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D-Effect Algebra Can Be Made into a D-Total Algebra

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Abstract: In this paper we prove that every *D-effect algebra* (E, Δ, 0, 1) can be made into a *D-total algebra* (E, $\boxed{\Delta}$, ¬, 1) in such a way that two elements are compatible in (E, Δ, 0, 1) if and only if they commute in(E, $\boxed{\Delta}$, ¬, 1) where x Δ y =(x' + y')'.

Keywords: D-Basic Algebra, Weak D-Basic Algebra, Antitone Involution, D-Effect Algebra, D-Total Algebra

1. Introduction

Chajda, I., Halas, R., Kuhr, J. (2009) introduced similar results for general effect algebras in the context of commutative directoids; they proved that every effect algebra (E,+, 0, 1) can be made into a total algebra (E, ⊕,¬, 0) in such a way that two elements are compatible in (E,+, 0, 1) if and only if they commute in (E, ⊕,¬, 0).

In the present paper we introduce and study the concept of a D-basic algebra, this being an algebra (A,$\boxed{\Delta}$, ¬, 1) of type (2, 1, 0) with the property that the underlying poset (A, ≤), defined by x ≥ y if and only if ¬x $\boxed{\Delta}$ y = ¬1, is a bounded lattice and, for each a ∊ A, the mapping (x → ¬ x $\boxed{\Delta}$ a) is an antitone involution on the principal ideal (a]= {x ∈ A | a ≥ x}. The name '*D-basic algebra*' is used because these algebras capture common features of many known structures such as Boolean algebras, orthomodular lattices, lattice D-effect algebras. we have special attention to lattice D-effect algebras, which were originally defined as partial algebras (E, Δ, 0, 1), but where the presence of the meet operation allowe one to replace partial Δ by D-total $\boxed{\Delta}$. The intent of the present paper is to establish similar results for D-effect algebras in the context of commutative directoids; we prove that every *D-effect algebra* (E, Δ, 0, 1) can be made into a D-total algebra (E, $\boxed{\Delta}$, ¬, 1).

We first recall several relevant notions.

Definition 1.1 [8]: A *commutative directoids* is a commutative, idempo-tent groupoid (A,.) satisfying the equation x. ((x. y). z) = (x. y). z.

Remark 1.2 For instance, every semilattice is a commutative directoid. It can easily be seen that the stipulation

$$x \geq y \text{ if and only if } x.\, y = y \tag{1}$$

defines a partial order on A such that, for every x, y ∊ A, x. y is a lower bound of {x, y}. Thus the poset (A, ≤) is downwards directed. *Conversely*, we may associate a commutative directoid to an arbitrary downwards directed set by letting x. y = y. x be some lower bound of {x, y}, such that whenever x, y are comparable, x. y = y. x is the least of x, y.

Like in semilattices, we could define the dual order by x ≥ y if and only if x. y = x, in which case x. y is an upper bound of {x, y}. But we shall be concerned with the partial order given by (1). Accordingly, we shall write ⊓ instead of. in order to emphasize that x ⊓ y is less than or equal to x, y.

Definition 1.3 [3] An *antitone involution* on a poset (P, ≤) is a mapping β: P → P such that, for all x, y ∊ P,

$$x \geq y \Rightarrow \beta(y) \geq \beta(x), \tag{2}$$

$$\beta(\beta(x)) = x. \tag{3}$$

By a commutative directoid with sectional antitone involutions we shall mean a system (A, ⊓, $(\beta_a)_{a\in A}$, 0, 1) where (A, ⊓) is a commutative directoid with a least element 0 and a greatest element 1, and every section (a] is equipped with an antitone involution β_a.

In particular, if (A, ⊓) is a semilattice, then the underlying poset is a lattice in which $\beta_1(\beta_1(x) \sqcap \beta_1(y))$ is the supremum of {x, y}, and hence we may say that (A, ⊓, $(\beta_a)_{a\in A}$, 0, 1) is a lattice with sectional antitone involutions.

2. Weak D-Basic Algebra

Definition 2.1 A *Weak D-basic algebra* is an algebra $(A, \boxed{\triangle}, \neg, 1)$ of type (2, 1, 0) satisfying the following identities and quasi-identity (where 1 is an abbreviation for $\neg 0$):

(DW1) $x \boxed{\triangle} 1 = x$,

(DW2) $\neg\neg x = x$,

(DW3) $\neg(\neg x \boxed{\triangle} y) \boxed{\triangle} y = \neg(\neg y \boxed{\triangle} x) \boxed{\triangle} x$,

(DW4) $x \boxed{\triangle} (\neg(\neg(\neg(x \boxed{\triangle} y) \boxed{\triangle} y) \boxed{\triangle} z) \boxed{\triangle} z) = 0$,

(DW5) $\neg x \boxed{\triangle} (y \boxed{\triangle} x) = 0$,

(DW6) $\neg x \boxed{\triangle} y = 0 \ \& \ \neg y \boxed{\triangle} z = 0 \Rightarrow \neg(\neg z \boxed{\triangle} x) \boxed{\triangle} (\neg y \boxed{\triangle} x) = 0$.

These algebras contains the equations $x \boxed{\triangle} 0 = 0 = 0 \boxed{\triangle} x$. Specifically, if $(A, \boxed{\triangle}, \neg, 0)$ is a *weak D-basic algebra* and if we put

$$x \sqcap y = \neg(\neg x \boxed{\triangle} y) \boxed{\triangle} y,$$

then $(A, \sqcap)$ is a commutative directoid with a least element 0 and a greatest element 1, such that the underlying order $\leq$ is given by:

$$x \geq y \text{ if and only if } x \sqcap y = y \text{ if and only if } \neg x \boxed{\triangle} y = 0 \quad (4)$$

For each $a \in A$, $(x \to \neg x \boxed{\triangle} a)$ is an antitone involution on $(a] = \{x \in A \mid a \geq x\}$. Conversely, if $(A, \sqcap, (\beta_a)_{a\in A}, 0, 1)$ is a commutative directoid with sectional antitone involutions, then we can define $\boxed{\triangle}$ and $\neg$ as $x \boxed{\triangle} y = \beta_y(\beta_1(x) \sqcap y)$ and $\neg x = \beta_1(x)$, respectively, and $(A, \boxed{\triangle}, \neg, 0)$ becomes a *weak D-basic algebra* in which $x \boxed{\triangle} y = \neg(\neg x \boxed{\triangle} y) \boxed{\triangle} y$ and $\beta_a(x) = \neg x \boxed{\triangle} a$. In every *Weak D-basic algebra*, in addition to the 'meat-like' operation $\sqcap$, we can introduce the dual 'join-like' operation $\sqcup$ by

$$x \sqcup y = \neg(\neg x \sqcap \neg y). \quad (5)$$

Then we have $x \geq y$ if and only if $x \sqcup y = x$, and the structure $(A, \sqcup, \sqcap)$ is a λ-lattice in the sense of [9], i.e., both $(A, \sqcap)$ and $(A, \sqcup)$ are commutative directoids and the absorption laws

$x \sqcap (x \sqcup y) = x = x \sqcup (x \sqcap y)$ are satisfied.

Definition 2.2 A *D-basic algebra* is an algebra $(A, \boxed{\triangle}, \neg, 0)$ of type (2, 1, 0) satisfying the identities (again, $1 = \neg 0$)

(DB1) $x \boxed{\triangle} 1 = x$,

(DB2) $\neg\neg x = x$,

(DB3) $\neg(\neg x \boxed{\triangle} y) \boxed{\triangle} y = \neg(\neg y \boxed{\triangle} x) \boxed{\triangle} x$,

(DB4) $\neg(\neg(\neg(x \boxed{\triangle} y) \boxed{\triangle} y) \boxed{\triangle} z) \boxed{\triangle} (x \boxed{\triangle} z) = 0$.

Originally, we required $x \boxed{\triangle} 0 = 0 = 0 \boxed{\triangle} x$.

Every *D-basic algebra* is a *Weak D-basic algebra*, and the above assignment between *weak D-basic algebras* and commutative directoids with sectional antitone involutions, restricted to *D-basic algebras*, furnishes a one-to-one correspondence between *D-basic algebras* and lattices with sectional antitone involutions. In other words, a weak *D-basic algebra* $(A, \boxed{\triangle}, \neg, 0)$ is a *D-basic algebra* if and only if $(A, \sqcap, (\beta_a)_{a\in A}, 0, 1)$ is a lattice with sectional antitone involutions.

The axioms (DW3) - (DW6) may be rewritten in terms of $\geq$ and $\sqcap$ as follows:

(DW3') $x \sqcap y = y \sqcap x$,

(DW4') $x \geq (x \sqcap y) \sqcap z$,

(DW5') $x \geq y \boxed{\triangle} x$,

(DW6') $x \geq y \ \& \ y \geq z \Rightarrow \neg z \boxed{\triangle} x \geq \neg y \boxed{\triangle} x$.

Moreover, in every weak D-basic algebra we have

$$x \boxed{\triangle} 0 = 0 = 0 \boxed{\triangle} x, \quad (6)$$

$$1 \boxed{\triangle} x = x, \quad (7)$$

$$\neg(x \sqcap y) \boxed{\triangle} y = \neg x \boxed{\triangle} y. \quad (8)$$

Indeed, $0 \boxed{\triangle} x = \neg 1 \boxed{\triangle} (x \boxed{\triangle} 1) = 0$ by (DW1) and (DW5), so $x \boxed{\triangle} 0 = x \boxed{\triangle} (0 \boxed{\triangle} \neg x) = 0$ by (DW2) and (DW5). Further, $1 \boxed{\triangle} x = \neg 0 \boxed{\triangle} x = (\neg 1 \boxed{\triangle} x) \boxed{\triangle} x = (\neg x \boxed{\triangle} 1) \boxed{\triangle} 1 = \neg\neg x = x$ and $\neg(x \sqcap y) \boxed{\triangle} y = \neg(\neg(\neg x \boxed{\triangle} y) \boxed{\triangle} y) \boxed{\triangle} y = (\neg x \boxed{\triangle} y) \sqcap y = \neg x \boxed{\triangle} y$.

Proposition 2.3 A *D-algebra* $A = (A, \boxed{\triangle}, \neg, 0)$ satisfying (DW1)–(DW4) is a *Weak D-basic algebra* if and only if it satisfies the identity

$$\neg(\neg((x \sqcap y) \sqcap z) \boxed{\triangle} x) \boxed{\triangle} (\neg y \boxed{\triangle} x) = 0. \quad (9)$$

Proof. Let A be a *Weak D-basic algebra*. We have $x \geq x \sqcap y$ and $x \sqcap y \geq (x \sqcap y) \sqcap z$, i.e., $\neg x \boxed{\triangle} (x \sqcap y) = 0$ and $\neg(x \sqcap y) \boxed{\triangle} ((x \sqcap y) \sqcap z) = 0$, which yields

$$\neg(\neg((x \sqcap y) \sqcap z) \boxed{\triangle} x) \boxed{\triangle} (\neg(x \sqcap y) \boxed{\triangle} x) = 0$$

by (DW6). But $\neg(x \sqcap y) \boxed{\triangle} x = \neg(y \sqcap x) \boxed{\triangle} x = \neg y \boxed{\triangle} x$ by (8), hence A fulfils (9). *Conversely*, assume that A satisfies (DW1)–(DW4) and (9). We first observe that $x \sqcap 1 = \neg(\neg x \boxed{\triangle} 1) \boxed{\triangle} 1 = \neg\neg x = x$, whence $0 \boxed{\triangle} x = \neg 1 \boxed{\triangle} ((1 \sqcap 1) \sqcap x) = 0$ by (DW4') and (8). This yields $1 \boxed{\triangle} x = \neg 0 \boxed{\triangle} x = \neg\neg 1 \boxed{\triangle} x) \boxed{\triangle} x = 1 \sqcap x = x$ and so $x \sqcap 0 = \neg(\neg 0 \boxed{\triangle} x) \boxed{\triangle} x = \neg x \boxed{\triangle} x = \neg x \boxed{\triangle} ((x \sqcap 1) \sqcap 1) = 0$. Further, $\neg x \boxed{\triangle} y = 0$ entails $x \sqcap y = \neg(\neg x \boxed{\triangle} y) \boxed{\triangle} y = \neg 0 \boxed{\triangle} y = 1 \boxed{\triangle} y = y$.

Now, if we substitute 0 and $\neg y$ for z and y, respectively, by (9) we obtain $0 = \neg(\neg((x \sqcap \neg y) \sqcap 0) \boxed{\triangle} x) \boxed{\triangle} (\neg\neg y \boxed{\triangle} x) = \neg(\neg 0 \boxed{\triangle} x) \boxed{\triangle} (y \boxed{\triangle} x) = \neg x \boxed{\triangle} (y \boxed{\triangle} x)$, which is (DW5). Finally, if $\neg x \boxed{\triangle} y = 0$ and $\neg y \boxed{\triangle} z = 0$, then $(x \sqcap y) \sqcap z = z$ and by (9) we have

$$0 = \neg(\neg((x \sqcap y) \sqcap z) \boxed{\triangle} x) \boxed{\triangle} (\neg y \boxed{\triangle} x)$$

$$= \neg(\neg z \boxed{\triangle} x) \boxed{\triangle} (\neg y \boxed{\triangle} x),$$

which settles (DW6). Thus A is a *Weak D-basic algebra*.

Another central concept is that of a *D-effect algebra*. We have a *D-effect algebra* is a system $(E, \Delta, 0, 1)$ where 0, 1 are distinguished elements of E and Δ is a partial binary operation on E such that

(DEA1) $x \Delta y = y \Delta x$ if one side is defined,

(DEA2) $(x \Delta y) \Delta z = x \Delta (y \Delta z)$ if one side is defined,

(DEA3) for every $x \in E$ there exists a unique $x' \in E$ with $x' \Delta x = 0$,

(DEA4) if $x \Delta 0$ is defined then $x = 1$.

Every *D-effect algebra* bears a natural partial order given by $x \geq y$ if and only if $y = x \Delta z$ for some $z \in E$.

The poset $(E, \leq)$ is bounded, 0 is the bottom element and 1 is the top element. If, moreover, $(E, \leq)$ is a lattice, then $(E, \Delta, 0, 1)$ is called a *lattice D-effect algebra*. In every *D-effect algebra*, a partial binary operation ∇ can be defined as follows:

$x \nabla y$ exists and equals z if and only if $z = (x' \Delta y)'$.

(Thus $x \nabla y$ is defined if and only if $y \geq x$.) The system $(E, \leq, \nabla, 0, 1)$ so obtained is called a *D-poset*.

When doing calculations, the following properties of *D-effect algebras* and *D-posets* will be useful:

Remark 2.4

(1) $x \Delta 1 = x$, $x \nabla 1 = x$, $x \nabla x = 1$, $0 \nabla x = x'$;

(2) $x \geq y$ if and only if $y' \geq x'$;

(3) $x \Delta y$ is defined if and only if $x \geq y'$ if and only if $y \geq x'$; in this case, $x \Delta y = (x' \nabla y)' = (y' \nabla x)'$;

(4) If $x \Delta y$ is defined, then so is $x_1 \Delta y_1$ for all $x_1 \geq x$ and $y_1 \geq y$;

(5) $x \Delta y = z$ if and only if $x' = y \Delta z'$ if and only if $y' = x \Delta z'$;

(6) If $x \geq y$, then $x \nabla y = (x' \Delta y)' \geq x$ and $x \nabla (x \nabla y) = y$;

(7) $x \geq y \geq z$ implies $z \nabla y \geq z \nabla x$ and $(z \nabla x) \nabla (z \nabla y) = y \nabla x$; in particular $x' \nabla y' = y \nabla x$;

(8) $x \geq y \geq z$ implies $y \nabla x \geq z \nabla x$ and $(z \nabla x) \nabla (y \nabla x) = z \nabla y$.

3. The Relation Between D-Effect Algebras and Weak D-Basic Algebras

Theorem 3.1 Let $A = (A, \boxed{\Delta}, \neg, 0)$ be a *Weak D-basic algebra*. Define the partial binary relation Δ on A as follows: $x \Delta y$ is defined if and only if $x \geq \neg y$, and in this case $x \Delta y = x \boxed{\Delta} y$. Then $\mathcal{E}(A) = (A, \Delta, 0, 1)$ is a D-effect algebra if and only if A satisfies the quasi-identity

$$x \geq \neg y \;\&\; x \boxed{\Delta} y \geq \neg z \Rightarrow (x \boxed{\Delta} y) \boxed{\Delta} z = x \boxed{\Delta} (z \boxed{\Delta} y). \quad (10)$$

Moreover, over *Weak D-basic algebras*, (10) is equivalent to the identity

$$(x \boxed{\Delta} y) \boxed{\Delta} (\neg (x \boxed{\Delta} y) \sqcup z) = (x \sqcup \neg y) \boxed{\Delta} ((\neg (x \boxed{\Delta} y) \sqcup z) \boxed{\Delta} y). \quad (11)$$

Proof. Suppose that $\mathcal{E}(A)$ is a *D-effect algebra*. If $x \geq \neg y$ and $x \boxed{\Delta} y \geq \neg z$, then $x \Delta y$ and $(x \Delta y) \Delta z$ exist, hence $y \Delta z$ and $x \Delta (y \Delta z)$ also exist and $(x \boxed{\Delta} y) \boxed{\Delta} z = (x \Delta y) \Delta z = x \Delta(y \Delta z) = x \Delta (z \Delta y) = x \boxed{\Delta} (z \boxed{\Delta} y)$.

Conversely, let A satisfy (10). We shall verify that $\mathcal{E}(A)$ is a *D-algebra*:

(EA1) Assume that $a \Delta b$ is defined. Since $a \geq \neg b$ if and only if $b \geq \neg a$, it follows that $b \Delta a$ is defined, too. By (10) we have $(1 \geq \neg a \;\&\; 1 \boxed{\Delta} a \geq \neg b) \Rightarrow a \boxed{\Delta} b = b \boxed{\Delta} a$, so that $a \geq \neg b$ entails $a \Delta b = b \Delta a$.

(EA2) Let $(a \Delta b) \Delta c$ be defined, i.e., $a \geq \neg b$ and $a \Delta b = a \boxed{\Delta} b \geq \neg c$. Since $b \geq a \boxed{\Delta} b \geq \neg c$, also $b \Delta c = c \Delta b$ exists. Further, by (W6), $b \geq a \boxed{\Delta} b \geq \neg c$ implies $c \boxed{\Delta} b = \neg\neg c \boxed{\Delta} b \geq \neg (a \boxed{\Delta} b) \boxed{\Delta} b = \neg a \sqcap b = \neg a$, so $a \Delta (c \Delta b) = a \Delta (b \Delta c)$ is defined. Analogously, if $a \Delta (b \Delta c)$ exists, then so does $(a \Delta b) \Delta c$. By (10) we have $(a \Delta b) \Delta c = (a \boxed{\Delta} b) \boxed{\Delta} c = a \boxed{\Delta} (c \boxed{\Delta} b) = a \Delta (c \Delta b) = a \Delta (b \Delta c)$.

(EA3) Clearly, we have $\neg a \Delta a = 0$. If $b \Delta a = 0$, then $b \geq \neg a$ since $b \Delta a$ is defined, and $\neg b \geq a$ (i.e., $b \leq \neg a$) since $b \boxed{\Delta} a = 0$. Thus $b = \neg a$.

(EA4) Finally, if $a \Delta 0$ is defined, then $a \geq \neg 0 = 1$, so $a = 1$.

It remains to show that (10) and (11) are equivalent over *weak D-basic algebras*.

In any *weak D-basic algebra*, $x \sqcup \neg y \geq \neg y$ and, using (8),

$$(x \sqcup \neg y) \boxed{\Delta} y = \neg (\neg x \sqcap y) \boxed{\Delta} y = x \boxed{\Delta} y \geq (x \boxed{\Delta} y) \sqcap \neg z = \neg (\neg (x \boxed{\Delta} y) \sqcup z).$$

Therefore, if (10) holds, then we have

$$(x \boxed{\Delta} y) \boxed{\Delta} (\neg(x \boxed{\Delta} y) \sqcup z) = ((x \sqcup \neg y) \boxed{\Delta} y) \boxed{\Delta} (\neg (x \boxed{\Delta} y) \sqcup z) = (x \sqcup \neg y) \boxed{\Delta} ((\neg(x \boxed{\Delta} y) \sqcup z) \boxed{\Delta} y),$$

which is (11). On the other hand, (11) evidently implies (10).

Corollary 3.2 Let $A = (A, \boxed{\Delta}, \neg, 0)$ be a *D-basic algebra* and let $\mathcal{E}(A) = (A, \Delta, 0, 1)$ be as in Theorem 3.1. Then $\mathcal{E}(A)$ is a *lattice D-effect algebra* if and only if A satisfies the quasi-identity (10).

However, as the following example shows, this is not true for *Weak D-basic algebras* since many different *Weak D-basic algebras* can determine the same *D-effect algebra*.

Example 3.3 Let $(A, \leq)$ be the poset

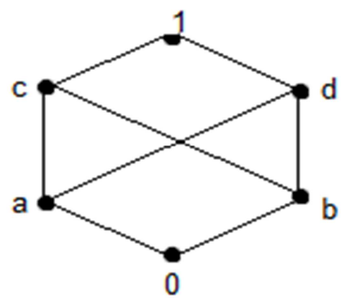

Figure 1. Weak D-basic algebra.

and let the sections (1] = A, (c] and (d] be equipped with the following antitone involutions:

β_1: $1 \to 0$, $0 \to 1$, $d \to a$, $a \to d$, $b \to c$, $c \to b$;

β_c: $c \to 0$, $0 \to c$, $b \to a$, $a \to b$,

β_d: $d \to 0$, $0 \to d$, $b \to b$, $a \to a$,

the other sections admit unique antitone involutions. There are three possible ways in which we can associate a commutative directoid to $(A, \geq)$, and consequently, there are three *weak D-basic algebras* with the underlying poset $(A, \leq)$:

In (*table 1.*) for $c \sqcap_1 d = a$ we get $A_1 = (A, \boxed{\Delta}_1, \neg, 1)$ where

Table 1. *Show the first way to weak D-basic algebra when $c \sqcap_1 d = a$.*

$\boxed{\Delta}_1$	0	a	b	c	d	1	¬
0	0	0	0	0	0	0	1
a	0	0	0	a	0	a	d
b	0	0	0	0	b	b	c
c	0	a	0	a	b	c	b
d	0	0	b	b	a	d	a
1	0	b	b	c	d	1	0

In (*table 2.*) for $c \sqcap_2 d = b$ we get $A_2 = (A, \boxed{\Delta}_2, \neg, 1)$ where

Table 2. *Show the second way to weak D-basic algebra when c* $\sqcap_2$ *d = b.*

$\boxed{\Delta}_2$	**0**	**A**	**B**	**c**	**d**	**1**	**¬**
0	0	0	0	0	0	0	1
a	0	0	0	b	0	a	d
b	0	0	0	0	a	b	c
c	0	a	0	a	b	c	b
d	0	0	B	b	a	d	a
1	0	a	B	c	d	1	0

In (*table 3.*) for c $\sqcap_3$ d = 0 we get A_3 = (A, $\boxed{\Delta}_3$, ¬, 1) where

Table 3. *Show the third way to weak D-basic algebra when c* $\sqcap_3$ *d = 0.*

$\boxed{\Delta}_3$	**0**	**a**	**B**	**c**	**d**	**1**	**¬**
0	0	0	0	0	0	0	1
a	0	0	0	C	0	a	d
b	0	0	0	0	d	b	c
c	0	a	0	a	b	c	b
d	0	0	b	b	a	d	a
1	0	a	b	c	d	1	0

All these *Weak D-basic algebras* induce the same *D-effect algebra* Ɛ (A_1) = Ɛ (A_2) = Ɛ (A_3) = (A, Δ, 0, 1) that clear in (*table 4.*) where

Table 4. *Show that the three ways in Weak D-basic algebras induce the same D-effect algebra.*

Δ	**0**	**A**	**b**	**c**	**d**	**1**	**¬**
0	.	.	.	.	.	0	1
a	.	.	.	.	0	a	d
b	.	.	.	0	.	b	c
c	.	.	0	a	b	c	b
d	.	0	.	b	a	d	a
1	0	a	b	c	d	1	0

Where for any x, y ∊ A we have x Δ y =. means x Δ y does not exist. Let E = (E, Δ, 0, 1) be a *D-effect algebra*. Since the underlying poset (E, ≤) is bounded, it can be organized into a commutative directoid (E, ⊓). We shall simply say that the pair (E, ⊓) is a *D-effect algebra* with an associated commutative directoid.

Theorem 3.4 Let (E, ⊓) be a *D-effect algebra* E = (E, Δ, 0, 1) with an associated commutative directoid. Define

x $\boxed{\Delta}$ y = (x' ⊓ y)' Δ y and ¬ x = x'.

Then DB (E, ⊓) = (E, $\boxed{\Delta}$, ¬, 1) is a *Weak D-basic algebra* satisfying (10). *Moreover*, Ɛ (DB (E, ⊓)), the *D-effect algebra* assigned to DB (E, ⊓) by *Theorem 3.1*, is just E.

Proof. First, we prove that for each a ∊ E, β_a: x → x' Δ a is an anititone involution on (a]. For all x ∊ (a], we have x' Δ a is defined since x' ≥ a', and x' Δ a ≤ a. Thus β_a is well defined. We also have β_a (β_a (x)) = (x' Δ a)' Δ a = x because, by (v), (x' Δ a)' Δ a = x if and only if (x' Δ a)'' = a Δ x'. Finally, if a ≥ x ≥ y, then β_a(y) = y' Δ a ≥ x' Δ a = β_a(x), proving that β_a is an anititone involution.

We know that if we put x$\boxed{\Delta}_1$y = β_y (β_1 (x) ⊓ y) and ¬ x = x', then (E, $\boxed{\Delta}_1$,¬, 1) becomes a *Weak D-basic algebra*. But x$\boxed{\Delta}_1$y = β_y (β_1 (x) ⊓ y) = ((x' Δ 1) ⊓ y)' Δ y = (x' ⊓ y)' Δ y = x $\boxed{\Delta}$ y. Therefore, DB (E, ⊓) is a *Weak D-basic algebra*.

Now, we prove that DB (E, ⊓) satisfies the quasi-identity (10). It is obvious that whenever x Δ y is defined in E (i.e., x ≥ y' = ¬y).

Then x $\boxed{\Delta}$ y = (x' ⊓ y)' Δ y = x Δ y. Hence if x ≥ ¬y and x $\boxed{\Delta}$ y ≥ ¬z, then (x $\boxed{\Delta}$ y) $\boxed{\Delta}$ z = (x Δ y) Δ z = x Δ (y Δ z) = x Δ (z Δ y) = x $\boxed{\Delta}$ (z $\boxed{\Delta}$ y), which settles (10). The last assertion is clear.

Example 3.5 Let E be the *D-effect algebra* we have obtained in Example 3.3. If we put c $\sqcap_1$ d = a then DB (E, $\sqcap_1$) is just the *Weak D-basic algebra* A_1 from Example 3.3. Analogously, if c $\sqcap_2$ d = b then DB (E, $\sqcap_2$) = A_2, and for c $\sqcap_3$ d = 0 we have DB (E, $\sqcap_3$) = A_3.

There is a one-to-one correspondence between *Weak D-basic algebras* satisfying (10) (respectively, (11)) and pairs (E, ⊓) where E = (E, Δ, 0, 1) is a *D-effect algebra* with an associated commutative directoid (E, ⊓). Namely, the assignment

A → (Ɛ (A), ⊓),

where Ɛ (A) is as in Theorem 3.1 and x ⊓ y = ¬ (¬ x $\boxed{\Delta}$ y) $\boxed{\Delta}$ y, is a bijection the inverse of which is

(E, ⊓) → DB (E, ⊓),

where DB (E, ⊓) is defined in Theorem 3.4.

Let E = (E, Δ, 0, 1) be a D-effect algebra. When constructing (E, ⊓), we so far have not taken care of existing infima we only required that x ⊓ y = y ⊓ x is min{x, y} provided x, y are comparable. Of course, this means that DB (E, ⊓) need not be a *D-basic algebra* even though E is a *lattice D-effect algebra*. The situation can be improved if we define ⊓ in such a way that the following condition holds:

If inf{x, y} exists, then x ⊓ y = y ⊓ x = inf{x, y}. (12)

Corollary 3.6 Let (E, ⊓) be a *D-effect algebra* with an associated commutative directoid that satisfies the condition (12). Then DB (E, ⊓) is a *weak D-basic algebra*, and if E is *a lattice D-effect algebra*, then DB (E, ⊓) is a *D-basic algebra*.

Proof. By *Theorem 3.4*, DB (E, ⊓) is a *Weak D-basic algebra*. Further, we know that DB (E, ⊓) is a *D-basic algebra* if and only if the corresponding commutative directoid with sectional antitone involutions (E, ⊓, $(\beta_a)_{a∊A}$, 0, 1), where β_a (x) = x' Δ a for x ≤ a, is actually a lattice with sectional antitone involutions, which is the case when (E, ⊓) is a semilattice. Hence, if E is a *lattice D-effect algebra*, then, owing to (12), (E, ⊓) is a semilattice, and it follows that DB (E, ⊓) is a *D-basic algebra*.

4. Compatibility in D-Effect Algebra

Difination 4.1 We shall say that two elements x, y in a *D-effect algebra* E are said to be *compatible* (in symbols x ↔ y) if there exist u, v ∊ E such that u ≥ x, y ≥ v and x ∇ u = v ∇ y. This is equivalent to the existence of z ∊ E with x, y ≥ z, z ∇ x ≥ y and z ∇ y ≥ x. But z ∇ x ≥ y ≥ z implies z ∇ y ≥ z ∇ (z ∇ x) = x, and conversely, z ∇ y ≥ x ≥ z entails z ∇ x ≥ z ∇ (z ∇ y) = y. Therefore,

x ↔ y if and only if there is z such that x, y ≥ z and z ∇ x ≥ y. (13)

In general we have:

Proposition 4.2 Let $(E, \sqcap)$ and DB $(E, \sqcap)$ be as in *Theorem 3.4*. For every $x, y \in E$, if $x \boxed{\Delta} y = y \boxed{\Delta} x$, then $x \leftrightarrow y$.

Proof. Let $z = x \boxed{\Delta} y = y \boxed{\Delta} x$, i.e., $(x' \sqcap y)' \Delta y = (y' \sqcap x)' \Delta x$. Then $x, y \geq z$ and $z \nabla x = ((y' \sqcap x)' \Delta x) \nabla x = (y' \sqcap x)' \geq y$, so that $x \leftrightarrow y$.

The reverse implication fails to be true. Let E be the *D-effect algebra* from Examples 3.3 and 3.5. It can easily be seen that every two elements are compatible, while $\boxed{\Delta}_i$ in A2 and A3 is not commutative (for instance, $a \leftrightarrow c$, but $a \boxed{\Delta}_i c \neq c \boxed{\Delta}_i a$ for $i = 2, 3$).

In order to overcome this disadvantage, we define the 'meet-like' operation $\sqcap$ in a *D-effect algebra* $E = (E, \Delta, 0, 1)$ in the following way:

$$\text{If } x \leftrightarrow y, \text{ then } x \sqcap y = y \sqcap x = z \text{ where } z \leq x, y \text{ and } z \nabla x \geq y. \text{ At the same time, } x' \sqcap y' = y' \sqcap x' = (x \nabla (z \nabla y))'. \tag{14}$$

We must show that the condition is correct.

If $z \leq x, y$ and $z \nabla x \geq y$, then $x \nabla (z \nabla y) = y \nabla (z \nabla x)$. Indeed, if we put $w = x \nabla (z \nabla y)$ then $w \Delta (z \nabla y) = x$ and $w \Delta (z \nabla y) \Delta (z \nabla x) = x \Delta (z \nabla x) = z$, whence $w \Delta (z \nabla x) = z \nabla (z \nabla y) = y$. So $w = y \nabla (z \nabla x)$ and $(x \nabla (z \nabla y))' = (y \nabla (z \nabla x))'$.

Obviously, $w' \leq x', y'$.

We have $w' \nabla x' = x \nabla w = x \nabla (x \nabla (z \nabla y)) = z \nabla y = y' \nabla z' \geq y'$. by using remark 2.4(ii)

Finally, $(x' \nabla (w' \nabla y'))' = z$. Indeed, $(x' \nabla (w' \nabla y'))' = x \Delta (w' \nabla y') = x \Delta (y \nabla w)$, thus $(x' \nabla (w' \nabla y'))' = z$ if and only if $y \nabla w = z \nabla x$, which is true since $y \nabla w = y \nabla (y \nabla (z \nabla x)) = z \nabla x$.

Also observe that the condition (14) is 'compatible' with (12) in the sense that we may take $z = \inf \{x, y\}$ whenever $\inf \{x, y\}$ exists. More precisely, if $x \sqcap y = \inf \{x, y\}$ and $x \leftrightarrow y$, then $(x \sqcap y) \nabla x \geq y$, and if, in addition, also $\inf \{x', y'\}$ exists, then $\inf \{x', y'\} = x' \sqcap y' = (x \nabla ((x \sqcap y) \nabla y))'$.

Indeed, $x \leftrightarrow y$ yields the existence of z with $z \leq x, y$ and $z \nabla x \geq y$. Since $x \sqcap y = \inf\{x, y\}$, we have $z \leq x \sqcap y \leq x$, whence $y \leq z \nabla x \geq (x \sqcap y) \nabla x$.

Further, assume that $\inf\{x', y'\}$ exists (equivalently, $\sup\{x, y\}$ exists). We have to show that $x \nabla((x \sqcap y) \nabla y) = \sup\{x, y\}$. Let $w = x \nabla ((x \sqcap y) \nabla y)$. By what we have established above we know that $w = y \nabla ((x \sqcap y) \nabla x)$ and $w \geq x, y$. Thus $w \geq \sup\{x, y\} \geq x, y$, whence

$$\sup \{x, y\} \nabla w \geq x \nabla w = x \nabla (x \nabla ((x \sqcap y) \nabla y)) = (x \sqcap y) \nabla y,$$

$$\sup \{x, y\} \nabla w \geq y \nabla w = y \nabla (y \nabla ((x \sqcap y) \nabla x)) = (x \sqcap y) \nabla x.$$

It is known that $\sup \{(x \sqcap y)\nabla y, (x \sqcap y)\nabla x\} = 1$ if $x \sqcap y$ is $\inf\{x, y\}$, and consequently, the above inequalities imply $\sup\{x, y\}\nabla w = 1$, so that $\sup\{x, y\} = w$ as desired.

Summarizing, we have proved that in every *D-effect algebra* $E = (E, \Delta, 0, 1)$, the operation $\sqcap$ can always be defined in such a way that it obeys the requirements of the condition (14). The next result says, that x, y are compatible in E if and only if x, y commute in DB $(E, \sqcap) = (E, \boxed{\Delta}, \neg, 0)$.

Theorem 4.3 Let $(E, \sqcap)$ be a *D-effect algebra* with an associated commutative directoid satisfying condition (14). Then DB $(E, \sqcap)$ is a *Weak D-basic algebra* such that, for all $x, y \in E$, the following are equivalent:

(i) $x \leftrightarrow y$,

(ii) $(x \sqcap y) \nabla y = x \nabla (x \sqcup y)$,

(iii) $x \boxed{\Delta} y = y \boxed{\Delta} x$.

Proof. (i) $\Leftrightarrow$ (ii): Let $x \leftrightarrow y$. Then $(x \sqcap y) \nabla y \geq x$ and $x \sqcup y = (x' \sqcap y')' = x \nabla ((x \sqcap y) \nabla y)$, whence $x \nabla (x \sqcup y) = x \nabla(x\nabla((x \sqcap y)\nabla y)) = (x \sqcap y) \nabla y$. *Conversely*, if $(x \sqcap y) \nabla y = x \nabla (x \sqcup y)$, then certainly $x \leftrightarrow y$ because $x \sqcap y \leq x, y$ and $(x \sqcap y) \nabla y \geq x$.

(i) $\Leftrightarrow$ (iii): By Proposition 4.2 we know that $x \boxed{\Delta} y = y \boxed{\Delta} x$ implies $x \leftrightarrow y$. Hence, Then alsox' $\leftrightarrow$ y, which means $(x' \sqcap y) \nabla y = x' \nabla (x' \sqcup y)$ by (ii). We then have $(x \boxed{\Delta} y)' = ((x' \sqcap y)' \Delta y)' = (x' \sqcap y) \nabla y = x' \nabla(x' \sqcup y) = (x' \sqcup y)' \nabla x = (y' \sqcap x) \nabla x = ((y' \sqcap x)' \Delta x)' = (y \boxed{\Delta} x)'$, thus $x \boxed{\Delta} y = y \boxed{\Delta} x$.

Definition 4.4 By a *block* of a *Weak D-basic algebra* $(A, \boxed{\Delta}, \neg, 0)$ we mean a subset DB of A which is maximal with respect to the property that $x \boxed{\Delta} y = y \boxed{\Delta} x$ for all $x, y \in$ DB. It is evident that every element of A is contained in a block.

Theorem 4.5 Let $(E, \sqcap)$ be a *D-effect algebra* with an associated commutative directoid satisfying the condition (14). Assume that for all $x, y, z \in E$, if $x \leftrightarrow y$, $x \leftrightarrow z$ and $y \Delta z$ is defined, then $x \leftrightarrow y \Delta z$. Then a block DB of DB $(E, \sqcap)$ is a subalgebra of DB $(E, \sqcap)$ if and only if $x \sqcap y \in$ DB for all $x, y \in$ DB.

Proof. Let DB be a block of DB $(E, \sqcap)$. In view of Theorem 4.3, DB is a maximal set of pairwise compatible elements (i.e., DB is a maximal subset of E such that $x \leftrightarrow y$ for all $x, y \in$ DB). Since $x \leftrightarrow 0$ and $x \leftrightarrow 1$ for each $x \in E$ (this follows at once from (13)), it is plain that $0, 1 \in$ DB. Also, $x \leftrightarrow y$ if and only if $x' \leftrightarrow y$, hence $x \in$ DB if and only if $\neg x = x' \in$ DB.

Suppose DB is closed under $\sqcap$. If $x, y \in$ DB, then also $(x' \sqcap y)' \in$ DB. Thus $(x' \sqcap y)' \leftrightarrow z$ and $y \leftrightarrow z$ for every $z \in$ DB, whence $x \boxed{\Delta} y = (x' \sqcap y)' \Delta y \in$ DB, proving that DB is a subalgebra of DB $(E, \sqcap)$. *Conversely*, if DB is a subalgebra of DB $(E, \sqcap)$, then it is automatically closed with respect to $\sqcap$ because $x \sqcap y = \neg (\neg x \boxed{\Delta} y) \boxed{\Delta} y$. The condition that $x \leftrightarrow y$ and $x \leftrightarrow z$ together yield $x \leftrightarrow y \Delta z$ (if $y \Delta z$ exists) holds in *lattice D-effect algebras*, however, the next example shows that the operation Δ in *Theorem 4.5* cannot be omitted.

Example 4.6 Let E be the set consisting of the following pairs of integers: $0 = (0, 0)$, $a = (1, 2)$, $b = (1, 1)$, $c = (2, 1)$, $d = (2, 3)$, $e = (3, 3)$, $f = (3, 2)$, $g = (2, 2)$ and $1 = (4, 4)$. If we equip E with Δ defined as the restriction to E of $\boxed{\Delta}$, then $E = (E, \Delta, 0)$ becomes a *D-effect algebra*. The underlying poset of E is as follows (notice that $(x, y) \geq (u, v)$ if and only if $(x, y) = (u, v)$, or $x > u$ & $y > v$) where

$(x, y) \boxed{\Delta} (u, v) = ((x, y)' + (u, v)')'$ and + is the usual point in addition and $((x, y)' = (4 - x, 4 - y))$:

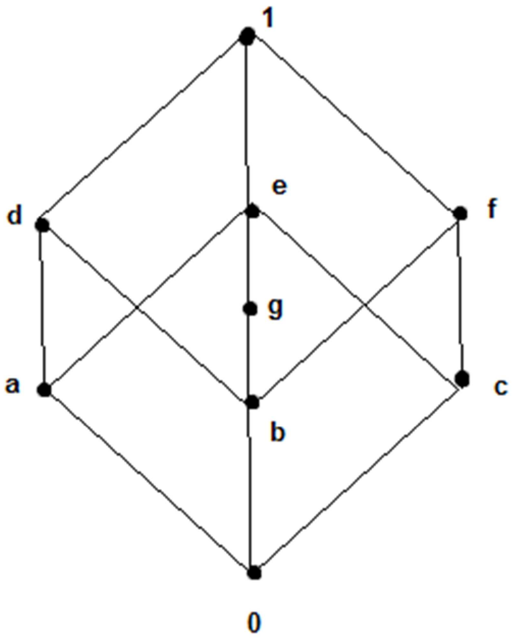

Figure 2. D-effect algebra.

Table 5. *Show that E \ {g} is a block of the assigned weak D-basic algebra DB (E, ⊓),*

$\boxed{\Delta}$	**0**	**A**	**b**	**C**	**d**	**e**	**f**	**g**	**1**	**¬**
0	0	0	0	0	0	0	0	0	0	1
a	0	a	0	0	a	a	0	b	a	f
b	0	0	0	0	b	0	b	0	b	e
c	0	0	0	C	0	c	c	b	c	d
d	0	A	b	0	d	a	b	g	d	c
e	0	A	0	C	a	g	c	b	e	b
f	0	0	b	C	b	c	f	b	f	a
g	0	A	0	C	a	b	b	0	g	G
1	0	A	b	C	d	e	f	g	1	0

It is obvious that a ↔ e, but a is not compatible with g = e Δ e. Indeed, the only common lower bound of a, g is 0, and 0 ∇ a = f ≱ g as well as 0 ∇ g = g ≱ a, thus a ↮ g by (13).

In accordance with the conditions (12) and (14), we put f ⊓ e = c (= f Δ e) and e ⊓ d = a (= e Δ d); in the other cases ⊓ coincides with inf. A direct inspection shows that E \ {g} is a block of the assigned *weak D-basic algebra* DB(E, ⊓) (see the table below (*table 5.*) which is closed under ⊓, but it is not closed under $\boxed{\Delta}$ as e Δ e = g. On the other hand, {0, b, e, g, 1} is both a block and a subalgebra of DB (E, ⊓).

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