

Asymptotic Behavior of n th Order Impulsive Differential Equations

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To cite this article:

Yuxin Zheng, Lijun Pan. Asymptotic Behavior of n th Order Impulsive Differential Equations. *Science Journal of Applied Mathematics and Statistics*, 12(3), 43-47. <https://doi.org/10.11648/j.sjams.20241203.12>

Received: 19 March 2024; **Accepted:** 8 May 2024; **Published:** 14 June 2024

Abstract: This paper devotes to the asymptotic behavior of all solutions of n th order impulsive differential equations. Based on impulsive differential inequality, boundedness and zero tendency of every solution for n th order impulsive differential equations are obtained. In addition, we derive globally uniformly exponential stability of every solution under Lyapunov function and impulsive technique, and these results are extend to n th order differential equations with periodic coefficient and periodic impulse. Meanwhile, an example with simulations are provided to verify the conclusion.

Keywords: Asymptotic, n th Order, Impulsive Differential Equation

1. Introduction

Due to the abrupt changes at certain moments, impulsive effects are common phenomena in natural world. Such phenomena are described by impulsive differential equations which have been used efficiently in modelling many real world problems that arise in the fields such as medicine, electronics, and network [1–7]. The behavior of all solutions is an important issue of impulsive differential equations. Therefore, much effort has been made to investigate the oscillation criteria and the asymptotic behavior of all solutions of impulsive differential equations [8–16]. For instance, in [9], the authors investigated the asymptotic behavior of solutions of second order nonlinear impulsive differential equations. The authors [14] obtained the results of Razumikhin and Krasovskii stability of impulsive stochastic delay systems via uniformly stable function method. However, the behavior results of solutions for impulsive differential equations mainly focus on first order impulsive differential equations and second order impulsive differential equations. For higher order impulsive differential equations, the asymptotic behavior and stability of solutions has been little discussed.

Motivated by the above discussions, this paper is to study the asymptotic behavior of n order impulsive differential equations. By constructing appropriate Lyapunov functions

and impulsive technique, the bounded property and zero convergence of every solution are obtained, which means that impulsive effects play an essential role in the behavior of n th order impulsive differential equations. Meanwhile, an example with simulations is provided to demonstrate the applicability of our results.

The rest of this paper is organized as follows. n th order impulsive differential equations is presented in Section 2. In Section 3, asymptotic behavior results of n th order impulsive differential equations are derived. A numerical example is given to demonstrate our results in Section 4 and Section 5 concludes the paper.

2. Model Description and Preliminaries

Consider the following n th order impulsive differential equations

$$\begin{cases} x^{(n)}(t) + p(t)x(t) = 0, t \geq 0, t \neq t_k, \\ x^{(i)}(t_k^+) = I_{ik}(x^{(i)}(t_k)), \\ i = 0, 1, \dots, n-1, k = 1, 2, \dots, \end{cases} \quad (1)$$

where $t_k, k = 1, 2, \dots$ are impulsive moments satisfying $0 = t_0 < t_1 < \dots < t_k < t_{k+1} < \dots, \lim_{k \rightarrow +\infty} t_k = +\infty$,

$x^{(i)}(t_k^+) = \lim_{h \rightarrow 0^+} \frac{x^{(i-1)}(t_k + h) - x^{(i-1)}(t_k^+)}{h}$, $x^{(i)}(t_k) = \lim_{h \rightarrow 0^-} \frac{x^{(i-1)}(t_k + h) - x^{(i-1)}(t_k)}{h}$, $p(t)$ is continuous and differentiable function on $[0, +\infty)$, $I_{ik}(\cdot)$ are continuous in R and there exist positive numbers d_k such that $|I_{ik}(x)| \leq d_k|x|$.

By a solution $x = x(t)$, we mean a real function on $[0, +\infty)$ satisfies that $x^{(n)}(t) + p(t)x(t) = 0$ at each point $t \in [0, +\infty)$ with the possible exception of the points $t \neq t_k$ and $x^{(i)}(t_k^+) = I_{ik}(x^{(i)}(t_k))$ for any t_k .

Lemma 1. ([9]) Assume that $l(t)$ is a continuous function on $[0, +\infty)$. If there exist positive numbers v_k , $k = 1, 2, \dots$ and constants $\gamma > 0$, $b > 0$, c such that for $t_{k+1} - t_k \leq \gamma$, $\ln v_k + \int_{t_k}^{t_{k+1}} l(s)ds \leq -b$ and

$$\ln v_k + \int_{t_k}^{t_k+\xi} l(s)ds \leq c, \forall \xi \in [0, \gamma] \quad (2)$$

then

$$\ln h(t) \leq -\frac{b}{\gamma}t + \max\{b + c, 0\}, t > 0, \quad (3)$$

where $h(t) = \prod_{0 \leq t_k < t} v_k \exp(\int_0^t l(s)ds)$.

Lemma 2. ([9]) Assume that $l(t)$ is a continuous function on $[0, +\infty)$. If there exist positive numbers v_k , $k = 0, 1, \dots$ and $i \in \{1, 2, \dots\}$ and $\omega > 0$, $c > 0$ such that for $t_{k+i} = t_k + \omega$, $v_{k+i} = v_k$, $l(t + \omega) = l(t)$ and

$$\ln\left(\prod_{t \leq t_k < t+\omega} v_k\right) + \int_t^{t+\omega} l(s)ds \leq -b, \forall t \geq 0, \quad (4)$$

then

$$\ln h(t) \leq -\frac{b}{\omega}t + b + \max_{t \in [0, \omega]} \{|\ln(\prod_{0 \leq t_k < t} v_k)|\} + \omega \max_{t \in [0, \omega]} \{|l(t)|\}, t > 0, \quad (5)$$

where $h(t) = \prod_{0 \leq t_k < t} v_k \exp(\int_0^t l(s)ds)$.

$$\alpha_n = d_1^2 \cdots d_{n-1}^2 \exp\left(\int_0^{t_n} \tilde{p}(s)ds\right), \tilde{p}(s) = \max\{2, 1 + p^2(s)\}.$$

Proof. For every solution $x(t)$ of (1), we construct a Lyapunov function

$$V(t) = \sum_{i=0}^{n-1} [x^{(i)}(t)]^2 \quad (6)$$

In view of system (1), we have

$$\begin{aligned} V'(t) &= 2 \sum_{i=0}^{n-1} [x^{(i)}(t)x^{(i+1)}(t)] = 2x^{(n-1)}(t)[-p(t)x(t)] + 2 \sum_{i=0}^{n-2} [x^{(i)}(t)x^{(i+1)}(t)] \\ &\leq \sum_{i=0}^{n-1} \{[x^{(i)}(t)]^2 + [x^{(i+1)}(t)]^2\} + p^2(t)[x^{(n-1)}(t)]^2 + x^2(t) \\ &= \sum_{i=0}^{n-2} 2[x^{(i)}(t)]^2 + [1 + p^2(t)][x^{(n-1)}(t)]^2 \leq \tilde{p}(t)V(t). \end{aligned} \quad (7)$$

When $t = t_k$, we have

$$\begin{aligned} V(t_k^+) &= \sum_{i=0}^{n-1} [x^{(i)}(t_k^+)]^2 = \sum_{i=0}^{n-1} [I_{ik}(x^{(i)}(t_k))]^2 \\ &\leq \sum_{i=0}^{n-1} d_k^2 [x^{(i)}(t_k)]^2 = d_k^2 V(t_k). \end{aligned} \quad (8)$$

For $t \in [0, t_1]$, by (7), it yields that

$$V(t) \leq V(0) \exp\left(\int_0^t \tilde{p}(s)ds\right). \quad (9)$$

Thus

$$V(t_1) \leq V(0) \exp\left(\int_0^{t_1} \tilde{p}(s)ds\right). \quad (10)$$

For $t \in [t_1, t_2]$, we have

$$\begin{aligned} V(t) &\leq V(t_1^+) \exp\left(\int_{t_1}^t \tilde{p}(s)ds\right) \\ &\leq d_1^2 V(t_1) \exp\left(\int_{t_1}^t \tilde{p}(s)ds\right) \\ &\leq V(0) d_1^2 \exp\left(\int_0^{t_1} \tilde{p}(s)ds + \int_{t_1}^t \tilde{p}(s)ds\right) \\ &= V(0) d_1^2 \exp\left(\int_0^t \tilde{p}(s)ds\right). \end{aligned} \quad (11)$$

3. Asymptotic Behavior Analysis

In this section, some criteria on asymptotic behavior results of (1) are established under Lyapunov functions and impulsive technique.

Theorem 1. If there exists $M > 0$ such that $\alpha_n \leq M$, $n = 1, 2, \dots$, then every solution $x(t)$ of (1) is bounded, where

By induction, for $t \in [t_{n-1}, t_n]$, we conclude that

$$\begin{aligned} V(t) &\leq V(0)d_1^2 \cdots d_{n-1}^2 \exp\left(\int_0^t \tilde{p}(s)ds\right) \\ &\leq V(0)d_1^2 \cdots d_{n-1}^2 \exp\left(\int_0^{t_n} \tilde{p}(s)ds\right) \\ &= V(0)\alpha_n. \end{aligned} \quad (12)$$

Since sequence $|\alpha_n| \leq M$, we can conclude that every solution $x(t)$ of (1) is bounded.

Theorem 2. If $\lim_{n \rightarrow \infty} \alpha_n = 0$, then every solution $x(t)$ of (1) satisfies $\lim_{t \rightarrow \infty} x(t) = 0$, where $\alpha_n = d_1^2 \cdots d_{n-1}^2 \exp\left(\int_0^{t_n} \tilde{p}(s)ds\right)$, $\tilde{p}(s) = \max\{2, 1 + p^2(s)\}$.

$$\begin{aligned} V'(t) &= p'(t)x^2(t) + 2p(t)x'(t)x(t) + 2 \sum_{i=1}^{n-1} [x^{(i)}(t)x^{(i+1)}(t)] \\ &= p'(t)x^2(t) + 2p(t)x'(t)x(t) + 2 \sum_{i=1}^{n-2} [x^{(i)}(t)x^{(i+1)}(t)] + 2x^{(n-1)}(t)[-p(t)x(t)] \\ &\leq p'(t)x^2(t) + p^2(t)x^2(t) + [x'(t)]^2 + \sum_{i=0}^{n-2} \{[x^{(i)}(t)]^2 + [x^{(i+1)}(t)]^2\} + [x^{(n-1)}(t)]^2 + p^2(t)x^2(t) \\ &= [2p^2(t) + p'(t)]x^2(t) + 2 \sum_{i=1}^{n-1} [x^{(i)}(t)]^2 \\ &= (2p(t) + \frac{p'(t)}{p(t)})p(t)x^2(t) + 2 \sum_{i=1}^{n-1} [x^{(i)}(t)]^2 \\ &\leq \bar{p}(t)V(t). \end{aligned} \quad (14)$$

According to the proof of Theorem 1, we can conclude that every solution $x(t)$ of (1) is bounded.

Theorem 4. If there exists $a > 0$ such that $p(t) \geq a$ and $\lim_{n \rightarrow \infty} \beta_n = 0$, then every solution $x(t)$ of (1) satisfies $\lim_{t \rightarrow \infty} x(t) = 0$, where $\beta_n = d_1^2 \cdots d_{n-1}^2 \exp\left(\int_0^{t_n} \bar{p}(s)ds\right)$, $\bar{p}(s) = \max\{2, 2p(s) + \frac{p'(s)}{p(s)}\}$.

Theorem 5. If there exist $c_1 > 0$, $c_2 \geq 0$ such that $\ln h_1(t) \leq -c_1 t + c_2$, then every solution $x(t)$ of (1) is globally uniformly exponentially stable, where $h_1(t) =$

$$\prod_{0 \leq t_k < t} d_k^2 \exp\left(\int_0^t \tilde{p}(s)ds\right), \tilde{p}(s) = \max\{2, 1 + p^2(s)\}.$$

Proof. For every solution $x(t)$ of (1), we construct a Lyapunov function

$$V(t) = \sum_{i=0}^{n-1} [x^{(i)}(t)]^2 \quad (15)$$

Based on the proof of Theorem 1, for $t \in [t_{n-1}, t_n]$, we have

$$\begin{aligned} V(t) &\leq V(0)d_1^2 \cdots d_{n-1}^2 \exp\left(\int_0^t \tilde{p}(s)ds\right) \\ &= V(0) \prod_{0 \leq t_k < t} d_k^2 \exp\left(\int_0^t \tilde{p}(s)ds\right) \\ &= V(0)h_1(t) \leq V(0)e^{c_2}e^{-c_1 t}, \end{aligned} \quad (16)$$

Theorem 3. If there exist $a > 0$, $\bar{M} > 0$ such that $p(t) \geq a$ and $\beta_n \leq \bar{M}$, $n = 1, 2, \dots$, then every solution $x(t)$ of (1) is bounded, where $\beta_n = d_1^2 \cdots d_{n-1}^2 \exp\left(\int_0^{t_n} \bar{p}(s)ds\right)$, $\bar{p}(s) = \max\{2, 2p(s) + \frac{p'(s)}{p(s)}\}$.

Proof. For every solution $x(t)$ of (1), we construct a Lyapunov function

$$V(t) = p(t)x^2(t) + \sum_{i=1}^{n-1} [x^{(i)}(t)]^2 \quad (13)$$

In view of system (1), we have

which implies that every solution $x(t)$ of (1) is globally uniformly exponentially stable.

Theorem 6. If there exist $a > 0$, $c_3 > 0$, $c_4 \geq 0$ such that $p(t) \geq a$ and $\ln h_2(t) \leq -c_3 t + c_4$, then every solution $x(t)$ of (1) is globally uniformly exponentially stable, where $h_2(t) =$

$$\prod_{0 \leq t_k < t} d_k^2 \exp\left(\int_0^t \bar{p}(s)ds\right), \bar{p}(s) = \max\{2, 2p(s) + \frac{p'(s)}{p(s)}\}.$$

By using Lemma 1 and Lemma 2, together with Theorem 5 and Theorem 6, we can obtain the following practical theorems.

Theorem 7. If there exist $\bar{\gamma}_1 > 0$, $\bar{b}_1 > 0$, \bar{c}_1 such that for $k = 1, 2, \dots, t_{k+1} - t_k \leq \bar{\gamma}_1$, $\ln d_k^2 + \int_{t_k}^{t_{k+1}} \tilde{p}(s)ds \leq -\bar{b}_1$ and

$$\ln d_k + \int_{t_k}^{t_k + \bar{\xi}_1} \tilde{p}(s)ds \leq \bar{c}_1, \forall \bar{\xi}_1 \in [0, \bar{\gamma}_1] \quad (17)$$

then every solution $x(t)$ of (1) is globally uniformly exponentially stable, where

$$\tilde{p}(s) = \max\{2, 1 + p^2(s)\}.$$

Theorem 8. If there exist $a > 0$, $\bar{\gamma}_2 > 0$, $\bar{b}_2 > 0$, \bar{c}_2 such that for $p(t) \geq a$, $t_{k+1} - t_k \leq \bar{\gamma}_2$, $\ln d_k^2 + \int_{t_k}^{t_{k+1}} \tilde{p}(s)ds \leq -\bar{b}_2$

and

$$\ln d_k^2 + \int_{t_k}^{t_k + \bar{\xi}_2} \bar{p}(s) ds \leq \bar{c}_2, \forall \bar{\xi}_2 \in [0, \bar{\gamma}_2], \quad (18)$$

then every solution $x(t)$ of (1) is globally uniformly exponentially stable, where

$$\bar{p}(s) = \max\{2, 2p(s) + \frac{p'(s)}{p(s)}\}.$$

Theorem 9. If there exist $i \in \{1, 2, \dots\}$ and $\omega > 0, \tilde{b}_1 > 0$ such that $t_{k+i} = t_k + \omega, d_{k+i} = d_k, \tilde{p}(t + \omega) = \tilde{p}(t)$ and

$$\ln\left(\prod_{t \leq t_k < t + \omega} d_k^2\right) + \int_t^{t+\omega} \tilde{p}(s) ds \leq -\tilde{b}_1, \forall t \geq 0, \quad (19)$$

then every solution $x(t)$ of (1) is globally uniformly exponentially stable, where

$$\tilde{p}(t) = \max\{2, 1 + p^2(t)\}.$$

Theorem 10. If there exist $i \in \{1, 2, \dots\}$ and $a > 0$

$\omega > 0, \tilde{b}_2 > 0$ such that $p(t) \geq a, t_{k+i} = t_k + \omega, d_{k+i} = d_k, \bar{p}(t + \omega) = \bar{p}(t)$ and

$$\ln\left(\prod_{t \leq t_k < t + \omega} d_k^2\right) + \int_t^{t+\omega} \bar{p}(s) ds \leq -\tilde{b}_2, \forall t \geq 0, \quad (20)$$

then every solution $x(t)$ of (1) is globally uniformly exponentially stable, where

$$\bar{p}(s) = \max\{2, 2p(s) + \frac{p'(s)}{p(s)}\}.$$

4. Numerical Simulations

This section presents an example to demonstrate theoretical results for asymptotic behavior of n th order impulsive differential equations.

Example 1. Consider the following second order impulsive differential equations:

$$\begin{cases} x''(t) + p(t)x(t) = 0, t \geq 0, t \neq t_k, \\ x(t_k^+) = d_k x(t_k), x'(t_k^+) = d_k x'(t_k), k = 1, 2, \dots, \end{cases} \quad (21)$$

where $p(t)$ is periodic function with periodic $\omega = 2$, and

$$p(t) = \begin{cases} 0, t \in [0, 0.5), \\ 2, t \in [0.5, 1], \end{cases}$$

the impulsive effects with periodic 1 are defined by

$t_{k+2} = t_k + 1, t_1 = 0.2, d_{2k} = 0.2, d_{2k+1} = 1.25, k = 0, 1, \dots$. By computation, we have

$$\tilde{p}(t) = \begin{cases} 2, t \in [0, 0.5), \\ 5, t \in [0.5, 1] \end{cases}$$

and

$$\ln\left(\prod_{t \leq t_k < t+2} d_k^2\right) + \int_t^{t+2} \tilde{p}(s) ds < 0, \forall t \geq 0. \quad (22)$$

It follows from Theorem 9 that every solution $x(t)$ of (1) is globally uniformly exponentially stable. Figure 1 depicts impulsive sequence with period 1. Figure 2 depicts state trajectory $x(t)$ of system (21).

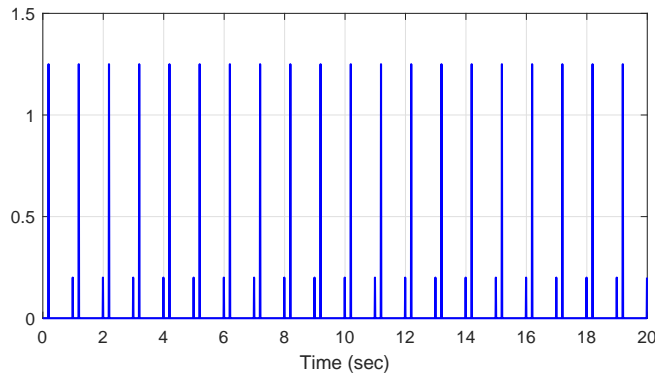


Figure 1. Impulsive sequence with periodic 1.

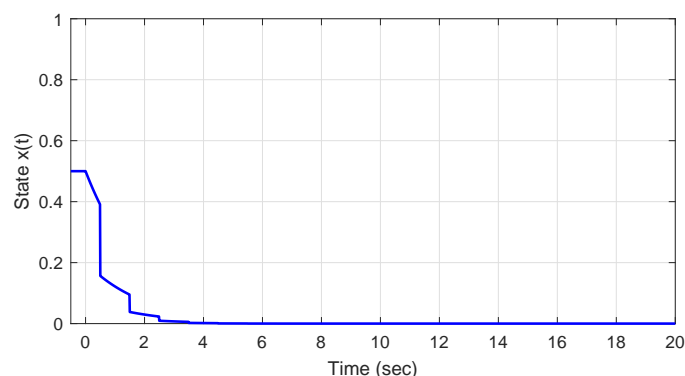


Figure 2. The state trajectory $x(t)$ of (21).

5. Conclusions

In this paper, asymptotic behavior of n order impulsive differential equations has been studied. By method of impulsive technique and Lyapunov function, some new behavior criteria have been derived. Finally, a standard example package illustrate that the new results are practical. Our future work will focus on the neural networks with impulse and consensus of multi-agent systems with impulse.

Funding

This research was supported by Guangdong Basic and Applied Basic Research Foundation (2022A1515010193).

Conflicts of Interest

The authors declare no conflicts of interest.

References

- [1] Lakshmikantham. V., Bainov. D., Simeonov. P. *Theory of Impulsive Differential Equations*. World Scientific. Singapor; 1989.
- [2] Bainov. D., Simeonov. P. *Oscillation Theorem of Impulsive Differential Equations*. International Publications; 1998.
- [3] Zhang. W., Tang. Y., Zheng. Z., Liu. Y Stability of time-varying systems with delayed impulsive effects. *International Journal of Robust and Nonlinear Control*. 2021, 31, 7825-7843.
- [4] Li. X., Ding. Y Razumikhin-type theorems for time-delay systems with persistent impulses. *Systems Control Letters*. 2017, 107, 22-27.
- [5] Wang., Duan. H. S., Li. C., Wang. L., Huang. T Globally exponential stability of delayed impulsive functional differential systems with impulse time windows. *Nonlinear Dynamics*. 2016, 84, 1-11.
- [6] Liu. B., Hill. D. Impulsive consensus for complex dynamical networks with nonidentical nodes and coupling time-delays. *SIAM J. Control Optim.* 2011, 49, 315-338.
- [7] Lu. J., Ho. D., Cao. J A unified synchronization criterion for impulsive dynamical networks. *Automatica*,. 2010, 46, 1215-1221.
- [8] Chen. Y., Feng. W Oscillations of second order nonlinear ODE with impulses. *J. Math. Anal. Appl.* 1997, 210, 150-169.
- [9] Wen. L., Chen. Y Razumikhin type theorems for functional differential equations with impulses. *Dynamics of Continuous, Discrete and Impulsive Systems*. 1999, 6, 389-400.
- [10] Jiao. J., Chen. L., Li. L Asymptotic behavior of solutions of second-order nonlinear impulsive differential equations. *J. Math. Anal. Appl.* 2008, 337, 458-463.
- [11] Wang. Q Oscillation and asymptotics for second order half linear differential equation. *Appl. Math. Comput.* 2001, 112, 253-266.
- [12] Wu. X., Chen.S., Ji. H Oscillation of a class of second-order nonlinear ODE with impulses. *Appl. Math. Comput.* 2003, 138, 181-188.
- [13] Chen. Y., Feng. W Asymptotic behavior of solutions of second-order nonlinear impulsive differential equations. *Journal of south normal university (Natural science)*. 2000, 1, 13-17.
- [14] Pan. L., Hu. J., Cao. J Razumikhin and Krasovskii stability of impulsive stochastic delay systems via uniformly stable function method. *Nonlinear Analysis: Modelling and Control*. 2023, 28, 1161-1181.
- [15] Mariton. M Oscillation properties of higher oeder linear impulsive delay differential equations. *Differential Equations Applications*. 2015, 7, 43-55.
- [16] Ren. W., Xiong. J Stability analysis of impulsive stochastic nonlinear systems. *IEEE Trans. Autom. Control*. 2017, 62, 4791-4797.