:

$$\frac{d^n}{dt^n}(t-a)^{k-\alpha}=\frac{\Gamma(k-\alpha+1)}{\Gamma(k-\alpha-n+1)}(t-a)^{k-\alpha-n}. \quad (19)$$

To complete the proof of formula (17), we need to verify that it holds also for $m=n+1$. For it, we have:

$$\frac{d^{n+1}}{dt^{n+1}}(t-a)^{k-\alpha}=\frac{d}{dt}\left[\frac{d^n}{dt^n}(t-a)^{k-\alpha}\right]. \quad (20)$$

Inserting the right-hand-side expression of (19) into the right-hand-side of the equation (20), we obtain:

$$\frac{d^{n+1}}{dt^{n+1}}(t-a)^{k-\alpha}=\frac{d}{dt}\left[\frac{\Gamma(k-\alpha+1)}{\Gamma(k-\alpha-n+1)}(t-a)^{k-\alpha-n}\right]=$$

$$\frac{\Gamma(k-\alpha+1)}{\Gamma(k-\alpha-n+1)}\cdot\frac{d}{dt}(t-a)^{k-\alpha-n}. \quad (21)$$

By the help of formula (18), the last equality of the equation (21) becomes:

$$\frac{d^{n+1}}{dt^{n+1}}(t-a)^{k-\alpha} =$$

$$\frac{\Gamma(k-\alpha+1)}{\Gamma(k-\alpha-n+1)} \cdot \frac{\Gamma(k-\alpha-n+1)}{\Gamma(k-\alpha-n)}(t-a)^{k-\alpha-(n+1)}$$

$$= \frac{\Gamma(k-\alpha+1)}{\Gamma(k-\alpha-n)}(t-a)^{k-\alpha-(n+1)}.$$

Since formula (17) holds also true for $m = n+1$, we conclude its proof.

Second, we prove the formula:

$$\frac{d^m}{dt^m}\left[\sum_{k=0}^{n-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(a)\right]$$

$$= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha-m+1)}(t-a)^{k-\alpha-m}.$$

To this end, by using formula (17), we have:

$$\frac{d^m}{dt^m}\left[\sum_{k=0}^{n-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(a)\right]$$

$$= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k+1-\alpha)} \frac{\Gamma(k-\alpha+1)}{\Gamma(k-\alpha-m+1)}(t-a)^{k-\alpha-m}$$

$$= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha-m+1)}(t-a)^{k-\alpha-m}. \quad (22)$$

Third, we demonstrate that:

$$\frac{d^m}{dt^m}\left[{}^C D_{at}^{\alpha} f(t)\right] = {}^C D_{at}^{\alpha+m} f(t).$$

To this purpose, we use the equation (5):

$$\frac{d^m}{dt^m}\left[{}^C D_{at}^{\alpha} f(t)\right] = \frac{d^m}{dt^m}\left[{}^{RL} D_{at}^{\alpha} f(t)\right] - \frac{d^m}{dt^m}\left[\sum_{k=0}^{n-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(a)\right]. \quad (23)$$

Using the last equality of the equation (22) and the formula given by equation (11), the equality (23) becomes:

$$\frac{d^m}{dt^m}\left[{}^C D_{at}^{\alpha} f(t)\right] = {}^{RL} D_{at}^{\alpha+m} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha-m+1)}(t-a)^{k-\alpha-m}. \quad (24)$$

Inserting $\alpha + m$ in the place of α in the equation (5), we obtain:

$${}^{RL} D_{at}^{\alpha+m} f(t) = {}^C D_{at}^{\alpha+m} f(t) + \sum_{k=0}^{n-1} \frac{(t-a)^{k-\alpha-m}}{\Gamma(k+1-\alpha-m)} f^{(k)}(a). \quad (25)$$

Now, inserting the right-hand-side expression of the equality (25) into the first term of the right-hand-side of the formula (24), we obtain:

$$\frac{d^m}{dt^m}\left[{}^C D_{at}^{\alpha} f(t)\right]$$

$$= {}^C D_{at}^{\alpha+m} f(t) + \sum_{k=0}^{n-1} \frac{(t-a)^{k-\alpha-m}}{\Gamma(k+1-\alpha-m)} f^{(k)}(a) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha-m+1)}(t-a)^{k-\alpha-m}$$

$$= {}^C D_{at}^{\alpha+m} f(t),$$

thus, we conclude the proof of the theorem.

Following, we prove formula (26), that is, an expression for the composition between Caputo derivative and Riemann-Liouville derivative:

Theorem 2.2. Let $a, b \in \mathbb{R}$, $a < b$, $n-1 < \alpha \leq n \in \mathbb{N}$, $m-1 < \beta \leq m \in \mathbb{N}$ and f an element of the space $AC^n[a,b]$. Then, the following holds:

$${}^C D_{at}^{\beta}\left[{}^{RL} D_{at}^{\alpha} f(t)\right] = (-1)^m \Gamma(\alpha+m+1)\Gamma(-\alpha-m) J_{at}^{-\alpha-\beta} f(t). \quad (26)$$

Proof. First, we verify that

$$\frac{d^n}{dt^n}\int_a^t (t-s)^{n-(\alpha+1)} f(s)ds = \frac{\Gamma(n-\alpha)}{\Gamma(-\alpha)}\int_a^t (t-s)^{-(\alpha+1)} f(s)ds. \quad (27)$$

To this end, we use the method of mathematical induction, as follows:

For $n = 1$, we have

$$\frac{d}{dt}\int_a^t (t-s)^{-\alpha} f(s)ds = -\alpha\int_a^t (t-s)^{-\alpha-1} f(s)ds$$

$$= \frac{(-\alpha)!}{(-\alpha-1)!}\int_a^t (t-s)^{-\alpha-1} f(s)ds$$

$$= \frac{\Gamma(1-\alpha)}{\Gamma(-\alpha)}\int_a^t (t-s)^{-(\alpha+1)} f(s)ds.$$

Let suppose that the formula (27) holds true for $n = k$, that is:

$$\frac{d^k}{dt^k}\int_a^t (t-s)^{k-(\alpha+1)} f(s)ds = \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)}\int_a^t (t-s)^{-(\alpha+1)} f(s)ds. \quad (28)$$

To complete the proof of the formula (27), we need to prove that it holds also for $n = k + 1$. To this purpose, we take:

$$\frac{d^{k+1}}{dt^{k+1}}\int_a^t (t-s)^{k-\alpha} f(s)ds = \frac{d^k}{dt^k}\left[\frac{d}{dt}\int_a^t (t-s)^{k-\alpha} f(s)ds\right]$$

$$= (k-\alpha)\frac{d^k}{dt^k}\int_a^t (t-s)^{k-(\alpha+1)} f(s)ds. \quad (29)$$

Using the formula (28), the last equality of (29) becomes:

$$\frac{d^{k+1}}{dt^{k+1}}\int_a^t (t-s)^{k-\alpha} f(s)ds$$

$$= (k-\alpha)\cdot\frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)}\int_a^t (t-s)^{-(\alpha+1)} f(s)ds$$

$$= \frac{\Gamma(k+1-\alpha)}{\Gamma(-\alpha)}\int_a^t (t-s)^{-(\alpha+1)} f(s)ds.$$

Since formula (27) holds also true for $n = k + 1$, we conclude its proof.

Second, we prove that:

$${}^{RL}D_{at}^{\alpha} f(t) = \frac{1}{\Gamma(-\alpha)}\int_a^t (t-s)^{-(\alpha+1)} f(s)ds. \quad (30)$$

To this purpose, we use formula (27):

$${}^{RL}D_{at}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-(\alpha+1)} f(s)ds$$

$$= \frac{1}{\Gamma(n-\alpha)}\left[\frac{\Gamma(n-\alpha)}{\Gamma(-\alpha)}\int_a^t (t-s)^{-(\alpha+1)} f(s)ds\right]$$

$$= \frac{1}{\Gamma(-\alpha)}\int_a^t (t-s)^{-(\alpha+1)} f(s)ds.$$

Third, we verify that:

$$\frac{d^m}{dt^m}\left[{}^{RL}D_{at}^{\alpha} f(t)\right]$$

$$= (-1)^m\Gamma(\alpha+m+1)\int_a^t (t-s)^{-(\alpha+1)-m} f(s)ds. \quad (31)$$

For it, we apply the method of mathematical induction in the formula (30):

For $m = 1$, we have

$$\frac{d}{dt}\left[{}^{RL}D_{at}^{\alpha} f(t)\right]$$

$$= \frac{1}{\Gamma(-\alpha)}\frac{d}{dt}\int_a^t (t-s)^{-(\alpha+1)} f(s)ds$$

$$= \frac{(-1)(\alpha+1)}{\Gamma(-\alpha)}\int_a^t (t-s)^{-(\alpha+1)-1} f(s)ds$$

$$= \frac{(-1)(\alpha+1)!}{\alpha!\Gamma(-\alpha)}\int_a^t (t-s)^{-(\alpha+1)-1} f(s)ds$$

$$= \frac{(-1)\Gamma(\alpha+2)}{\Gamma(\alpha+1)\Gamma(-\alpha)}\int_a^t (t-s)^{-(\alpha+1)-1} f(s)ds$$

$$= (-1)^1\Gamma(\alpha+1+1)\int_a^t (t-s)^{-(\alpha+1)-1} f(s)ds.$$

Let suppose that the formula (31) holds true for $m = k$, that is:

$$\frac{d^k}{dt^k}\left[{}^{RL}D_{at}^{\alpha} f(t)\right]$$

$$= (-1)^k\Gamma(\alpha+k+1)\int_a^t (t-s)^{-(\alpha+1)-k} f(s)ds. \quad (32)$$

To complete the proof of the formula (31), we need to prove that it holds also for $m = k + 1$. To this purpose, we take:

$$\frac{d^{k+1}}{dt^{k+1}}\left[{}^{RL}D_{at}^{\alpha} f(t)\right] = \frac{d}{dt}\left[\frac{d^k}{dt^k}\left({}^{RL}D_{at}^{\alpha} f(t)\right)\right]$$

$$= \frac{d}{dt}\left[(-1)^k\Gamma(\alpha+k+1)\int_a^t (t-s)^{-(\alpha+1)-k} f(s)ds\right]$$

$$= (-1)^k\Gamma(\alpha+k+1)\frac{d}{dt}\int_a^t (t-s)^{-(\alpha+1)-k} f(s)ds$$

$$= (-1)^k\Gamma(\alpha+k+1)[-(\alpha+1)-k]\int_a^t (t-s)^{-(\alpha+1)-(k+1)} f(s)ds$$

$$= (-1)^{k+1}(\alpha+k+1)\Gamma(\alpha+k+1)\int_a^t (t-s)^{-(\alpha+1)-(k+1)} f(s)ds$$

$$= (-1)^{k+1}\Gamma(\alpha+k+1+1)\int_a^t (t-s)^{-(\alpha+1)-(k+1)} f(s)ds,$$

as we required.

Fourth, we prove that:

$${}^{C}D_{at}^{\beta}\left[{}^{RL}D_{at}^{\alpha} f(t)\right]$$

$$= (-1)^m\Gamma(\alpha+m+1)\Gamma(-\alpha-m)J_{at}^{-\alpha-\beta} f(t).$$

To do it, we use formula (31):

$$
\begin{aligned}
&{}^{C}D_{at}^{\beta}\left[{}^{RL}D_{at}^{\alpha}f(t)\right]\\
&=\frac{1}{\Gamma(m-\beta)}\int_a^t(t-\tau)^{m-(\beta+1)}\frac{d^m}{d\tau^m}\left[{}^{RL}D_{a\tau}^{\alpha}f(\tau)\right]d\tau\\
&=\frac{1}{\Gamma(m-\beta)}\int_a^t(t-\tau)^{m-(\beta+1)}[(-1)^m\Gamma(\alpha+m+1)\cdot\\
&\quad\cdot\int_a^{\tau}(\tau-\Psi)^{-(\alpha+1)-m}f(\Psi)d\Psi]d\tau\\
&=\frac{(-1)^m\Gamma(\alpha+m+1)}{\Gamma(m-\beta)}\int_a^t f(\Psi)d\Psi\int_{\Psi}^t(t-\tau)^{m-(\beta+1)}\cdot\\
&\quad\cdot(\tau-\Psi)^{-(\alpha+1)-m}d\tau. \qquad (33)
\end{aligned}
$$

Using the last equality of the equation (10) in the formula (33), we obtain:

$$
\begin{aligned}
&{}^{C}D_{at}^{\beta}\left[{}^{RL}D_{at}^{\alpha}f(t)\right]\\
&=\frac{(-1)^m\Gamma(\alpha+m+1)}{\Gamma(m-\beta)}\cdot\\
&\quad\cdot\frac{\Gamma(-\alpha-m)\Gamma(m-\beta)}{\Gamma(-\alpha-\beta)}\int_0^1(t-\Psi)^{-\beta-\alpha-1}f(\Psi)d\Psi\\
&=(-1)^m\Gamma(\alpha+m+1)\Gamma(-\alpha-m)J_{at}^{-\alpha-\beta}f(t).
\end{aligned}
$$

Thus, the formula (26) of the Theorem 2.2 is demonstrated.

The following theorem is about finding an expression for the composition between Caputo fractional derivatives, as it is seen in the formula (15).

Theorem 2.3. Let $a,b\in\mathbb{R}$, $a<b$, $n-1<\alpha\leq n\in\mathbb{N}$, $m-1<\beta\leq m\in\mathbb{N}$ and f an element of the space $AC^n[a,b]$. Then, the following holds:

$$
\begin{aligned}
&{}^{C}D_{at}^{\alpha}\left({}^{C}D_{at}^{\beta}f(t)\right)\\
&=(-1)^n\Gamma(\beta+n+1)\Gamma(-\beta-n)I_{at}^{-\beta-\alpha}f(t)\\
&\quad-\sum_{k=0}^{m-1}\frac{f^{(k)}(a)}{\Gamma(k-\beta-\alpha+1)}(t-a)^{k-\beta-\alpha}.
\end{aligned}
$$

Proof. Using the relationship (5), we have:

$$
\begin{aligned}
&{}^{C}D_{at}^{\alpha}\left[{}^{C}D_{at}^{\beta}f(t)\right]\\
&={}^{C}D_{at}^{\alpha}\left[{}^{RL}D_{at}^{\beta}f(t)-\sum_{k=0}^{m-1}\frac{(t-a)^{k-\beta}}{\Gamma(k+1-\beta)}f^{(k)}(a)\right]\\
&={}^{C}D_{at}^{\alpha}\left[{}^{RL}D_{at}^{\beta}f(t)\right]-\sum_{k=0}^{m-1}\frac{f^{(k)}(a)}{\Gamma(k+1-\beta)}\left[{}^{C}D_{at}^{\alpha}(t-a)^{k-\beta}\right]. \qquad (34)
\end{aligned}
$$

Using (4) and (26) in the last equation of (34), we obtain:

$$
\begin{aligned}
&{}^{C}D_{at}^{\alpha}\left({}^{C}D_{at}^{\beta}f(t)\right)\\
&=(-1)^n\Gamma(\beta+n+1)\Gamma(-\beta-n)J_{at}^{-\beta-\alpha}f(t)\\
&\quad-\sum_{k=0}^{m-1}\frac{f^{(k)}(a)}{\Gamma(k-\beta-\alpha+1)}(t-a)^{k-\beta-\alpha}.
\end{aligned}
$$

Lastly, we prove the formula(16), which is a Leibniz rule for Caputo fractional differentiation.

Theorem 2.4. If $\lim_{n\to\infty}R_n^{\alpha}(t)=0$, where $R_n^{\alpha}(t)$ is given by formula (7), and $f(\tau)$ and $\Phi(\tau)$ along with all its derivatives are continuous in $[a,t]$. Then:

$$
\begin{aligned}
{}^{C}D_{at}^{\alpha}\left(\Phi(t)f(t)\right)&=\sum_{k=0}^{\infty}\binom{\alpha}{k}\Phi^{(k)}(t)\,{}^{RL}D_{at}^{\alpha-k}f(t)\\
&\quad-\sum_{k=0}^{n-1}\frac{(t-a)^{k-\alpha}}{\Gamma(k+1-\alpha)}\left(\Phi(t)f(t)\right)^{(k)}(a). \qquad (35)
\end{aligned}
$$

Proof. Using the formula (5), we can write

$$
\begin{aligned}
{}^{C}D_{at}^{\alpha}\left(\Phi(t)f(t)\right)&={}^{RL}D_{at}^{\alpha}\left(\Phi(t)f(t)\right)\\
&\quad-\sum_{k=0}^{n-1}\frac{(t-a)^{k-\alpha}}{\Gamma(k+1-\alpha)}\left(\Phi(t)f(t)\right)^{(k)}(a). \qquad (36)
\end{aligned}
$$

Now, using formula (8) in the equation (36), we obtain finally equality (35).

3. Conclusion

One of the fractional differential operator with properties better developed is Riemann-Liouville fractional derivative. The goal of this paper has been to obtain some of such properties, but in the sense of Caputo fractional derivative. To do this, we have used, as a fundamental tool, the relationship between Caputo derivative and Riemann-Liouville derivative.

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Conflicts of Interest

The authors declare that he has no conflict of interest.

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