

# Minimum Time Problem for Co-operative Parabolic System with Control-State Constraints

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**To cite this article:**

Mohammed Shehata. Minimum Time Problem for Co-operative Parabolic System with Control-State Constraints. *Mathematical Modelling and Applications*. Vol. 1, No. 1, 2016, pp. 1-7. doi: 10.11648/j.mma.20160101.11

**Received:** September 6, 2015; **Accepted:** September 18, 2015; **Published:** October 12, 2016

**Abstract:** In this paper, the minimum time problem for differential systems of parabolic type with distributed control and control - state constraints are considered. The minimum time problem is replaced by an equivalent one with fixed time and the necessary optimality conditions of time-optimal control are obtained by using the generalized Dubovitskii-Milyutin Theorem (see [1]).

**Keywords:** Time-Optimal Control Problem, Parabolic System, Dubovitskii - Milyutin Method, Canonical Approximations, Optimality Conditions

## 1. Introduction

The most widely studies of the problems in the mathematical theory of control are the "time optimal" control problems. The simple version, is the following optimization problem :

find  $(u, y) \in C([0, \infty); U \times Y) :$

$$\left. \begin{aligned} 0 < t \rightarrow \min \\ u(t) \in U_{ad} \\ y(t) \in Y_{ad} \end{aligned} \right\}$$

where  $U_{ad}$ ,  $Y_{ad}$  are spaces of admissible control and states respectively. In order to explain the results we have in mind, it is convenient to consider the abstract form of the Dubovitskii-Milyutin theorem. At the first we recall some definitions of conical approximations [2], [3] and cones of the same sense or of the opposite sense [3]. Let  $A$  be a set contained in a Banach space  $X$  and  $F : X \rightarrow \mathfrak{R}$  be a given functional.

*Definition 1 A set*

$$\begin{aligned} TC(A, x^0) &:= \{h \in X : \exists \varepsilon_0 > 0, \forall \varepsilon \in (0, \varepsilon_0), \\ &\exists r(\varepsilon) \in X; x^0 + \varepsilon h + r(\varepsilon) \in A\}, \end{aligned}$$

where  $\frac{r(s)}{s} \rightarrow 0$  as  $\varepsilon \rightarrow 0$  is called tangent cone to the set

$A$  at the point  $x^0 \in A$ .

*Definition 2 A set*

$$AC(A, x^0) := \{h \in X : \exists \varepsilon_0 > 0, \exists U(h),$$

where  $U(h)$  is a neighborhood of  $h$ , is called the admissible cone to the set

$A$  at the point  $x^0 \in A$ .

*Definition 3 A set*

$$FC(A, x^0) := \{h \in X : \exists \varepsilon_0 > 0, \exists U(h), \forall \varepsilon \in (0, \varepsilon_0),$$

is called the cone of decreases of the functional  $F$  at the point

$x^0 \in A$ .

All the cones defined above are cones with vertices at the origin. The cones  $AC(A, x^0)$ ,  $FC(A, x^0)$ , are open while the  $TC(A, x^0)$ , is closed. If  $\text{int } A = \Phi$ , then  $AC(A, x^0)$  does not exist. Moreover, if  $A_1, \dots, A_n \in X$ ,  $x^0 \in \bigcap_{i=1}^n A_i$  then

$$\begin{aligned} \bigcap_{i=1}^n TC(A_i, x^0) &\supset TC(\bigcap_{i=1}^n A_i, x^0), \\ \bigcap_{i=1}^n AC(A_i, x^0) &\supset AC(\bigcap_{i=1}^n A_i, x^0). \end{aligned}$$

If the cones  $TC(A, x^0)$ ,  $AC(A, x^0)$  and  $FC(A, x^0)$  are convex, then they are called regular cones and we denote them by  $RTC(A, x^0)$ ,  $RAC(A, x^0)$  and  $RFC(A, x^0)$  respectively.

Let  $C_i, i=1,2,\dots,n$  be a system of cones and  $B_M$  be a ball with center 0 and radius  $M > 0$  in the space  $X$ .

**Definition 4** The cones  $C_i, i=1,2,\dots,n$  are of the same sense if  $\forall M > 0, \exists M_1, \dots, M_n > 0$  so that  $\forall x \in B_M \cap \bigcap_{i=1}^n C_i, x = \sum_{i=1}^n x_i, x_i \in C_i, i=1,2,\dots,n$ , we have  $x_i \in B_{M_i} \cap C_i, i=1,2,\dots,n$  ( or equivalently the inequality  $\|x\| \leq M$  implies the inequality  $\|x_i\| \leq M_i, i=1,2,\dots,n$ ).

**Definition 5** The cones  $C_i, i=1,2,\dots,n$  are of the opposite sense if  $\exists (x_1, \dots, x_n) \neq (0, \dots, 0), C_i, i=1,2,\dots,n$  so that  $0 = \sum_{i=1}^n x_i$ .

**Remark 1** Form definitions 4 and 5 it follows that the set of cones of the same sense is disjoint with the set of cones of the opposite sense. If a certain subsystem of cones is of the opposite sense, then the whole system is also of the opposite sense.

**Definition 6** Let  $K$  be a cone in  $X$ . The adjoint cone  $K^*$  of  $K$  is defined as

$$K^* := \{f \in X^*; f(x) \geq 0, \forall x \in K\}$$

where  $X^*$  denotes the dual space of  $X$

Let  $X$  be Banach space,  $Q_k \subset X, \text{int } Q_k \neq \Phi, k=1,2,\dots,p$  represent inequality constraints,  $Q_k \subset X, k=p+1,\dots,m$  represent equality constraints and  $I: X \rightarrow \mathfrak{R}$  is given functional.

**Theorem 1 ([1])** Assume that :

- (i)  $I: X \rightarrow \mathfrak{R}$  is convex and continuous,
- (ii) the cones  $Q_k \subset X, k=1,\dots,m$  are convex,
- (iii) the cones  $[RTC(Q_k, x^0)]^*, k=p+1,\dots,m$  are either of the same sense or of the opposite sense, then  $x^0$  is a solution of the problem

$$\min \{I(x), x \in \bigcap_{k=1}^p Q_k\}$$

if and only if the following equation ( Euler -Lagrange equation ) must hold:

$$f_0 + \sum_{k=1}^m f_k = 0,$$

where  $f_0 \in [RFC(I, x^0)]^*$ ,

$f_k \in [RAC(Q_k, x^0)]^* k=1,\dots,p$  and  $f_k \in [RTC(Q_k, x^0)]^* k=p+1,\dots,m$  with not all functionals equal to zero simultaneously.

The above generalization of the Dubovitskii- Milyutin theorem is based on the definitions of the regular cones RTC, RFC, RAC and cones of the same sense and of the opposite sense which are introduced above. But for the purpose of our problems we are going to use the following sufficient

condition for two cones to be of the same sense .

**Theorem 2 ([3])** Let  $C_1$  be a cone of the form  $C_1 = \{(x_1, y_1) \in X \times Y : x_1 = My_1\}, C_2 = X \times \hat{C}_2$ , where  $\hat{C}_2$  is a cone in  $Y$  ( $X, Y$  - normed spaces ). If the operator  $M$  is linear and continuous, then

$$C_1^* = \{(x_1^*, y_1^*) \in X \times Y : y_1^* = -M^* x_1^*\}$$

$$C_2^* = \{(0, y_2^*) \in X \times Y : y_2^* \in \hat{C}_2\}$$

and the cones  $C_1^*, C_2^*$  are of the same sense.

Various optimization problems associated with the optimal control of distributed parameter systems have been studied in [6]-[7],[10]-[13]. The problem of time-optimal control associated with the parabolic systems have been discussed in some papers. In [6] the existence of a time-optimal control of system governed by a parabolic equation has been discussed. In [5], the maximum principles for the time optimal control for parabolic equation is given. All these results concerned the time optimal control problems of systems governed by only one parabolic equation and only control constraints. In [14]-[25], the above results for systems governed by one parabolic equation are extended to the case of  $n \times n$  co-operative parabolic or hyperbolic systems with only distributed control constraints. In the present paper, we will consider time-optimal distributed control problem for the following  $n \times n$  co-operative linear parabolic system with control-state constraints (here and everywhere below the index  $i=1,2,\dots,n$ ):

$$\left. \begin{aligned} \frac{\partial y_i}{\partial t} &= (A(t)y)_i + u_i(x, t) && \text{in } Q = \Omega \times (0, T), \\ y_i(x, 0) &= y_{i,0}(x) && \text{in } \Omega, \\ y_i(x, t) &= 0 && \text{on } \Sigma = \Gamma \times (0, T), \end{aligned} \right\} \quad (1)$$

where  $\Omega \subset R^N$  is a bounded open domain with smooth boundary  $\Gamma$ ,  $y_{i,0}$  is a given functions,  $u_i$  represents a distributed control and  $A(t)$  ( $t \in ]0, T[$ ) are a family of  $n \times n$  continuous matrix operators,

$$A(t)y = \begin{pmatrix} \Delta + a_1 & a_{12} & a_{1n} \\ a_{21} & \Delta + a_2 & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \Delta + a_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

with co-operative coefficient functions  $a_i, a_{ij}$  satisfying the following conditions:

$$\left. \begin{aligned} a_i, a_{ij} &\text{ are positive functions in } L^\infty(Q), \\ a_{ij}(x, t) &\leq \sqrt{a_i(x, t)a_j(x, t)}. \end{aligned} \right\} \quad (2)$$

## 2. Solutions of Co-operative Parabolic Systems

This section is devoted to the analysis of the existence and uniqueness of solutions of system (1). Let  $H_0^1(\Omega)$  be the usual Sobolev space (see [4]) of order one which consists of all  $\phi \in L^2(\Omega)$  whose distributional derivatives  $\frac{\partial \phi}{\partial x_i} \in L^2(\Omega)$  and  $\phi|_{\Gamma} = 0$  with the scalar product

$$\langle y, \phi \rangle_{H_0^1(\Omega)} = \langle y, \phi \rangle_{L^2(\Omega)} + \langle \nabla y, \nabla \phi \rangle_{L^2(\Omega)}, \quad \nabla = \sum_{k=1}^N \frac{\partial}{\partial x_k}.$$

We have the following dense embedding form (see [4]) :

$$H_0^1(\Omega) \subseteq L^2(\Omega) \subseteq H_0^{-1}(\Omega)$$

where  $H_0^{-1}(\Omega)$  is the dual of  $H_0^1(\Omega)$

For  $\mathbf{y} = (y_1, y_2, \dots, y_n)^T$ ,  $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \dots, \varphi_n)^T \in (H_0^1(\Omega))^n$  and  $t \in ]0, T[$ , let us define a family of continuous bilinear forms

$$\pi(t; \cdot, \cdot) : (H_0^1(\Omega))^n \times (H_0^1(\Omega))^n \rightarrow \mathfrak{R} \text{ by}$$

$$\begin{aligned} \pi(t; \mathbf{y}, \boldsymbol{\varphi}) &= \sum_{i=1}^n \int_{\Omega} [(\nabla y_i)(\nabla \varphi_i) - a_i(x, t)y_i \varphi_i] dx \\ &\quad - \sum_{i,j=1}^n \int_{\Omega} a_{ij}(x, t)y_j \varphi_i dx \\ &= \sum_{i=1}^n \int_{\Omega} [(-\Delta y_i) - a_i(x, t)y_i] \varphi_i dx - \sum_{i,j=1}^n \int_{\Omega} a_{ij}(x, t)y_j \varphi_i dx \\ &= \sum_{i=1}^n \langle -(A(t)\mathbf{y})_i, \varphi_i \rangle_{L^2(\Omega)} \end{aligned} \tag{3}$$

Lemma 1 If  $\Omega$  is a regular bounded domain in  $R^N$ , with boundary  $\Gamma$ , and if  $m$  is positive on  $\Omega$  and smooth enough (in particular  $m \in L^\infty(\Omega)$ ) then the eigenvalue problem:

$$\left. \begin{aligned} -\Delta y &= \lambda m(x)y && \text{in } \Omega, \\ y &= 0 && \text{on } \Gamma \end{aligned} \right\}$$

possesses an infinite sequence of positive eigenvalues:

$$0 < \lambda_1(m) < \lambda_2(m) \leq \dots \lambda_k(m) \dots; \lambda_k(m) \rightarrow \infty, \text{ as } k \rightarrow \infty.$$

Moreover  $\lambda_1(m)$  is simple, its associate eigenfunction  $e_m$  is positive, and  $\lambda_1(m)$  is characterized by:

$$\lambda_1(m) \int_{\Omega} m y^2 dx \leq \int_{\Omega} |\nabla y|^2 dx \tag{4}$$

*Proof.* See[5].

Now, let

$$\lambda_1(a_i) \geq n - 1, \quad i = 1, 2, \dots, n \tag{5}$$

Lemma 2 If (2) and (5) hold then, the bilinear form (3) satisfy the Gårding inequality

$$\pi(t; \mathbf{y}, \mathbf{y}) + c_0 \|\mathbf{y}\|_{L^2(\Omega)^n}^2 \geq c_1 \|\mathbf{y}\|_{(H^1(\Omega))^n}^2, \quad c_0, c_1 > 0 \tag{6}$$

*Proof.* In fact

$$\begin{aligned} \pi(t; \mathbf{y}, \mathbf{y}) &= \sum_{i=1}^n \int_{\Omega} [|\nabla y_i|^2 - a_i(x, t)y_i^2] dx \\ &\quad - \sum_{i,j=1}^n \int_{\Omega} a_{ij}(x, t)y_i y_j dx \\ &\geq \sum_{i=1}^n \int_{\Omega} [|\nabla y_i|^2 - a_i(x, t)y_i^2] dx \\ &\quad - 2 \sum_{i>j}^n \int_{\Omega} \sqrt{a_i(x, t)a_j(x, t)} y_i y_j dx. \end{aligned}$$

By Cauchy Schwarz inequality and (4), we obtain

$$\begin{aligned} \pi(t; \mathbf{y}, \mathbf{y}) &\geq \sum_{i=1}^n \left(1 - \frac{1}{\lambda_1(a_i)}\right) \int_{\Omega} |\nabla y_i|^2 dx - \sum_{i=1}^n \int_{\Omega} a_i y_i^2 dx \\ &\quad - 2 \sum_{i>j}^n \int_{\Omega} \left(\frac{1}{\sqrt{\lambda_1(a_i)\lambda_1(a_j)}}\right) \left(\int_{\Omega} |\nabla y_i|^2 dx\right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla y_j|^2 dx\right)^{\frac{1}{2}} \\ &\geq \sum_{i=1}^n \left(\frac{\lambda_1(a_i) - n + 1}{\lambda_1(a_i)}\right) \int_{\Omega} |\nabla y_i|^2 dx - \sum_{i=1}^n \int_{\Omega} a_i y_i^2 dx. \end{aligned}$$

Finally, from (5) we have (6).

Under the above lemma (Lemma 2) and using the results of Lions [6] and Lions and Magenes [7] we can prove the following theorems:

Theorem 3 Assume that (2) and (5) hold. Then, problem (1) has a unique weak solution if  $u_i \in L^2(0, T; H^{-1}(\Omega))$  and  $y_{i,0} \in L^2(\Omega)$ . Moreover, the mapping  $t \rightarrow y(t; u)$  is continuous from  $[0, T] \rightarrow (L^2(\Omega))^n$ .

## 3. Control Problem

Let us consider the following optimization problem

$$T \rightarrow \min, \tag{7}$$

under the following constraints:

$$\left. \begin{aligned} \frac{\partial y_i}{\partial t} + (A(t)y)_i &= u_i && \text{in } Q, \\ y_i(x, 0) &= y_{i,0}(x) && \text{in } \Omega, \\ y_i &= 0 && \text{on } \Sigma, \\ y(x, T) &\in K, && u \in U_{ad}, \end{aligned} \right\} \tag{8}$$

$$\left. \begin{aligned} U_{ad} \text{ is a closed, convex subset of } (L^2(Q))^n, \\ K \text{ is a closed, convex subset of } (L^2(\Omega))^n, \text{ int } K \neq \Phi \end{aligned} \right\} \quad (9)$$

*Notation 1* We will call the problem (7)-(9), problem I

The optimization problem I can be replaced by another equivalent one with a fixed time  $T$ . To show that we need two auxiliary lemmas.

**Lemma 3** Let  $T^0 > 0$  be the optimal time for the problem I. If  $\text{int } K \neq \Phi$  then  $y(x, T^0) \in \partial K$  (boundary of  $K$ )

for any set  $y$  satisfying (7)-(8)

*Proof.* Any solution of (8) is continuous with respect to  $t$ . If  $y(x, T^0) \in \partial K$  is not true, then there exists an admissible state  $y$  such that the observation  $y(x, T^0) \in \text{int } K$ . Thus a  $\hat{T} < T^0$  exists so that  $y(x, \hat{T}) \in \partial K$ . This contradicts the optimality of  $T^0$ .

**Lemma 4** Let  $T^0 > 0$  be the optimal time for the problem I, let  $u^0$  and  $y^0$  be an optimal control and corresponding state, respectively. Then there exist a non-trivial vector  $g(x) = (g_i(x))^n \in (L^2(\Omega))^n$  so that the pair  $(u^0, y^0)$  is the optimal for the following control problem with the fixed time  $T^0$ :

$$I(y, u) := \langle g(x), y(x, T^0) \rangle_{(L^2(\Omega))^n} \rightarrow \min \quad (10)$$

*Proof.* The linearity of the equations (8) implies that the endpoints  $y(x, T^0)$  of all admissible states  $y$  form a convex set  $Y_{T^0}$ . From Lemma 3 we have  $Y_{T^0} \cap \text{int } K = \Phi$  and  $y(x, T^0) \in \partial K$ .

Since  $\text{int } K \neq \Phi$  thus there exists a closed hyperplane separating  $Y_{T^0}$  and  $K$  containing  $y(x, T^0)$ , i.e. there is a nonzero vector  $g \in (L^2(\Omega))^n$  such as [8]

$$\begin{aligned} \sup_{y \in Y_{T^0}} \langle g(x), y(x, T^0) \rangle &\leq \langle g(x), y^0(x, T^0) \rangle_{(L^2(\Omega))^n} \\ \leq \inf_{y \in K} \langle g(x), y \rangle. \end{aligned}$$

This completes the proof.

**Remark 2** The method fails if  $\text{int } K \neq \Phi$ , e.g. in the case when  $K$  consists of a single point.

**Remark 3** If the set  $K$  has a special form i.e

$$K = \{y \in (L^2(\Omega))^n : \|y_i - y_{id}\|_{L^2(\Omega)} \leq \varepsilon\} \quad (11)$$

where  $\varepsilon > 0$  and  $y_{id} \in L^2(\Omega)$  are given, then  $g_i$  is known explicitly and is expressed by

$$g_i(x) = y_i^0(x, T^0) - y_{id}$$

According to Lemma 4, problem I is equivalent to the one with the fixed time  $T^0$  and the performance index in the form (10).

Let us denote by  $Q_1, Q_2, Q_3$  the sets in the space  $E = Y \times U$  as follows

$$Q_1 := \left\{ \begin{aligned} &\frac{\partial y_i}{\partial t} + (A(t)y)_i = u_i \quad \text{in } Q, \\ (y, u) \in E : &y_i(x, 0) = y_{i,0}(x) \quad \text{in } \Omega, \\ &y_i' = 0 \quad \text{on } \Sigma. \end{aligned} \right\} \quad (12)$$

$$Q_2 := \{(y, u) \in E : y \in Y, u \in U_{ad}\}, \quad (13)$$

$$Q_3 := \{(y, u) \in E : y(x, T^0) \in K, u \in U_{ad}\}. \quad (14)$$

Thus the optimization problem I may be formulated in such a form

$$I(y, u) \rightarrow \min \quad \text{subject to } Q_1 \cap Q_2 \cap Q_3. \quad (15)$$

We approximate the sets  $Q_1$  and  $Q_2$  by the regular tangent cones (RTC),  $Q_3$  by the regular admissible cone (RAC) and the performance functional by the regular cone of decrease (RFC).

The tangent cone to the set  $Q_1$  at  $(y^0, u^0)$  has the form

$$\begin{aligned} &RTC(Q_1, (y^0, u^0)) = \{(\bar{y}, \bar{u}) \in E : \\ &P'(y^0, u^0)(\bar{y}, \bar{u}) = 0 \\ &= \left\{ \begin{aligned} &\frac{\partial \bar{y}_i}{\partial t} + (A(t)\bar{y})_i = \bar{u}_i \quad \text{in } Q, \\ (\bar{y}, \bar{u}) \in E : &\bar{y}_i(x, 0) = \bar{y}_{i,0}(x) \quad \text{in } \Omega, \\ &\bar{y}_i = 0 \quad \text{on } \Sigma, \end{aligned} \right\} \end{aligned} \quad (16)$$

where  $P'(y^0, u^0)(\bar{y}, \bar{u})$  is the Fréchet differential of the operator

$$P(y, u) := \left( \frac{\partial y}{\partial t} + A(t)y - u, y(x, 0) - y_0(x) \right)$$

mapping from the space  $E$  into the space  $F$  where  $F = L^2(0, T; (H^{-1}(\Omega))^n) \times (L^2(\Omega))^n$ . According to Theorem 3 on the existence of solution to the equation (8) it is easy to prove that  $P'(y^0, u^0)$  is the mapping from the space  $E$  onto the space  $F$  as required in the Lusternik Theorem ([2]).

According to (13) the tangent cone  $RTC(Q_2, (y^0, u^0))$  to the set  $Q_2$  at  $(y^0, u^0)$  has the form

$$RTC(Q_2, (y^0, u^0)) = Y \times RTC(U_{ad}, u^0) \quad (17)$$

where  $RTC(U_{ad}, u^0)$  is the tangent cone to the set  $U_{ad}$  at the point  $u^0$ . From [2] it is known that tangent cones are closed.

Applying the same arguments as in Section 2.2 from [9] we can show that

$$RTC(Q_1 \cap Q_2, (y^0, u^0)) = RTC(Q_1, (y^0, u^0)) \cap RTC(Q_2, (y^0, u^0))$$

We have to use Theorem 2, to show that  $[RTC(Q_1, (y^0, u^0))]^*$  and  $[RTC(Q_2, (y^0, u^0))]^*$  are of the same sense. (Note that we do not need to determine the explicit form of  $[RTC(Q_1, (y^0, u^0))]^*$  in order to derive this conclusion.) It is enough to use the Theorem 3 about the existence and uniqueness of the solution for parabolic system (8) which determine  $RTC(Q_1, (y^0, u^0))$  in (16). According to this theorem the solution of such a system depends continuously on the right side; i.e., in our case on  $\bar{u}$  so we can rewrite the cone given by (16) in the form

$$[RTC(Q_1, (y^0, u^0))] = \{(\bar{y}, \bar{u}) \in Y \times U : \bar{y} = -M \bar{u}\} \quad (18)$$

where  $M : U \rightarrow Y$  is a linear and continuous operator. Then, applying Theorem 2, to the cones given by (17) and (18), we get the assumption (iv) of Theorem 1 is satisfied.

The admissible cone  $RAC(Q_3, (y^0, u^0))$  to the set  $Q_3$  at  $(y^0, u^0)$  has the form

$$RAC(Q_3, (y^0, u^0)) = RAC(K, (y^0(T^0))) \times U \quad (19)$$

where,  $RAC(K, (y^0(T^0)))$  is the admissible cone to the set  $K$  at the point  $y^0(x, T^0)$ .

Using Theorem 7.5 [2], the regular cone of decrease for the performance functional  $I$  is given by

$$RFC(I, (y^0, u^0)) = \{(\bar{y}, \bar{u}) \in E : I'(y^0, u^0)(\bar{y}, \bar{u}) < 0\} \quad (20)$$

where  $I'(y^0, u^0)(\bar{y}, \bar{u})$  is the fréchet differential of the performance functional  $I$ .

If  $RFC(I, (y^0, u^0)) \neq \Phi$  then the adjoint cone consists of the elements of the form (Theorem 10.2 in [2])

$$f_0((y^0, u^0)) = -\lambda_0 I'(y^0, u^0)(\bar{y}, \bar{u}) \text{ where } \lambda_0 \geq 0$$

The functionals belonging to  $[RTC(Q_1, (y^0, u^0))]^*$  have the form (Theorem 10.1 [2])

$$f_2(\bar{y}, \bar{u}) = 0, \forall (\bar{y}, \bar{u}) \in RTC(Q_1, (y^0, u^0)).$$

The functionals  $f_3(\bar{y}, \bar{u}) \in [RTC(Q_2, (y^0, u^0))]^*$  and  $f_1(\bar{y}, \bar{u}) \in [RAC(Q_3, (y^0, u^0))]^*$  can be expressed as follows

$$f_3(\bar{y}, \bar{u}) = f_3^1(\bar{y}) + f_3^2(\bar{u}), \quad f_1(\bar{y}, \bar{u}) = f_1^1(\bar{y}) + f_1^2(\bar{u}),$$

where  $f_3^1(\bar{y}) = 0 \forall \bar{y} \in Y$  and  $f_1^2(\bar{u}) = 0 \forall \bar{u} \in U$  (Theorem 10.1 in [2]),  $f_3^2(\bar{u})$  is the support functional to the set  $U_{ad}$  at the point  $u^0$  and,  $f_1^1(\bar{y})$  is the support functional to the

set  $K$  at the point  $y^0(x, T^0)$  (Theorem 10.5 [2]).

Since all assumptions of Theorem 1 are satisfied and we know suitable adjoint cones then we ready to write down the Euler -Lagrange Equation in the following form.

$$f_3^2(\bar{u}) + f_1^1(\bar{y}) = I'(y^0, u^0)(\bar{y}, \bar{u}) \quad \forall (\bar{y}, \bar{u}) \in RTC(Q_1, (y^0, u^0)) \quad (21)$$

### 4. Special Case

Since  $I$  depended on the target set  $K$ , we shall interpret (21) after choosing  $K$  in a less form fashion (11).

sNotation 2 We will call the problem I with  $K$  is given by (11), problem  $I_1$

In the present case, according to Remark 3 (23)

Introducing the adjoint variable  $p$  by the solution of the following systems

$$\left. \begin{aligned} -\frac{\partial p_i}{\partial t} + (A(t)p)_i &= 0, & x \in \Omega, t \in ]0, T^0[ \\ p_i(x, T^0) &= (y_i^0(x, T^0) - y_{id}) & x \in \Omega, \\ p_i &= 0, & x \in \Gamma, t \in ]0, T^0[. \end{aligned} \right\}$$

The existence of a unique solution for the above equation can be proved using Theorem 3 with an obvious change of variables)

Taking into account that  $\bar{y}$  is the solution of  $P'(y^0, u^0)(\bar{y}, \bar{u}) = 0$  for any fixed  $\bar{u}$ , we obtain;

$$\begin{aligned} 0 &= \int_0^{T^0} \int_{\Omega} \left[ -\frac{\partial p_i}{\partial t} + (A^*(t)p)_i \right] \bar{y}_i dx dt \\ &= \int_0^{T^0} \int_{\Omega} \left[ \frac{\partial \bar{y}_i}{\partial t} + (A(t)\bar{y})_i \right] p_i dx dt - \\ &\quad - \int_{\Omega} (y_i^0(x, T^0) - y_{id}) \bar{y}_i(x, T^0) dx \end{aligned}$$

Hence

$$\int_{\Omega} (y_i^0(x, T^0) - y_{id}) \bar{y}_i(x, T^0) dx = \int_0^{T^0} \int_{\Omega} p_i \bar{u}_i dx dt.$$

So, the Euler -Lagrange Equation (21) takes the form:

$$\begin{aligned} f_3^2(\bar{u}) + f_1^1(\bar{y}) &= \frac{1}{2} \lambda_0 \sum_{i=1}^n \int_0^{T^0} \int_{\Omega} p_i \bar{u}_i dx dt. \\ &+ \frac{1}{2} \lambda_0 \sum_{i=1}^n \int_{\Omega} (y_i^0(x, T^0) - y_{id}) \bar{y}_i(x, T^0) dx. \end{aligned} \quad (22)$$

A number  $\lambda_0$  cannot be equal to 0 because in such a case all

functionals in the Euler -Lagrange Equation would be zero which is impossible according to the DM Theorem. Using the definition of the support functional and dividing both members of the obtained inequalities by  $\lambda_0$  from (22) we obtain the maximum conditions:

If  $RFC(I, (y^0, u^0)) = \Phi$ , then the optimality conditions are fulfilled with equality in the maximum conditions. We have thus proved:

Theorem 4 *Assuming that  $T^0 > 0$  is the optimal time for the problem  $I_1$  and  $u^0$  and  $y^0$  are the optimal control and corresponding state respectively. Then, there exists the adjoint state  $p = (p_i)_{i=1}^n \in L^2(0, T; (H^1(\Omega))^n)$  so that the following system of equations must be satisfied:*

State equations;

$$\left. \begin{aligned} \frac{\partial y_i^0}{\partial t} + (A(t)y^0)_i &= u_i^0, & x \in \Omega, t \in ]0, T^0[, \\ y_i^0(x, 0) &= y_{i,0}(x) & x \in \Omega, \\ y_i^0 &= 0, & x \in \Gamma, t \in ]0, T^0[, \\ u_i^0 &\in U_{ad}, \\ \|y_i^0 - y_{id}\|_{L^2(\Omega)} &\leq \varepsilon. \end{aligned} \right\} \quad (23)$$

Adjoint equations;

$$\left. \begin{aligned} -\frac{\partial p_i}{\partial t} + (A(t)p)_i &= 0, & x \in \Omega, t \in ]0, T^0[, \\ p_i(x, T^0) &= (y_i^0(x, T^0) - y_{id}) & x \in \Omega, \\ \dot{p}_i &= 0, & x \in \Gamma, t \in ]0, T^0[. \end{aligned} \right\} \quad (24)$$

Maximum conditions;

$$\sum_{i=1}^n \int_0^{T^0} \int_{\Omega} p_i (u_i - u_i^0)_i dx dt \geq 0 \forall u \in U_{ad} \quad (25)$$

$$\sum_{i=1}^n \int_{\Omega} (y_i^0(x, T^0) - y_{id})(y_i - y_i^0) dx \geq 0 \forall y \in K \quad (26)$$

### 5. Conclusion

In this study, we have derived the optimality conditions to a special co-operative parabolic systems with control-state conditions. Most of the results we described in this paper apply, without any change on the results, to more general parabolic systems involving the following second order operator:

$$L(x, \cdot) = \sum_{i,j=1}^n b_{ij}(x, \cdot) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^n b_j(x, \cdot) \frac{\partial}{\partial x_j} + b_0(x, \cdot)$$

with sufficiently smooth coefficients (in particular,  $b_{ij}, b_j, b_0 \in L^\infty(Q), b_j, b_0 > 0$ ) and under the Legendre-Hadamard ellipticity condition

$$\sum_{i,j=1}^n \eta_i \eta_j \geq \sigma \sum_{i=1}^n \eta_i \quad \forall (x, t) \in Q,$$

for all  $\eta_i \in \Re$  and some constant  $\sigma > 0$ .

In this case, we replace the first eigenvalue of the Laplace operator by the first eigenvalue of the operator  $L$  (see [5]).

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