
Analytical Results of the Motion of Oscillating Dumbbell in a Viscous Fluid

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Abstract: The aim of this paper is to investigate analytically the motion of oscillating dumbbell, two micro-spheres connected by a spring, in a viscous incompressible fluid at low Reynolds number. The oscillating dumbbell consists of one conducting sphere and assumed to be actively in motion under the action of an external oscillator field while the other is non-conducting sphere. As result, the oscillating dumbbell moves due to the induced flow oscillation of the surrounding fluid. The fluid flow past the spheres is described by the Stokes equation and the governing equation in the vector form for the oscillating dumbbell is solved asymptotically using the two-timing method. For illustrations, applying a simple oscillatory external field, a systematic description of the average velocity of the oscillating dumbbell is formulated. The trajectory of the oscillating dumbbell was found to be inversely proportional to the frequency of the external field, and the results demonstrated that the oscillating dumbbell moves in a circular path with a speed that decreases inversely with the length of the spring.

Keywords: Fluid Dynamics, Low Reynolds Number, Oscillatory Motion, Stokes Equation, Two-Timing Method

1. Introduction

The study of particle motion in a viscous fluid is a key to understanding the physical processes associated with particle suspensions at low Reynolds number. It has been of interest to scientists for many years and is still an active area of research [2, 3, 7, 8, 24, 25, 27]. The interactions between particles in the viscous fluid plays important roles in many applications in medicine and technology, such as minimizing surgical invasion and controlling drug delivery [13, 14, 16, 18, 20].

To successfully design a micro-robot, the motion of linked two (or more) micro-spheres must be studied and formulated. For example, the motion of dumbbells, in which two particles linked with a thin rod of a fixed length, has been studied by many researchers [1, 6, 12, 22]. In this article, we studied an extended case, in which the length is no longer fixed but varied between two lengths.

A conducting or active micro-sphere suspended in an external oscillatory force tends to accelerate in the direction of the applied external field, while the dumbbell moves due to

the local surrounding fluid velocity generated by the motion of the active sphere. There is, however, a secondary effect arising from the fact that the oscillating dumbbell has not solely moved due to the external field, but its motion is also distorted by the presence of the other sphere. The key question is how does the surrounding fluid influence the motion of a oscillating dumbbell; will it also change the direction the active sphere moves, and if so, what trajectory does the oscillating dumbbell follow? Does the size of the micro-spheres or the length of the spring between the spheres have any effects on the motion of the oscillating dumbbell? In an attempt to answer these questions, we develop an analytical framework to study the motion of oscillating dumbbell in Stokes flow driven by an external oscillatory force. This analytical theory serves as a preliminary investigation on the effect of an external oscillatory field on the motion of a suspension of conducting particles.

This paper aims to provide a systematic and explicit description of a two-dimensional motion of oscillating

dumbbell in Cartesian coordinates (X, Y) and calculate its average velocities. Two micro-spheres of radii $R_\nu < 10^{-6}$ meter are considered submerged in a viscous fluid at low Reynolds number, $Re \ll 1$, and suppose that one sphere is passive and the other is active which is accelerated by an external oscillatory field such as magnetic field [4, 9, 11, 17], electric field *e.g.* [10] and molecular Brownian force [23].

Formulation of the problem leads us to study the motion of an oscillating dumbbell with time-periodic forces. The problem is described by the Stokes equation, where the fluid inertial effect is neglected, and is solved by employing the two-timing method [11]. The results present in general form and discuss it in detail through an example. Treatment of the problem is simple, but it can be considered as the basis for the development of a full theory of suspension mechanics.

2. Formulation of Problem

Oscillating dumbbell, two solid spherical spheres connected by a spring of stiffness k and length l is considered, see Figure 1. The oscillating dumbbell involved two homogeneous spheres; one of them is active and positively buoyant and the

other is passive and negatively buoyant. The active sphere is driven by an external field which oscillates periodically with constant frequency ω . Geometrically, the centers of the spheres and the length of the spring are denoted by

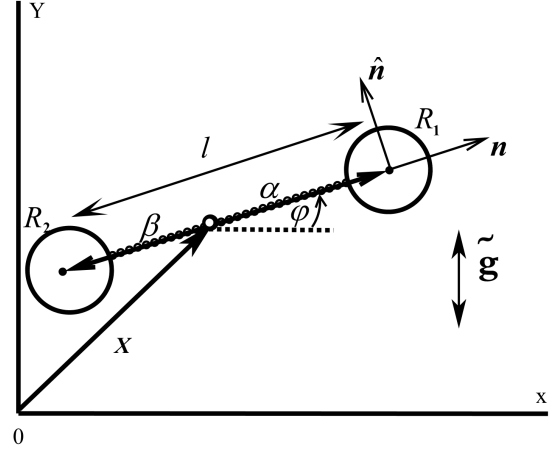


Figure 1. Diagram of oscillating dumbbell.

$$\mathbf{x}^{(1)} = \mathbf{X} + \boldsymbol{\alpha}, \quad \mathbf{x}^{(2)} = \mathbf{X} + \boldsymbol{\beta}, \quad l = \mathbf{x}^{(1)} - \mathbf{x}^{(2)}, \quad (1)$$

where $\mathbf{X} = (X, Y)$ is the displacement vector of the system and it is considered a reaction center of the spring's ends. We use the subscript $i, j = 1, 2$ for Cartesian components of vectors and tensors and superscript $\mu, \nu = 1, 2$ to identify the spheres. The vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are define as

$$\boldsymbol{\alpha} = \alpha \mathbf{n} = r_1 l \mathbf{n}, \quad \boldsymbol{\beta} = \beta \mathbf{n} = -r_2 l \mathbf{n}, \quad r_1 = R_2 / (R_1 + R_2), \quad r_2 = R_1 / (R_1 + R_2) \quad (2)$$

where \mathbf{n} is unit vector along l such that $\mathbf{n} \equiv l / l$ and $l = |l|$. The unit vectors $\mathbf{n}, \hat{\mathbf{n}}$ and the angle φ are given by

$$\mathbf{n} \equiv \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}, \quad \hat{\mathbf{n}} \equiv \begin{pmatrix} -\sin \varphi \\ \cos \varphi \end{pmatrix} \quad (3)$$

The oscillating dumbbell moves due to the fluid flow only; where the forces exerted by the spring on the spheres can be considered negligible, since the spring thickness is smaller in comparison with the radius of spheres. Hence,

$$R_1 \boldsymbol{\alpha} + R_2 \boldsymbol{\beta} = 0. \quad (4)$$

The length of spring l is changing periodically due to the oscillations of an active sphere which is given by

$$l = L + \epsilon \tilde{l}(\tau), \quad \tau = \omega t, \quad \epsilon = \text{const} \quad (5)$$

where L is a constant (averaged value) and \tilde{l} is a 2π -periodic function of τ with zero average value. Throughout the paper, a 'tilde' above a function denotes a 2π -periodic function with zero mean value. We consider a high-frequency

asymptotic problem of oscillating dumbbell in a viscous incompressible fluid which oscillates as a rigid body due to an external field. The rigid-body oscillations are prescribed as a two-dimensional transitional spatial displacement of fluid particles, $\tilde{\boldsymbol{\xi}}(\tau) = (\xi_1(\tau), \xi_2(\tau))$, through the related induced acceleration $\tilde{\boldsymbol{\xi}}_{tt} = \omega^2 \tilde{\boldsymbol{\xi}}_{\tau\tau}$, where the subscripts t, τ stand for related partial time derivatives. This problem is classified as an oscillating (non-inertial) system of reference, in which a fluid at infinity is termed in a state of rest. In this frame, according to Einstein's principle of equivalence, there is an equivalence of gravitational and inertial mass, and so there is no influence of gravity field on the micro-spheres such that the equations of fluid motion are standard [21]. Hence, the micro-spheres float by the buoyancy force

$$\mathbf{f}_b^{(\nu)} = -M^{(\nu)} \tilde{\mathbf{g}}(\tau), \quad (6)$$

which produces the potential energy of the spheres [22]

$$\Pi^{(\nu)} = -M^{(\nu)} \tilde{\mathbf{g}} \cdot \mathbf{x}^{(\nu)}, \quad \Pi_{sp} = k(l - L)^2 / 2 \quad (7)$$

where $M^{(\nu)}$ is equal to the difference in the mass of a sphere and that of displaced fluid and $\tilde{\mathbf{g}}(\tau)$ is equivalent to the induced acceleration

$$\tilde{\mathbf{g}}(\tau) = \omega^2 \tilde{\boldsymbol{\xi}}_{\tau\tau}. \quad (8)$$

The potential energy of the spring [11], is defined as

$$\Pi_{sp} = k(l - L)^2/2 \quad (9)$$

The total potential energy of the oscillating dumbbell is given by

$$\begin{aligned} \Pi &= \Pi^{(1)} + \Pi^{(2)} + \Pi_{sp} \\ &= M \tilde{\mathbf{g}} \cdot \mathbf{l} + k(l - L)^2/2 \end{aligned} \quad (10)$$

where $M \equiv M^{(1)} = -M^{(2)}$.

The micro-spheres experience Stokes force which is described by

$$\mathbf{f}_h^{(\nu)} = 6\pi\eta R_\nu(\mathbf{x}_t^{(\nu)} - \mathbf{u}_{\mu\nu}), \quad \mu \neq \nu \quad (11)$$

where η is a viscosity of the fluid and \mathbf{u} is the velocity of the flow due to the movement of other sphere [15] which is defined as

$$\mathbf{u}_{\mu\nu} = \frac{3R_\mu}{4} \left[\frac{\mathbf{x}_t^{(\mu)} + \mathbf{n}(\mathbf{x}_t^{(\mu)} \cdot \mathbf{n})}{l} \right], \quad (12)$$

such that \mathbf{u}_{12} represents the fluid disturbance generated by the movement of the first sphere on the second, while \mathbf{u}_{21} represents the fluid disturbance generated by the movement of the second sphere on the first. We assume that the fluid flow past the micro-spheres is solely described by the Stokes equation where all inertial terms are neglected, hence, kinetic energy $\mathcal{K} \equiv 0$.

The geometric configuration of the problem contains two characteristic lengths: the length of the spring L and the radius of the spheres R . Using the reference scales T , G , M and F ; characteristic time-scale, constant of gravity, mass and Stokes force where

$$R = (R_1 + R_2)/2, \quad T = 1/\omega, \quad G = \max |\tilde{\mathbf{g}}(\tau)|, \quad F = 6\pi\eta RL/T \quad (13)$$

The dimensionless variables (marked with asterisks) are chosen as

$$x = Lx^*, l = Ll^*, R_\nu = RR_\nu^*, t = Tt^*, g = Gg^*, \quad (14)$$

$$f_b^{(\nu)} = FF_b^*, \quad f_h^{(\nu)} = FF_h^{*(\nu)}. \quad (15)$$

Four independent small parameters of the problem are

$$\epsilon \equiv 1/\omega^*, \quad \delta \equiv 3R/4L, \quad \gamma \equiv MG/F, \quad K = kL/F; \quad \epsilon, \delta \ll 1, \quad (16)$$

where ω^* is the dimensionless frequency of the oscillation.

The substituting (13), (14) and (15) into (11), (6) and (10) yield the dimensionless form (all asterisks are omitted for brevity)

$$\mathbf{F}_h^{(\nu)} = -R_\nu \mathbf{x}_t^{(\nu)} + \delta R_\mu R_\nu \mathbb{S} \mathbf{x}_t^{(\mu)}, \quad (17)$$

$$\mathbf{F}_b = -\gamma \tilde{\mathbf{g}}, \quad (18)$$

$$\Pi = \gamma \tilde{\mathbf{g}} \cdot \mathbf{l} + K(l - 1)^2/2, \quad (19)$$

$$\mathbb{S} = S_{ij} \equiv \frac{1}{l}(\delta_{ij} + n_i n_j), \quad (20)$$

where δ_{ij} is a Kronecker delta; $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$.

The oscillating dumbbell represents a mechanical system which can be described by choosing generalized coordinates as

$$\mathbf{q} = (q_1, q_2, q_3, q_4) \equiv (X, Y, l, \varphi). \quad (21)$$

To describe the motion of the oscillating dumbbell, we use the Lagrangian function $\mathcal{L} = \mathcal{L}(\mathbf{q}, \mathbf{q}_t)$ which is defined as [11]

$$\mathcal{L}(\mathbf{q}, \mathbf{q}_t) = \mathcal{K} - \Pi = -\Pi = -\gamma \tilde{\mathbf{g}} \cdot \mathbf{l} - K(l-1)^2/2. \quad (22)$$

The Lagrange equations are

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \mathbf{q}_{mt}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}_m} = \mathbf{Q}_m, \quad \mathbf{Q}_m = \sum_{\nu=1}^2 \sum_{j=1}^2 F_{jh}^{(\nu)} \frac{\partial x_j^{(\nu)}}{\partial \mathbf{q}_m} \quad (23)$$

where $\mathbf{Q} = (Q_1, Q_2, Q_3, Q_4)$ is the generalized external viscous force exerted by the fluid on the oscillating dumbbell and subscript $m = 1, 2, 3, 4$ denoted the generalized coordinates.

Using the left part of (23) yield

$$Q_1 = 0, \quad Q_2 = 0, \quad Q_3 = \gamma \tilde{\mathbf{g}} \cdot \mathbf{l}_l + K(l-1), \quad Q_4 = \gamma \tilde{\mathbf{g}} \cdot \mathbf{l}_\varphi. \quad (24)$$

Using the right part of (23) yield

$$Q_1 = F_{1h}^{(1)} + F_{1h}^{(2)}, \quad Q_2 = F_{2h}^{(1)} + F_{2h}^{(2)}, \quad (25)$$

$$Q_3 = r_1 \cos \varphi F_{1h}^{(1)} + r_1 \sin \varphi F_{2h}^{(1)} - r_2 \cos \varphi F_{1h}^{(2)} - r_2 \sin \varphi F_{2h}^{(2)}, \quad (26)$$

$$Q_4 = -r_1 l \sin \varphi F_{1h}^{(1)} + r_1 l \cos \varphi F_{2h}^{(1)} + r_2 l \sin \varphi F_{1h}^{(2)} - r_2 l \cos \varphi F_{2h}^{(2)}. \quad (27)$$

$$-r_2 l \cos \varphi F_{2h}^{(2)}. \quad (28)$$

Combining (24), (25), (26) and (28) yield:

$$F_{1h}^{(1)} + F_{1h}^{(2)} = 0, \quad F_{2h}^{(1)} + F_{2h}^{(2)} = 0, \quad (29)$$

$$r_1 \cos \varphi F_{1h}^{(1)} + r_1 \sin \varphi F_{2h}^{(1)} - r_2 \cos \varphi F_{1h}^{(2)} - r_2 \sin \varphi F_{2h}^{(2)} = \gamma \tilde{\mathbf{g}} \cdot \mathbf{l}_l + K(l-1), \quad (30)$$

$$-r_1 l \sin \varphi F_{1h}^{(1)} + r_1 l \cos \varphi F_{2h}^{(1)} + r_2 l \sin \varphi F_{1h}^{(2)} - r_2 l \cos \varphi F_{2h}^{(2)} = \gamma \tilde{\mathbf{g}} \cdot \mathbf{l}_\varphi. \quad (31)$$

Integrating (30) with respect to l and then simple simplification yield

$$l r_1 \begin{pmatrix} F_{1h}^{(1)} \\ F_{2h}^{(1)} \end{pmatrix} \cdot \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix} - l r_2 \begin{pmatrix} F_{1h}^{(2)} \\ F_{2h}^{(2)} \end{pmatrix} \cdot \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix} = \gamma \tilde{\mathbf{g}} \cdot \mathbf{l} + K(l-1)^2/2, \quad (32)$$

which can be written in the form

$$\left(r_1 \mathbf{F}_h^{(1)} - r_2 \mathbf{F}_h^{(1)} \right) \cdot \mathbf{n} = \gamma \tilde{\mathbf{g}} \cdot \mathbf{n} + K(l-1)^2/2l. \quad (33)$$

Similarly, (31) takes the following form

$$\left(r_1 \mathbf{F}_h^{(1)} - r_2 \mathbf{F}_h^{(1)} \right) \cdot \hat{\mathbf{n}} = \gamma \tilde{\mathbf{g}} \cdot \hat{\mathbf{n}}. \quad (34)$$

For future use, we write (29), (33) and (34) as

$$\mathbf{F}_h^{(1)} + \mathbf{F}_h^{(2)} = 0 \quad (35)$$

$$\mathbf{F}^- \cdot \mathbf{n} = \gamma \tilde{\mathbf{g}} \cdot \mathbf{n} + K(l-1)^2/2l, \quad (36)$$

$$\mathbf{F}^- \cdot \hat{\mathbf{n}} = \gamma \tilde{\mathbf{g}} \cdot \hat{\mathbf{n}}, \quad (37)$$

where

$$\mathbf{F}^- = \left(R_2 \mathbf{F}_h^{(1)} - R_1 \mathbf{F}_h^{(2)} \right) / 2 R. \quad (38)$$

The use of (17) into (38) yield

$$\mathbf{F}^- = - \left(R_1 R_2 \mathbf{l}_t + \delta R_1 R_2 \mathbb{S} \left(\hat{R} \mathbf{X}_t + R_1 R_2 \mathbf{l}_t / R \right) \right) / 2 R. \quad (39)$$

The use of (17) into (35) and the use of (39) into (36) and (37) yield:

$$\mathbf{X}_t - \delta R_1 R_2 \mathbb{S} \left(\mathbf{X}_t - \hat{R} \mathbf{l}_t / 4 R \right) / R = 0 \quad (40)$$

$$\left(\mathbf{l}_t + \delta \mathbb{S} \left(\hat{R} \mathbf{X}_t + R_1 R_2 \mathbf{l}_t / R \right) \right) \cdot \hat{\mathbf{n}} = -2 R \gamma \tilde{\mathbf{g}} \cdot \hat{\mathbf{n}} / R_1 R_2 \quad (41)$$

$$\left(\mathbf{l}_t + \delta \mathbb{S} \left(\hat{R} \mathbf{X}_t + R_1 R_2 \mathbf{l}_t / R \right) \right) \cdot \mathbf{n} = -R \left(2 \gamma \tilde{\mathbf{g}} \cdot \mathbf{n} + K(l-1)^2 / l \right) / R_1 R_2 \quad (42)$$

where $\hat{R} = R_1 - R_2$. An approximate solution to the equations (40), (41) and (42) is presented in the next section using an asymptotic procedure containing the two-timing method.

$$\tau = \omega t, \quad s = t / \omega^\alpha; \quad \alpha > -1. \quad (43)$$

For different values of α there are multi paths for the asymptotic solution. In a rigorous asymptotic procedure $\omega \rightarrow \infty$, there is a unique path can be lead to valid solution. If such a limit brings valid an asymptotic result, then it is called a distinguished limit. In this paper, we have chosen $\alpha = 1$; hence,

$$\tau = \omega t, \quad s = t / \omega, \quad (44)$$

such that they lead to a valid asymptotic procedure of successive approximations [5, 26].

3. Two-timing Method and Asymptotic Procedure

3.1. Definition

Two-timing method constructs an asymptotic solution to the equation of motion (40), (41) and (42) by introducing two dependent time-scales s and τ as mutually independent variables, called slow and fast times, respectively, see Figure 2. This method converts (40), (41) and (42) from an ODE with the only independent variable t into a PDE with two independent variables s, τ .

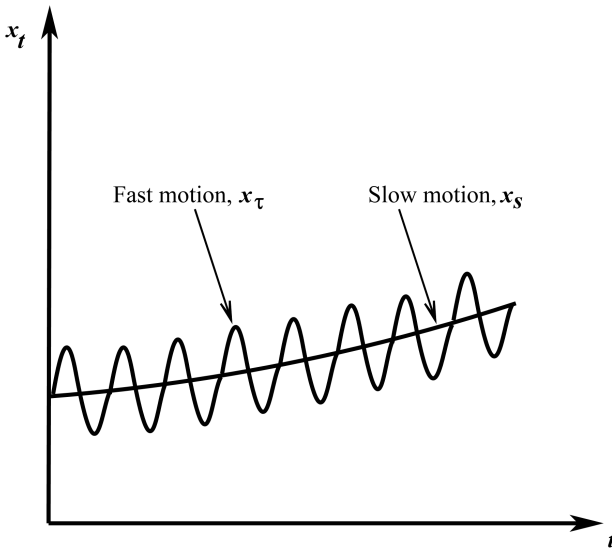


Figure 2. Diagram of the fast and slow motions.

The proper relations between s, τ and t is defined by [19]

3.2. Functions and Notation

For making further progress analytically, we introduce a few convenient notation. We assume that any dimensionless functions $f(s, \tau)$ (which could be scalar, vectorial, or tensorial one) [2], has the following properties:

1. Subscripts τ and s stand for related partial time derivatives.
2. $f = f(s, \tau)$ belongs to class $O(1)$ such that $f = O(1)$, and all partial s and τ derivatives of f (required for our consideration) are also $O(1)$.
3. (iii) We consider only *periodic function* in τ $\{f \in \mathcal{P} : f(s, \tau) = f(s, \tau + 2\pi)\}$, where s -dependence is not specified.
4. (iv) For arbitrary $f \in \mathcal{P}$, the *averaging-operation* is

$$\langle f \rangle \equiv \frac{1}{2\pi} \int_{\tau_0}^{\tau_0 + 2\pi} f(s, \tau) d\tau \equiv \bar{f}(s), \quad \forall \tau_0. \quad (45)$$

The *bar-function* $\bar{f} = \bar{f}(s)$ (or mean-function) does not depend on τ .

5. (v) The *tilde-function*, $\tilde{f}(\tau)$ (or purely oscillating function) represents a special case of \mathcal{P} -function with

zero average

$$\langle \tilde{f} \rangle = 0. \quad (46) \quad \left(\tilde{f}^\tau \right)_\tau \equiv \left(\tilde{f}_\tau \right)^\tau \equiv \tilde{f}. \quad (49)$$

A unique decomposition is valid

$$f = \bar{f} + \tilde{f}. \quad (47)$$

3.3. Successive Approximations

The choices $\tau = \omega t$ and $s = t/\omega$ lead to the following chain rule:

$$\begin{aligned} d/dt &= \omega \partial/\partial\tau + 1/\omega \partial/\partial s \\ &= \omega (\partial/\partial\tau + \epsilon^2 \partial/\partial s); \quad \epsilon = 1/\omega. \end{aligned} \quad (50)$$

6. (vi) A special notation \tilde{f}^τ is the *tilde-integration* of $\tilde{f}(\tau)$,

$$\tilde{f}^\tau = \int_0^\tau \tilde{f}(s, \sigma) d\sigma - \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^\mu \tilde{f}(s, \sigma) d\sigma \right) d\mu. \quad (48)$$

The *tilde-integration* is inverse to the τ -differentiation

However, we consider series expansions in ϵ , at most $O(\epsilon^2)$ and keep at most linear in δ terms. The unknown \mathbf{X} is written as series in ϵ

$$\mathbf{X}(s, \tau) = (\bar{\mathbf{X}}_0 + \tilde{\mathbf{X}}_0) + \epsilon (\bar{\mathbf{X}}_1 + \tilde{\mathbf{X}}_1) + \epsilon^2 (\bar{\mathbf{X}}_2 + \tilde{\mathbf{X}}_2) + O(\epsilon^2) \quad (51)$$

Differentiate (51) with respect to t gives

$$\mathbf{X}_t = (\bar{\mathbf{X}}_{0t} + \tilde{\mathbf{X}}_{0t}) + \epsilon (\bar{\mathbf{X}}_{1t} + \tilde{\mathbf{X}}_{1t}) + \epsilon^2 (\bar{\mathbf{X}}_{2t} + \tilde{\mathbf{X}}_{2t}) + O(\epsilon^2) \quad (52)$$

The use of (50) into (52) yield

$$\begin{aligned} \mathbf{X}_t = & \omega \left((\bar{\mathbf{X}}_{0\tau} + \epsilon^2 \bar{\mathbf{X}}_{0s}) + (\tilde{\mathbf{X}}_{0\tau} + \epsilon^2 \tilde{\mathbf{X}}_{0s}) \right) + \omega \epsilon \left((\bar{\mathbf{X}}_{1\tau} + \dots) + (\tilde{\mathbf{X}}_{1\tau} + \dots) \right) \\ & + \omega \epsilon^2 \left((\bar{\mathbf{X}}_{2\tau} + \dots) + (\tilde{\mathbf{X}}_{2\tau} + \dots) \right) + \dots \end{aligned} \quad (53)$$

and with similar expressions for \mathbf{l} . The two-timing method study only the class of solutions with

$$\tilde{\mathbf{X}}_0(s, \tau) = 0, \quad \tilde{\mathbf{l}}_0 = 0 \quad \text{while} \quad \bar{\mathbf{X}}_0(s, \tau) \neq 0 \quad \text{and} \quad \bar{\mathbf{l}}_0(s, \tau) \neq 0. \quad (54)$$

Physically, this constraint means that the amplitude of oscillations is small compared with the amplitude of the averaged solution such that when $\tilde{\mathbf{X}}_0(s, \tau) \neq 0$, the main term of velocity grows to infinity as $\omega \rightarrow \infty$.

Hence, (53) is given in the form

$$\mathbf{X}_t = \omega \left(\epsilon \tilde{\mathbf{X}}_{1\tau} + \epsilon^2 (\tilde{\mathbf{X}}_{2\tau} + \bar{\mathbf{X}}_{0s}) + O(\epsilon^2) \right). \quad (55)$$

Taylor series expansion of the tensor \mathbb{S}_{ij} (20) about $\mathbf{l} = \mathbf{l}_0$ takes the form

$$\mathbb{S}_{ij}(\mathbf{l}) = \mathbb{S}_{ij}(\mathbf{l}_0) + \mathbb{S}'_{ij}(\mathbf{l}_0) (\mathbf{l} - \mathbf{l}_0) + O(\epsilon^2), \quad (56)$$

where $\mathbf{l} - \mathbf{l}_0 = \epsilon \mathbf{l}_1 + \epsilon^2 \mathbf{l}_2 + O(\epsilon^2)$. Hence, (56) can be written as

$$\mathbb{S}_{ij}(\mathbf{l}) = \mathbb{S}_{ij}(\mathbf{l}_0) + \epsilon \mathbf{l}_{1k} \frac{\partial \mathbb{S}_{ij}(\mathbf{l}_0)}{\partial x_k} + O(\epsilon^2), \quad (57)$$

where $\mathbf{l}_0, \mathbf{l}_{1k}, \mathbb{S}_{ij}(\mathbf{l}_0)$ and $\partial \mathbb{S}_{ij}(\mathbf{l}_0)/\partial x_k$ are given by

$$\mathbf{l}_0 = \mathbf{x}_0^{(1)} - \mathbf{x}_0^{(2)}, \quad \mathbf{l}_{1k} = \mathbf{x}_{1k}^{(1)} - \mathbf{x}_{1k}^{(2)}, \quad (58)$$

$$\mathbb{S}_{ij}(\mathbf{l}_0) = \frac{1}{l_0} [\delta_{ij} + \mathbf{n}_{0i} \mathbf{n}_{0j}], \quad \mathbf{n}_0 = \mathbf{l}_0 / l_0; \quad l_0 = |\mathbf{l}_0|, \quad (59)$$

$$\frac{\partial \mathbb{S}_{ij}(\mathbf{l}_0)}{\partial x_k} = \frac{1}{l_0^2} (-\delta_{ij} \mathbf{n}_{0k} + \delta_{ik} \mathbf{n}_{0j} + \delta_{jk} \mathbf{n}_{0i} - 3 \mathbf{n}_{0i} \mathbf{n}_{0j} \mathbf{n}_{0k}). \quad (60)$$

For future use, we write the first and second terms in the right-hand side of (57) as

$$\mathbb{S}_0 = \mathbb{S}_{ij}(\mathbf{l}_0), \quad \mathbb{S}_1 = \mathbb{S}_{1ij}(\mathbf{l}_0) = l_{1k} \frac{\partial \mathbb{S}_{ij}(\mathbf{l}_0)}{\partial x_k}, \quad (61)$$

The use of (55) and (57) into the system (40), (41) and (42) yield

$$\begin{aligned} & \epsilon \widetilde{\mathbf{X}}_{1\tau} + \epsilon^2 (\widetilde{\mathbf{X}}_{2\tau} + \overline{\mathbf{X}}_{0s}) + O(\epsilon^2) - \delta \frac{R_1 R_2}{R} (\mathbb{S}_0 + \epsilon \mathbb{S}_1 + O(\epsilon^2)) (\epsilon \widetilde{\mathbf{X}}_{1\tau} \\ & + \epsilon^2 (\widetilde{\mathbf{X}}_{2\tau} + \overline{\mathbf{X}}_{0s}) + O(\epsilon^2)) - \frac{\hat{R}}{4R} (\epsilon \widetilde{\mathbf{l}}_{1\tau} + \epsilon^2 (\widetilde{\mathbf{l}}_{2\tau} + \overline{\mathbf{l}}_{0s}) + O(\epsilon^2)) = 0 \end{aligned} \quad (62)$$

$$\begin{aligned} & [\epsilon \widetilde{\mathbf{l}}_{1\tau} + \epsilon^2 (\widetilde{\mathbf{l}}_{2\tau} + \overline{\mathbf{l}}_{0s}) + O(\epsilon^2) + \delta (\mathbb{S}_0 + \epsilon \mathbb{S}_1 + O(\epsilon^2)) (\hat{R} (\epsilon \widetilde{\mathbf{X}}_{1\tau} + \epsilon^2 (\widetilde{\mathbf{X}}_{2\tau} + \overline{\mathbf{X}}_{0s}) + O(\epsilon^2)) \\ & + \frac{R_1 R_2}{R} (\epsilon \widetilde{\mathbf{l}}_{1\tau} + \epsilon^2 (\widetilde{\mathbf{l}}_{2\tau} + \overline{\mathbf{l}}_{0s}) + O(\epsilon^2)))] \cdot \hat{\mathbf{n}}_0 = \epsilon \frac{-2R\gamma}{R_1 R_2} \widetilde{\mathbf{g}} \cdot \hat{\mathbf{n}}_0 \end{aligned} \quad (63)$$

$$\begin{aligned} & [\epsilon \widetilde{\mathbf{l}}_{1\tau} + \epsilon^2 (\widetilde{\mathbf{l}}_{2\tau} + \overline{\mathbf{l}}_{0s}) + O(\epsilon^2) + \delta (\mathbb{S}_0 + \epsilon \mathbb{S}_1 + O(\epsilon^2)) (\hat{R} (\epsilon \widetilde{\mathbf{X}}_{1\tau} + \epsilon^2 (\widetilde{\mathbf{X}}_{2\tau} + \overline{\mathbf{X}}_{0s}) + O(\epsilon^2)) \\ & + \frac{R_1 R_2}{R} (\epsilon \widetilde{\mathbf{l}}_{1\tau} + \epsilon^2 (\widetilde{\mathbf{l}}_{2\tau} + \overline{\mathbf{l}}_{0s}) + O(\epsilon^2)))] \cdot \mathbf{n}_0 = \epsilon \frac{-R}{R_1 R_2} (2\gamma \widetilde{\mathbf{g}} \cdot \mathbf{n}_0 + K(l-1)^2/l) \end{aligned} \quad (64)$$

The successive approximations of (62), (63) and (64) lead to:
terms of order ϵ^0 give the identities $0=0$. Terms of order ϵ yield

$$\widetilde{X}_{1\tau} = \delta R_1 R_2 \mathbb{S}_0 (\widetilde{X}_{1\tau} - \hat{R} \widetilde{\mathbf{l}}_{1\tau} / 4R) / R, \quad (65)$$

$$[\widetilde{\mathbf{l}}_{1\tau} + \delta \mathbb{S}_0 (\hat{R} \widetilde{X}_{1\tau} + R_1 R_2 \widetilde{\mathbf{l}}_{1\tau} / R)] \cdot \hat{\mathbf{n}}_0 = \frac{-2R\gamma}{R_1 R_2} \widetilde{\mathbf{g}} \cdot \hat{\mathbf{n}}_0, \quad (66)$$

$$[\widetilde{\mathbf{l}}_{1\tau} + \delta \mathbb{S}_0 (\hat{R} \widetilde{X}_{1\tau} + R_1 R_2 \widetilde{\mathbf{l}}_{1\tau} / R)] \cdot \mathbf{n}_0 = \frac{-2R}{R_1 R_2} (\gamma \widetilde{\mathbf{g}} \cdot \mathbf{n}_0 + K(l-1)^2/2l). \quad (67)$$

The use of (65) into (66) and (67), keep at most linear in δ , yield

$$[\widetilde{\mathbf{l}}_{1\tau} + \delta \mathbb{S}_0 R_1 R_2 \widetilde{\mathbf{l}}_{1\tau} / R] \cdot \hat{\mathbf{n}}_0 = \frac{-2R\gamma}{R_1 R_2} \widetilde{\mathbf{g}} \cdot \hat{\mathbf{n}}_0, \quad (68)$$

$$[\widetilde{\mathbf{l}}_{1\tau} + \delta \mathbb{S}_0 R_1 R_2 \widetilde{\mathbf{l}}_{1\tau} / R] \cdot \mathbf{n}_0 = \frac{-2R}{R_1 R_2} (\gamma \widetilde{\mathbf{g}} \cdot \mathbf{n}_0 + K(l-1)^2/2l). \quad (69)$$

Terms of order ϵ^2 yield

$$\widetilde{\mathbf{X}}_{2\tau} + \overline{\mathbf{X}}_{0s} - \delta \frac{R_1 R_2}{R} \mathbb{S}_0 (\widetilde{\mathbf{X}}_{2\tau} + \overline{\mathbf{X}}_{0s} - \frac{\hat{R}}{4R} (\widetilde{\mathbf{l}}_{2\tau} + \overline{\mathbf{l}}_{0s})) - \delta \frac{R_1 R_2}{R} \mathbb{S}_1 (\widetilde{\mathbf{X}}_{1\tau} - \frac{\hat{R}}{4R} \widetilde{\mathbf{l}}_{1\tau}) = 0 \quad (70)$$

$$[\tilde{\mathbf{l}}_{2\tau} + \bar{\mathbf{l}}_{0s} + \delta \mathbb{S}_0(\hat{R}(\tilde{\mathbf{X}}_{2\tau} + \bar{\mathbf{X}}_{0s}) + \frac{R_1 R_2}{R}(\tilde{\mathbf{l}}_{2\tau} + \bar{\mathbf{l}}_{0s})) + \delta \mathbb{S}_1(\hat{R}\tilde{\mathbf{X}}_{1\tau} + \frac{R_1 R_2}{R}\tilde{\mathbf{l}}_{1\tau})] \cdot \hat{\mathbf{n}}_0 = 0 \quad (71)$$

$$[\tilde{\mathbf{l}}_{2\tau} + \bar{\mathbf{l}}_{0s} + \delta \mathbb{S}_0(\hat{R}(\tilde{\mathbf{X}}_{2\tau} + \bar{\mathbf{X}}_{0s}) + \frac{R_1 R_2}{R}(\tilde{\mathbf{l}}_{2\tau} + \bar{\mathbf{l}}_{0s})) + \delta \mathbb{S}_1(\hat{R}\tilde{\mathbf{X}}_{1\tau} + \frac{R_1 R_2}{R}\tilde{\mathbf{l}}_{1\tau})] \cdot \mathbf{n}_0 = 0 \quad (72)$$

The use of averaging procedure (45) and (46) into (70), (71) and (72) yield

$$\bar{\mathbf{X}}_{0s} - \delta \frac{R_1 R_2}{R} \mathbb{S}_0 \bar{\mathbf{X}}_{0s} - \delta \frac{R_1 R_2 \hat{R}}{4R^2} \mathbb{S}_0 \bar{\mathbf{l}}_{0s} + \delta \frac{R_1 R_2 \hat{R}}{4R^2} \langle \mathbb{S}_1 \tilde{\mathbf{l}}_{1\tau} \rangle = 0 \quad (73)$$

$$\left(\bar{\mathbf{l}}_{0s} + \delta \hat{R} \mathbb{S}_0 \bar{\mathbf{X}}_{0s} + \delta \frac{R_1 R_2}{R} \mathbb{S}_0 \bar{\mathbf{l}}_{0s} + \delta \hat{R} \langle \mathbb{S}_1 \tilde{\mathbf{X}}_{1\tau} \rangle + \delta \frac{R_1 R_2}{R} \langle \mathbb{S}_1 \tilde{\mathbf{l}}_{1\tau} \rangle \right) \cdot \hat{\mathbf{n}}_0 = 0 \quad (74)$$

$$\left(\bar{\mathbf{l}}_{0s} + \delta \hat{R} \mathbb{S}_0 \bar{\mathbf{X}}_{0s} + \delta \frac{R_1 R_2}{R} \mathbb{S}_0 \bar{\mathbf{l}}_{0s} + \delta \hat{R} \langle \mathbb{S}_1 \tilde{\mathbf{X}}_{1\tau} \rangle + \delta \frac{R_1 R_2}{R} \langle \mathbb{S}_1 \tilde{\mathbf{l}}_{1\tau} \rangle \right) \cdot \mathbf{n}_0 = 0 \quad (75)$$

The equation (73) can be written as

$$\bar{\mathbf{X}}_{0s} = \delta \frac{R_1 R_2}{R} \mathbb{S}_0 \bar{\mathbf{X}}_{0s} + \delta \frac{R_1 R_2 \hat{R}}{4R^2} \mathbb{S}_0 \bar{\mathbf{l}}_{0s} - \delta \frac{R_1 R_2 \hat{R}}{4R^2} \langle \mathbb{S}_1 \tilde{\mathbf{l}}_{1\tau} \rangle \quad (76)$$

The use of (65) and (76) into (74) and (75), keep at most linear in δ , yield

$$\bar{\mathbf{l}}_{0s} = -\delta \frac{R_1 R_2}{R} \langle \mathbb{S}_1 \tilde{\mathbf{l}}_{1\tau} \rangle \quad (82)$$

$$\left(\bar{\mathbf{l}}_{0s} + \delta \frac{R_1 R_2}{R} \mathbb{S}_0 \bar{\mathbf{l}}_{0s} + \delta \frac{R_1 R_2}{R} \langle \mathbb{S}_1 \tilde{\mathbf{l}}_{1\tau} \rangle \right) \cdot \hat{\mathbf{n}}_0 = 0 \quad (77)$$

The substituting (82) into (76), keep at most linear in δ , yield

$$\left(\bar{\mathbf{l}}_{0s} + \delta \frac{R_1 R_2}{R} \mathbb{S}_0 \bar{\mathbf{l}}_{0s} + \delta \frac{R_1 R_2}{R} \langle \mathbb{S}_1 \tilde{\mathbf{l}}_{1\tau} \rangle \right) \cdot \mathbf{n}_0 = 0 \quad (78)$$

$$\bar{\mathbf{X}}_{0s} = -B^{-1} \delta \frac{R_1 R_2 \hat{R}}{4R^2} \langle \mathbb{S}_1 \tilde{\mathbf{l}}_{1\tau} \rangle \quad (83)$$

where the matrix $B = I - \delta R_1 R_2 \mathbb{S}_0 / R$. The inverse matrix of B is given by

$$\bar{\mathbf{l}}_{0s} + \delta \frac{R_1 R_2}{R} \mathbb{S}_0 \bar{\mathbf{l}}_{0s} + \delta \frac{R_1 R_2}{R} \langle \mathbb{S}_1 \tilde{\mathbf{l}}_{1\tau} \rangle = 0 \quad (79)$$

$$B_{ik}^{-1} = (1 + \delta \frac{R_1 R_2}{Rl}) \delta_{ik} + \delta \frac{R_1 R_2}{Rl} \mathbf{n}_{0i} \mathbf{n}_{0k} \quad (84)$$

which can be written as

The use of (84) into (83), keep at most linear in δ , yield

$$\bar{\mathbf{l}}_{0s} = -A^{-1} \delta \frac{R_1 R_2}{R} \langle \mathbb{S}_1 \tilde{\mathbf{l}}_{1\tau} \rangle \quad (80)$$

$$\bar{\mathbf{X}}_{0s} = -\delta \frac{R_1 R_2 \hat{R}}{4R^2} \langle \mathbb{S}_1 \tilde{\mathbf{l}}_{1\tau} \rangle \quad (85)$$

where the matrix $A = I + \delta R_1 R_2 \mathbb{S}_0 / R$ such that I is the identity matrix. Using simple algebraic transformation, the inverse matrix of A is given by

Expressions (85) still contain unknown functions $\langle \mathbb{S}_1 \tilde{\mathbf{l}}_{1\tau} \rangle$, which can be determined from (60) and (61). Hence,

$$A_{ik}^{-1} = (1 - \delta \frac{R_1 R_2}{Rl}) \delta_{ik} - \delta \frac{R_1 R_2}{Rl} \mathbf{n}_{0i} \mathbf{n}_{0k} \quad (81)$$

$$\bar{\mathbf{X}}_{0s} = -\delta \frac{\hat{R} \gamma^2}{l^2 R_1 R_2} \langle \tilde{\mathbf{g}}^T \tilde{\mathbf{g}} \rangle \mathbf{n}_0 \quad (86)$$

The use of (81) into (80), keep at most linear in δ , yield

which represents the main result of this paper.

Equation (86) shows that the oscillating dumbbell moves with a constant velocity in the fixed direction \mathbf{n}_0 . It is clear

that $\bar{\mathbf{X}}_{0s}$ is inversely proportional to the length of the spring l and it does not depend on the spring stiffness k ; and in the limit as length $l \rightarrow \infty$, then $\bar{\mathbf{X}}_{0s} \rightarrow 0$.

Integrating of (86) lead to the displacement vector

$$\bar{\mathbf{X}}_0 = -\delta U \hat{\mathbf{n}}_0 + \mathbf{C} \quad \text{or} \quad (\bar{\mathbf{X}}_0 - \mathbf{C})^2 = \delta^2 U^2 \quad (87)$$

where $U = \hat{R} \gamma^2 \langle \tilde{\mathbf{g}}^\tau \tilde{\mathbf{g}} \rangle / l^2 R_1 R_2$ and \mathbf{C} is a vectorial constant of integration. The equality (87) represents that $\bar{\mathbf{X}}_0$ changes along a circular path of radius δU .

It is interesting that the average velocity $\bar{\mathbf{X}}_{0s}$ can be arranged by appropriate choice of the induced acceleration, $\tilde{\mathbf{g}}(\tau)$.

3.4. An Illustrative Example

Let us consider a particular example

$$\tilde{\mathbf{g}}(\tau) = \begin{pmatrix} a \sin \tau \\ b \cos \tau \end{pmatrix} \quad (88)$$

where a and b are constants. The use of the integral (48) gives

$$\tilde{\mathbf{g}}^\tau(\tau) = \begin{pmatrix} -a \cos \tau \\ b \sin \tau \end{pmatrix}, \quad (89)$$

and thus

$$\tilde{\mathbf{g}}_i^\tau \tilde{\mathbf{g}}_k = \begin{pmatrix} -a^2 \sin \tau \cos \tau & -ab \cos^2 \tau \\ ab \sin^2 \tau & b^2 \sin \tau \cos \tau \end{pmatrix}. \quad (90)$$

Calculating $\langle \tilde{\mathbf{g}}_i^\tau \tilde{\mathbf{g}}_k \rangle$ using the definition of average (45) yields

$$\langle \tilde{\mathbf{g}}_i^\tau \tilde{\mathbf{g}}_k \rangle = \frac{ab}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (91)$$

Substituting (91) into (86) gives

$$\bar{\mathbf{X}}_{0s} = -\delta \frac{\hat{R} \gamma^2 a b}{2 R_1 R_2 l^2} \mathbf{n}_0 \quad (92)$$

Integrating of (92) leads to

$$\bar{\mathbf{X}}_0 = -\delta \lambda \hat{\mathbf{n}}_0 + \mathbf{C} \quad (93)$$

where $\lambda = U a b / 2$.

In this paper, we consider motions (40) with large ω where all functions and its derivatives belongs to class $O(1)$. Hence,

$$\mathbf{C} = (a - \delta P, b - \delta P), \quad \text{where } p = \hat{R} \gamma^2 a b / 2 R_1 R_2 l^2. \quad (94)$$

The use of (58) and (59) lead to

$$\hat{\mathbf{n}}_0 = (-\sin \phi, \cos \phi) \quad \text{where } \phi = \varphi_0. \quad (95)$$

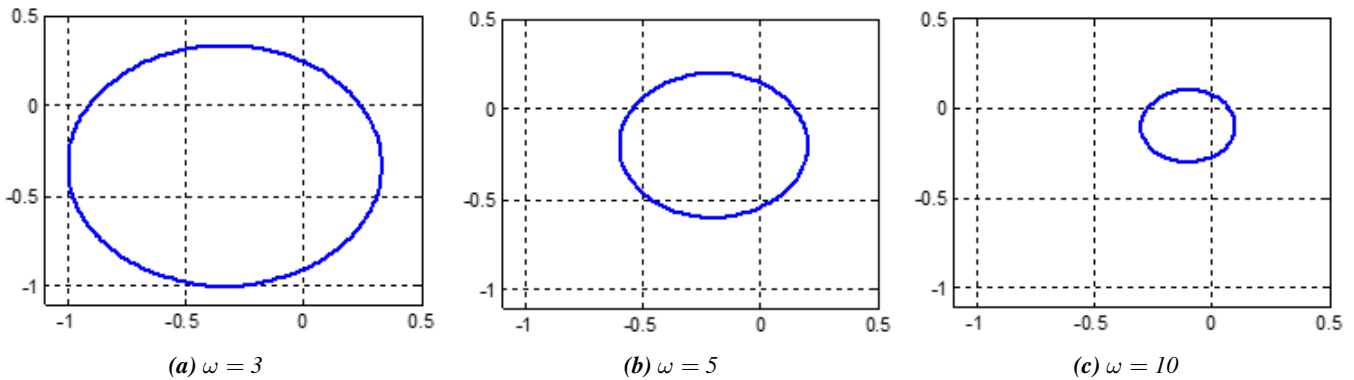


Figure 3. Trajectory of the oscillating dumbbell for different ω .

The slow time scale $s = t/\omega$ implies that in order to obtain dimensionless trajectory of the oscillating dumbbell we have to multiply $\bar{\mathbf{X}}_0$ by $1/\omega$.

The trajectory of oscillating dumbbell for different frequencies is plotted in Figure 3. It is clearly shown that the circular path of the oscillating dumbbell is decreased as the frequency increased which a small oscillation leads to a large motion of the surrounding fluid. This result agrees with the classical studies of the oscillatory motion of particle suspensions at low Reynolds numbers [12, 13].

4. Conclusion

In this paper, we investigated analytically the dynamics of two micro-spheres, that are elastically connected by a spring, one of them is magnetized and driven by an external oscillator field. We constructed an asymptotic procedure with the dimensionless inverse frequency $\epsilon = 1/\omega$ and derived the average velocity of the system using the two-timing method. Our choice of slow time $s = \epsilon t$ and fast time $\tau = t/\epsilon$

lead to a result that agrees with the experimental studies of an oscillating sphere in a viscous fluid [7, 12]. It shows the strength and analytical simplicity to describe the oscillatory motion of the dumbbell at low Reynolds number. It is worth noting that the average velocities of the oscillating dumbbell is given in the most general form which could be applicable to study a full three-dimensional problem.

Conflicts of Interest

The authors declare no conflicts of interest.

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