Pure and Applied Mathematics Journal
2014; 3(3): 49-65
Published online June 10, 2014 (http://www.sciencepublishinggroup.com/j/pamj)
doi: 10.11648/j.pamj.20140303.11

Stairs of natural set theories

Maydim A. Malkov

Russian Research Center for Artificial Intelligence, Moscow, Russia

Email address:
mamalkov@gmail.com

To cite this article:
Maydim A. Malkov. Stairs of Natural Set Theories. *Pure and Applied Mathematics Journal.* Vol. 3, No. 3, 2014, pp. 49-65.
doi: 10.11648/j.pamj.20140303.11

Abstract: All contemporary set theories have intersected classes. We have built the stairs of set theories with disjoint classes. We call such theories *natural*. We numerate these theories by ordinals. The first set theory is T_0. We build the theory from the set N of natural numbers by using the operations of direct products and of power set by finite times. The theory contains all results of Cantor's theory. We argue that the theory can satisfy all needs of applied mathematics. We build theory T_1 by using the universe set of all sets of T_0 and by using the operations of direct products and of power set by finite times. We build theory T_α+1 from the set of previous by using the operations of direct products and of power set by finite times, too. We build theory T_ω from the set of all sets of T_α with α < ω again by using the operations of direct products and of power set by finite times. And so on for every theory T_α, if theory T_α-1 does not exists. We use the join of all these sets to build theory T_On without operation of power set. We call members of T_On *families*, members of families *sets*, families, which are not members of families, *up-sets*. Families are an analog of classes of the MK set theory and up-sets are an analog of proper classes of MK theory. The theory T_On is more strong than MK theory because we use more strong axiom of comprehension. The last theory T_On+1 is external to T_On. We use T_On+1 to prove those theorems of T_On that are unproved in T_On.

Keywords: Set theory, Classifications, Disjoint classes of sets, Cantor ordinals, Cantor cardinals, Comprehension axiom.

1. Introduction

The main problem of every theory is the classification of all objects of the theory by using their properties based on theorem results. Every object must belong to a class and classes must be disjoint. We call such classification *natural*. We have constructed the stairs of set theories with the natural classification of all its objects. All existed set theories have not such classification. We will name set theories with natural classification *natural*.

The first set theory was created by Cantor (1874, [1]). It contains modern theory of cardinal and ordinal numbers and the theory of well-ordered sets almost whole. Cantor also pointed out the paradox in his theory, later it was called Cantor's paradox.

The first set theory without paradoxes was the type theory T (shortening of Type), created by Russel (1903, [2]). Types in the theory form a hierarchy of objects. The first level of the hierarchy has atoms, the second level contains sets of atoms, the next level has sets of sets and so on. Objects of a level have the same type, sets of such objects have type that is more that the type of their objects. So the set $\{a,\{b,c\}\}$ is absent in the theory because objects of the set have different types.

Then Russel created the new type theory RT (shortening of Ramified Theory, 1908, [3]). A set in the theory can have objects of different types if their types are less than the type of the set. Non-predicative formulas exist but only if they can be replaced by a sequence of predicative formulas.

The other way to exclude paradoxes was introduce by E. Zermelo (1908,[4]) in the theory Z. The two ways of excluding paradoxes are basic now.

The type theory T was investigated by K. Gödel (1931, [5]) and A. Tarski (1931, [6]).

The theories NF (shortening of New Foundation, W. Quine, 1937, [7]), ML (shortening of Mathematical Logic, W. Quine, 1940. [8]) and STT (shortening of Simple Theory of Types, A. Church, 1940, [9]) are extensions of T theory. The theory Σ (Hao Wang, 1954, [10]) was an extension of the theory RT.

The theory Z was extended by A. Fraenkel (1922, [11]). The name of the theory is ZF. With the choice axioms the theories are ZC and ZFC. The theories have infinite number of axioms.

The theory NBG (J. Neumann, 1925-1928, [12], R. Robinson, 1937, [13], P. Bernays, 1937-1954, [14], K. Gödel, 1940, [15]) replaces infinite number of axioms of ZFC by finite number.

The next extension of NBG is MK theory (J. Kelley, 1955, [16], A. Morse, 1965, [17]). In the theory domain of variables can contain all classes including proper (the variables are bounded to quantifiers).

The theory ICS (shortening of Iterative Conception of Set, G. Boolos, 1971, [18]) is an extension of ZF and RT. The signature of the theory is added by symbols of two-place predicates E, F and by symbols of new variables $s_0, s_1, s_2, \ldots$. The variables are interpreted as steps of construction of sets. The predicate $s_1 E s_2$ is interpreted as a step s_1 is before a step s_2. The predicate $x_1 F s_1$ is interpreted as a set x_1 created at a step s_1.

Some works are dedicated to large cardinals as an extension to ZFC theory (P. Maddy, 1988, [19], W. Woodin, 2001, [20], A. Kanamori, 2003, [21]).

One more extension of ZFC by adding new axioms was presented by J. Steel, (2000, [22]).

More information can be found in Handbook of set theory (M. Foreman, A. Kanamory, ed., 2010, [23].

Type theories. We use a type theory but the type theory is not our starting point. The type theory is the corollary of our natural classification of sets.

Now the term *type theory* does not have a rigidly fixed value. In this theory its objects belong to layers forming an hierarchy, every layer has a type. This hierarchy is not necessarily linear and does not necessarily countable. Values of variables are objects of the same type. Superscripts of variables equal types of their objects. The axiom of compression has the form

$$\exists x_0^{t_0} \forall x_1^{t_1}, \ldots, x_n^{t_n} \langle x_1^{t_1}, \ldots, x_n^{t_n} \rangle \in x_0^{t_0}$$

$$\leftrightarrow t_0 > \max(\mathrm{t}_1, \ldots, t_n) \wedge x_0^{t_0} = \{x_1^{t_1}, \ldots, x_n^{t_n} : \varphi(x_1^{t_1}, \ldots, x_n^{t_n})\}$$

where φ is any well formed (wf) formula (see 3.12). The formula is predicative. The type t_0 of x_0 is greater than types of the other variables.

This formula can be replaced by non-predicative. Then axiom of compression is

$$\exists x_0^{t_0} \forall x_1^{t_1}, \ldots, x_n^{t_n} \langle x_1^{t_1}, \ldots, x_n^{t_n} \rangle \in x_0^{t_0}$$

$$\leftrightarrow t_0 > \max(\mathrm{t}_1, \ldots, t_n) \wedge x_0^{t_0} = \{x_1^{t_1}, \ldots, x_n^{t_n} : \varphi(x_0^{t_0}, x_1^{t_1}, \ldots, x_n^{t_n})\}$$

where $x_0^{t_0}$ is constructed step-by-step: $x_0^{t_0}$ in φ is empty at step 0 and at step $i+1$ φ uses value of $x_0^{t_0}$ at step i.

There are type theories with finite types and with types that are any initial interval of ordinals. There are type theories with quantifiers on sets of types. And there are type theories such that their wfs have types.

In the article, types are corollary of the natural classification. And so all variables have the same domain (in type theories number of domains equals number of types). In the article, types belong to sets but do not belong to variables. This means we use new type theory which is free from difficulties of type theories and which has all properties of MK set theory. We use predicative and non-predicative wfs. We use linear hierarchy of types. Values of types are ordinals $0 \le \alpha < \mathrm{On} + \omega$. If all members of a set have the same type then the type of the set is more than the type of members by a finite number. Members of some sets have infinite spectrum of types but the such sets have types that are next to all types of their members.

*About article.*The article consists of 5 sections.

In section 1 we give an informal construction of the theories. This construction begins from the theory T_0 and comes to the end with the external theory $T_{\mathrm{On}+1}$. We state that the theories T_0 and T_{On} are basic. The theory T_1 and $T_{\mathrm{On}+1}$ are external for T_0 and T_{On} respectively.

In section 2 we construct the theory T_0. All objects of the theory are sets. We take definition of natural numbers from arithmetics, then construct the set of natural numbers, and then construct all other sets by using finite times the operation of direct product and the function of power set. We construct ordinals and cardinals. We construct the classification of sets of the theory on the basis of an equivalence relation. This classification has a denumerable number of disjoint classes (not to be confuse with classes of MK theory). We argue that the theory T_0 can satisfy almost all needs of applied mathematics.

In section 3 we construct the set theory T_1 as external to T_0. The sets of this theory are: all sets of T_0, the set U_1 of all these sets, and sets that are results of applying the operations of direct product and the power set to U_1 by finite times. We give the definition of the sets of T_0 to distinguish them from the other sets of the theory T_1. This definition cannot be given in the theory T_0. By the definition, all sets constructed in T_0 exist. We prove theorem for sets of T_0 that cannot be proved in T_0. In particular, we prove that the classification of sets of T_0 covers all sets of T_0. We find the natural classification of all sets of the theory T_1, too. This classification differs a bit from the natural classification of sets of T_0.

In section 4 we construct the natural set theory T_{On}. We call objects of the theory *families*. Families simulate classes of MK theory. Families can be sets or *up-sets*. The families are: all sets of previous theories, the family U of all these sets (an analog of universe V of MK theory), and the families constructed by the operations of direct product and subfamily. The operation (function) of power set is not used in order to limit powers of families. The maximal power of families is $\aleph_{\mathrm{On}}$. Up-sets are not members of any families. Some up-sets are empty. Then we construct the natural classification of families.

In section 5 we construct the theory $T_{\mathrm{On}+1}$ as external to the natural theory. We name objects of the theory *up-families*. Up-families are: the collection of all families of T_{On} , sub-collections, and direct products of these collections. We prove some theorems that are not proved in the natural theory.

*Notation and agreements.*We call variables, values of which are names of sets, briefly *sets* and denote them by the letter x with subscripts.

We denote a power set by $P(x)$. The $P^n(x)$ means the n -times application of the operation P . In particular,

$$P^0(x) = x, P^1(x) = P(x), P^2(x) = P(P(x)) \text{ , etc.}$$

The use of quantifiers is a bit simplified by reducing the number of brackets. And universal quantifiers are absent, if they must be in the beginning of formulas.

Using inductive (non-predicative) definitions we construct any collections step-by-step. In particular, the definition of classes in T_0 is presented by the formula:

$$\text{class } x_0 \leftrightarrow x_0 = N \vee \forall x_1 \forall x_2 \text{ class } x_1 \wedge \text{class } x_2$$

$$\rightarrow x_0 = x_1 \times x_2 \vee x_0 = P(x_1)$$

The class of natural numbers N exists at step 0 of the formula realization. At step 1 the class $N \times N$ is generated first, then the class $P(N)$ is generated. At step 2, ten more classes are generated sequentially. And so on. This sequence of the realization is taken from the algorithm theory[1].

Some theorems are presented as statements without proofs, if these proofs are obvious.

Ordinals and cardinals. Hear we give informal definitions of ordinals and cardinals. The formal definitions are given in 2.14-2.16 and 4.6, 4.7.

We use Cantor's definitions of ordinals and cardinals. We call them *Cantor ordinals* and *Cantor cardinals* and we call representatives of Cantor ordinals and Cantor cardinals briefly *ordinals* and *cardinals*.

The construction of ordinals differs from the generally accepted constructions because, by Cantor, every ordinal must be well ordered set, i.e. the ordered pair $\langle$ a set, its ordering relation $\rangle$. But we call an ordering relation a *well ordered set* because a set can have several ordering relations but an ordering relation has only one set. Hence the ordered pair can be replaced by only ordering relation. This allows to simplify the construction of ordinals.

Ordinals[2] 0 and 1 are 0 and $\{0\}$. We state that 0-member and 1-member sets are well ordered.

But 2-member sets are not well ordered. Every 2-member set has two well ordered sets. For example the set $\{0,\{0\}\}$ has two well ordered sets (well ordering relations) $\{\langle 0,\{0\}\rangle\}$ and $\{\langle\{0\},0\rangle\}$. This means $0 < \{0\}$ and $\{0\} < 0$ respectively. But we use only the well ordering relation $\{\langle 0,\{0\}\rangle\}$ and we call the relation *ordinal 2*.

For simplification we denote 0 by $\dot{0}$, $\{0\}$ by $\dot{1}$, $\{0,\{0\}\}=\{\dot{0},\dot{1}\}$ by $\dot{2}$ and so on. We denote $\{\dot{0},\dot{1},\dot{2},...\}$ by $\dot{\omega}$ and $\{\dot{0},\dot{1},\dot{2},...,\dot{\omega}\}$ by $\omega \dot{+} 1$.

Then *ordinal 3* is the ordering relation $\{\langle\dot{0},\dot{1}\rangle,\langle\dot{0},\dot{2}\rangle,\langle\dot{1},\dot{2}\rangle\}$. This means $\dot{0} < \dot{1} < \dot{2}$.

We will give in 2.15 and 2.16 the rule to construct ordinals. This rule does not limit the construction. But the collection of all constructed ordinals exists, it is On. And On is the next to all the ordinals. We call On *up-ordinal*.

The construction of cardinals (representatives of Cantor cardinals) does not differ from the generally accepted. Cardinal 0 is $\dot{0}$ =0, cardinal 1 is $\dot{1}$ ={ $\dot{0}$ }, cardinal 2 is $\dot{2}$ ={ $\dot{0}$, $\dot{1}$ }, and so on.

2. Stairs of Theories

There are next stairs of theories and their objects. All definitions are inductive.

Sets of theory T_0 are:

- set N of natural numbers and members of N ,
- a direct product of sets and its members,
- a power set of a set and members of the power set.

The theory is simple but powerful to satisfy all needs of applied mathematics. It contains all Cantor's results without any paradoxes because there is an axiom system. Arithmetic is the sub-theory of T_0 .

Sets of theory T_1 are:

- universe U_1 (the set of all sets of T_0) and its members,
- a direct product of sets and its members,
- a power set of a set and members of the power set.

We include sets of T_0 in T_1 because some subsets of U are sets of T_0 .

The theory T_1 is external for T_0 . And we use T_1 to search some properties of sets of T_0 and to prove theorems of T_0 which are not proved in T_0 .

Sets of theory $T_{\alpha+1}$ are constructed similarly.

Sets of theory T_ω are:

- the universe U_ω (the set of all sets of previous

[1] The formal definition of the rule for inductive (non-predicative) definitions and proofs belongs to logic. We exclude all formal definitions belonging to logic because logic must be up theories but not be repeated in every theory.

[2] We use the same notations for finite ordinals, for finite cardinals, and for natural numbers in hope this does not lead to misunderstanding.

theories) and its members,
- a direct product of sets and its members,
- a power set of a set and members of the power set.

Sets of theory T_α, for which theory $T_{\alpha-1}$ is absent, are constructed similarly.

Objects of theory T_{On} are families:
- universe U (the collection of sets of all previous theories) and its members,
- a direct product of families and its members,
- a sub-families of families and its members.

We do not use the power set operation in order to restrict cardinals of families by $\aleph_{\text{On}}$.

We call families that are not members of families *up-sets*. We call the other families *sets*. All sets are members of U.

Families are an analog of classes of MK set theory. Up-sets are an analog of proper classes of MK set theory. Some up-sets are empty.

Objects of last theory $T_{\text{On}+1}$ are up-families:
- universe U_+ (the collection of all families) and its members,
- a direct product of up-families and its members,
- sub-collections of up-families and its members.

We can research in the theory the other characteristics of families and can prove some theorem of T_{On} which can not be proved in T_{On}.

The more interesting set theories are T_0, T_{On}, and their external theories T_1, $T_{\text{On}+1}$.

3. Theory T_0

3.1. Signature and Variable Notation

The theory T_0 is ordered four: $(0,',\langle\rangle,\in)$, where 0 is constant, ' is the one-place function of the succession (this function is partial[3] and its domain is the set of natural numbers), $\langle\rangle$ is the two-place function of the ordered pair, and $\in$ is the two-place predicate of membership. This means that the symbols are undefinable. But 0 is a set and $',\langle\rangle,\in$ generate sets, because all objects of T_0 are sets.

We take the first two symbols of the signature from arithmetic. This allows to use the standard definition of natural numbers and not to invent something new. The third symbol allows not to invent a definition of the ordered pair. The fourth symbol is standard.

We add symbols $x_0, x_1, x_2, \ldots$ to the alphabet of the theory and use them as variables (of functions and predicates). Sometimes we use x instead x_0. The domain of these variables is the universe U_1 of the external theory T_1.

We must define ordered n-tuples as a generalization of ordered pairs.

Definition. $\langle x_1\rangle = x_1$,

$$\langle x_1,\ldots,x_n\rangle = \langle\langle x_1,\ldots,x_{n-1}\rangle, x_n\rangle$$

But $\langle\langle x_1,x_2\rangle,\langle x_3,x_4\rangle\rangle \neq \langle x_1,x_2,x_3,x_4\rangle$.

3.2. Axiom 1

We will present the system of axioms of the theory later. Now we present only one axiom. The axiom states inequality of natural numbers and ordered pairs. But preliminary we shall give the definition of natural numbers.

Definition. A natural number is 0 or a result of the success operation:

$$\text{nat}\, x_0 \leftrightarrow x_0 = 0 \vee \exists x_1\, \text{nat}\, x_1 \wedge x_0 = x_1'$$

where nat is a predicate for natural numbers.

Axiom 1. $\neg\text{nat}\,\langle x_1,x_2\rangle$.

Now we can define atoms.

Definition. An atom is a natural number or an ordered pair:

$$\text{atom}\, x_0 \leftrightarrow \text{nat}\, x_0 \vee \exists x_1 \exists x_2\, x_0 = \langle x_1,x_2\rangle$$

Atoms are sets. And so they present in the domain of variables because the domain is the collection of all sets. Atoms can be components of ordered pairs because values of x_1 and x_2 in the definition can be atoms.

3.3. Set of Natural Numbers and Direct Product

We can define the set of natural numbers and the direct product of sets[4].

Definition. The set of natural numbers N is:

$$x_0 \in N \leftrightarrow \text{nat}\, x_0$$

Definition. A set $x_1 \times x_2$ is a direct product of sets x_1 and x_2, if $x_1 \times x_2$ contains ordered pairs which first components are members of x_1 and second components are members of x_2:

$$\neg\text{atom}\, x_1 \wedge \neg\text{atom}\, x_2 \rightarrow (x_0 \in x_1 \times x_2 \\ \leftrightarrow \exists x_3 \exists x_4\, x_3 \in x_1 \wedge x_4 \in x_2 \wedge x_0 = \langle x_3,x_4\rangle)$$

By definition the operation $\times$ is partial because it is not defined for atoms. And by definition x_0 is an empty set {}, if x_1 or x_2 are empty.

3.4. Power Set

First we shall define subsets.

[3] This function is partial because we use the other way to construct ordinals $\alpha+1$, see 3.14.

[4] The collection of natural numbers and results of direct product are sets but the proof of this belongs to T_1.

Definition. Set x_1 is a subset of set x_2, if both sets are not atoms and if all members of x_1 are members of x_2:

$$x_1 \subseteq x_2 \leftrightarrow \neg \text{atom}\, x_1 \wedge \neg \text{atom}\, x_2 \\ \wedge \forall x_3\; x_3 \in x_1 \rightarrow x_3 \in x_2$$

By definition atoms are not subsets of sets and have no subsets. The definition allows a set to have several empty subsets and an empty subset to be the subset of several sets. The other axioms do not allow a set to have several empty subsets.

And $x_1 \subset x_2$ if $x_1 \subseteq x_2 \wedge x_1 \neq x_2$. Then x_1 is the proper subset of x_2.

Now we can define the power set[5].

Definition. The set of all subsets of x_1 is the power set $P(x_1)$:

$$\neg \text{atom}\, x_1 \rightarrow (x_0 \in P(x_1) \leftrightarrow x_0 \subseteq x_1)$$

By definition the operation P is partial. This operation is not defined, if x_1 is an atom. If x_1 is an empty subset then x_0 is an empty subset, too. Atoms are not members of power set due to the definition of subsets.

Remark. The definition $x_0 \in P(x_1) \leftrightarrow x_0 \subseteq x_1$ is not well because a predicate exists even if a part of it is undefined, in this case the predicate is false. So if x_1 is an atom then $x_0 \subseteq x_1$ is false and then (for all x_0) $x_0 \in P(x_1)$ is false, i.e. $P(x_1)$ is empty but exists.

3.5. Classes of Sets

We shall define the natural classification of sets. Every set must belongs to some class and classes must be disjoint.

Definition. A class is the set N or a set constructed from N by the direct product and power set operations:

$$\text{class}\, x_0 \leftrightarrow x_0 = N \;\vee\; \exists x_1 \exists x_2\; \text{class}\, x_1 \\ \wedge \text{class}\, x_2 \wedge (x_0 = x_1 \times x_2 \vee x_0 = P(x_1))$$

By definition, each class is both a set and a member of some other class. This is natural because the class does not contain itself as a member. In particular, class N contains all natural numbers but it is a member of the other class $P(N)$.

Classes can be well ordered in the sequence of their construction

$$N;\; N \times N, P(N);\; N \times (N \times N), N \times P(N), \\ N \times N \times N, P(N \times N), N \times N \times (N \times N), \\ N \times N \times P(N), P(N) \times N, P^2(N), \\ P(N) \times (N \times N), P(N) \times P(N);\; \ldots$$

Here the semicolons separate sets constructed at the next step.

It is useful to present this sequence as *parenthesis-free record*:

$$N;\; \times NN, PN; \times N \times NN, \times NPN, \times\times NNN, \\ P\times NN, \times\times NN \times NN, \times\times NNPN, \\ \times PNN, PPN, \times PN \times NN, \times PNPN;\; \ldots$$

We will say that a class begins at symbol $\times$ or P, if the first symbol in the record is $\times$ or P. And only class N begins at N.

The classes can be numbered in the sequence of their construction. This means that T_0 is a type theory. Members of a class have a type that equals the ordinal number of the class. A class has a type greater than the type of members of the class. In particular, members of N have type 0, members of $N \times N$ have type 1, the set N and the other members of $P(N)$ have type 2.

So we define at first classes then, as a corollary, types. Classes are not types because types are not sets. Types are superscripts in names of sets (but not of variables). For simplification we do not use the superscripts but we mean them.

3.6. Finite Descent Rule

Because the definition of classes is inductive then the induction rule is used for proving of theorems. The finite descent rule is used, too[6]. Here this rule is given informally.

The finite descent rule uses finiteness of a parenthesis-free record of classes. The structure of a parenthesis-free record of classes is analyzed at the first symbol of the record and to the end of the record at proving of theorems. The transition from analysis of a symbol to analysis of the next symbol is called a *descent*. The *finite descent rule* confirms the end of these descents.

Remark. Classes of sets have the next property: descents finish at the symbol N.

3.7. Main Theorem

We must prove that two different classes have no common members.

Lemma. The sets N and $P(x_1)$ have no common members: $x_0 \notin N \vee x_0 \notin P(x_1)$.

Proof. Members of $P(x_1)$ are not atoms but members of

---

[5] Results of power set are sets but proof of this belongs to T_1, too.

[6] The rule was introduced in 17-th century by Pierre Fermat. He named the rule *infinite descent*. In 19-th century Adrien-Marie Legendre used the rule and renamed it *finite descent*. This rule is used to prove theorems. And so it belongs to logic.

N are. □

Lemma. The sets $x_1 \times x_2$ *and* $P(x_3)$ *have no common members:* $x_0 \notin x_1 \times x_2 \vee x_0 \notin P(x_3)$.

Proof. Members of $x_1 \times x_2$ are atoms but members of $P(x_3)$ are not. □

Lemma. The sets N *and* $x_1 \times x_2$ *have no common members:* $x_0 \notin N \vee x_0 \notin x_1 \times x_2$.

Proof. Members of N are natural numbers. Members of $x_1 \times x_2$ are ordered pairs. By axiom 1, ordered pairs are not natural numbers. □

Theorem. Two different classes have no common members:

$$\text{class}\, x_1 \wedge \text{class}\, x_2 \wedge x_1 \neq x_2$$
$$\rightarrow \forall x_0\ x_0 \notin x_1 \vee x_0 \notin x_2$$

Proof. We use the symbol-by-symbol comparison of parenthesis-free records of classes. If the first compared pair of symbols has equal symbols then we pass to compare the next pair. By the finite descent rule there will be a pair of unequal symbols. It follows from the lemmas that sets generated by a pair of unequal symbols have no common members. □

3.8. Completeness of Set Classification

The classification of sets covers all sets: every set is a member of a class.

This statement holds for members of N, for members of direct products, and for members of power sets. But this list of sets is not complete because theory T_0 does not give the answer to the question: what object is a set and what object is not a set. The answer to this question is only in the external theory T_1. In the theory the definition of sets will be done (see 4.2) and completeness of the set classification will be proved (see 4.3).

3.9. Non-Transitivity and Foundation (Regularity)

The transitivity relation is not realized in the theory: if one (the first) set is a member of a second set, and the second set is a member of a third set, then the first set is not a member of the third set. This statement is an addition to the main theorem.

The other addition is the statement of existence of foundations for sets.

Both statements will be proved in the theory T_1 because these proofs use the completeness of set classification (see 4.4).

3.10. Atoms and Null-Sets

We continue to list the axioms of the theory. First of all we must add the axiom of the emptiness of atoms.

Axiom 2. $\text{atom}\, x_0 \rightarrow \forall x_1\ x_1 \notin x_0$

The ordered pair $\langle\rangle$ and the succession ' are functions. This axiom states that the functions construct only empty sets (see 3.2).

Definition. An empty set is a null-set, if this empty set is not an atom:

$$\text{null}\, x_0 \leftrightarrow \neg\text{atom}\, x_0 \wedge \forall x_1\ x_1 \notin x_0$$

Null-sets exist because they are subsets of every set (not atom). The next axiom states that every set (not atom) has only one empty subset.

Axiom 3. $\text{null}\, x_1 \wedge \text{null}\, x_2 \wedge$

$$\wedge(\exists x_3\ x_1 \subseteq x_3 \wedge x_2 \subseteq x_3) \rightarrow x_1 = x_2$$

If a parenthesis-free record of a class begins at a symbol P then any member of this class has an empty subset in the class. The next theorem states that the number of such subsets in the class equals 1.

Theorem. Each class, which parenthesis-free record begins at symbol P*, contains only one null-set:*

$$\text{null}\, x_1 \wedge \text{null}\, x_2$$
$$\wedge (\exists x_3 \text{class}\, x_3 \wedge x_1 \in P(x_3) \wedge x_2 \in P(x_3)) \rightarrow x_1 = x_2$$

Proof. By definition each member of a class $P(x_3)$ is a subset of the class x_3. Hence each empty subset from the class $P(x_3)$ is a subset of x_3 but this class x_3 has only one empty subset (by axiom 3). □

Atoms have the next property. They have no braces. The other sets have braces. In particular, null sets are $\{\}$. We will point out types of null sets. For example, $\{\}^2$ is the empty subset of N because the types of N and of all subsets of N are 2.

Remark. The proved theorem is true only in T_0.

3.11. Additional Axioms and Definitions

The axiom system of the theory includes two axioms of arithmetic[7].

Axiom 4. $\text{nat}\, x_0 \rightarrow x_0' \neq 0$.

Axiom 5. $\text{nat}\, x_1 \wedge \text{nat}\, x_2 \wedge x_1' = x_2' \rightarrow x_1 = x_2$

This axiom states equalities for natural numbers. Two natural numbers equal if their preeccessors equal. Then we use the finite descent rule.

We must add the equality axioms for the other empty sets.

Ordered pairs equal if their components equal:

Axiom 6. $\langle x_1, x_2\rangle = \langle x_3, x_4\rangle \rightarrow x_1 = x_3 \wedge x_2 = x_4$

Null sets equal if its classes equal:

Axiom 7. $\text{null}\, x_1 = \text{null}\, x_2$

$$\rightarrow \exists x_3\ \text{class}\, x_3 \wedge x_1 \in x_3 \wedge x_2 \in x_3$$

---

[7] We have excluded the induction axiom because the axiom belongs to logic as for definitions as for proofs. The other two Peano axioms are presented in 2.1: 0 is a natural number, and x' is a natural number if x is a natural number.

The extensional axiom states that nonempty sets equal, if their members equal:

Axiom 8. $(\exists x_3\ x_3 \in x_1)$

$$\wedge(\exists x_3\ x_3 \in x_2) \wedge (\forall x_3\ x_3 \in x_1 \leftrightarrow x_3 \in x_2) \rightarrow x_1 = x_2$$

If $x_1 = x_2$ then values of x_1 and x_2 are different names of the same object of T_0. In particular, a natural number has different names in different number systems but they are names of the same object.

Many definitions of the theory coincide with definitions of set theory MK or a bit differ from them. The definition of the set complement is an exception.

Definition. The x_1 is complement (com) to x_2, if x_1 and x_2 are disjoint and contain all members of a class:

$$x_1 \text{com}\, x_2 \leftrightarrow \exists x_3 \text{ class}\, x_3 \wedge x_1 \subseteq x_3$$
$$\wedge\, x_2 \subseteq x_3 \wedge \forall x_4\ x_4 \in x_3 \rightarrow (x_4 \in x_1 \leftrightarrow x_4 \notin x_2)$$

We use $x_1 = \bar{x}_2$ if $x_1 \text{com}\, x_2$. And $\bar{\bar{x}} = x$.

The set operations of union and intersection are partial. Both sets in these operations should belong to the same class.

3.12. Comprehension Axiom and Russel Paradox

There are two atomic formulas in T_0 : $x_1 \in x_2$ and $x_1 = x_2$, the last formula belongs to logic. A formula is well formed (wf), if it is constructed from atomic formulas by logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ and quantifiers $\forall$, $\exists$ (the formal definition of wf belongs to logic, for example [24], p. 54). A wf can be true or false for some values of variables. If some operations of wf are partial then wf is false for undefined values of variables. Then a collection of true values of any wf is a collection of objects of T_0 (these collections can be not sets of T_0 but then they are sets of T_1).

So every wf constructs some collection of sets. In particular, the wf $x_1 \subseteq x_2$ constructs the collection of $\langle x_1, x_2 \rangle$, where for all x_2, which are not atoms, values of x_1 are all subsets of x_2. The collection is not a set in T_0 but it is the set in T_1. The wf $0 \subseteq 0$ constructs the null set in the class $P(N \times N)$ because $0 \in N$.

It is generally accepted to present a collection constructed by a wf in the form

$$x_0 = \{\langle x_1,...,x_n \rangle \mid \varphi(x_0, x_1,..., x_n)\}$$

where $\langle x_1,...,x_n \rangle \in x_0$, φ is a wf, and x_0 can be the fictive variable in φ.

We call wf *admissible*, if it constructs only sets. But any wf cannot construct atoms because wfs construct only members of some collection.

Definition. A wf is admissible (adm), if it constructs a set (not atom),

$$\text{adm}\,\varphi \leftrightarrow \exists x_0\ \neg\text{atom}\, x_0$$
$$\wedge\, \forall x_1,...,x_n\ x_0 = \{\langle x_1,...,x_n \rangle \mid \varphi(x_0, x_1,..., x_n)\}$$

By comprehension axiom any set (not atom) is constructed by some admissible wf (inverse, all admissible wf construct a set, is true by definition).

Axiom 9. $\neg\text{atom}\, x_0 \rightarrow$

$$\rightarrow \exists \varphi\, \forall x_1,..., x_n\ x_0 = \{\langle x_1,...,x_n \rangle \mid \varphi(x_0, x_1,..., x_n)\}$$

The wf $x \notin x$ is well known as Russel paradox. This wf generates all sets of T_0 but the collection of these sets is not a set of T_0. So the wf is not admissible.

3.13. Choice Axiom

The choice axiom is last. It makes the theory more informative. Cantor used this axiom implicitly.

We must give the exact definition of well ordering predicate because the predicate is used to formulate the choice axiom.

Definition. A predicate $x_1 \text{We}\, x_0$ *is '* x_1 *well orders* x_0*':*

$$x_1 \text{We}\, x_0 \leftrightarrow (|\, x_0 \,|= 0 \vee |\, x_0 \,|= 1 \rightarrow x_1 = x_0) \vee x_1 \subseteq x_0 \times x_0$$
$$\wedge\, (\forall x_2, x_3\ \langle x_2, x_3 \rangle \in x_1 \leftrightarrow \langle x_3, x_2 \rangle \notin x_1) \wedge \forall x_2\ x_2 \subseteq x_0 \wedge \neg\text{null}\, x_2$$
$$\rightarrow \exists x_3\ x_3 \in x_2 \wedge \forall x_4\ x_4 \in x_2 \wedge x_4 \neq x_3 \rightarrow \langle x_3, x_4 \rangle \in x_1$$

where

$$|\, x_0 \,|= 0 \leftrightarrow \forall x_1\ x_1 \notin x_0, \ \ |\, x_0 \,|= 1 \leftrightarrow \exists_1 x_1\ x_1 \in x_0 .$$

The first operand of the disjunction in the right part of definition (in parentheses) states that 0-member and 1-member sets are well ordered. The second operand of the disjunction contains three operands of the conjunction. The first operand deletes superfluous members of x_1 . The second operand excludes from x_1 ordered pairs $\langle x_3, x_2 \rangle$ if $\langle x_2, x_3 \rangle \in x_1$. This means that we put the strict order. The third operand (in last line) provides the existence of the least (in sense of the relation x_1) member in each subset of set x_0.

We call a set that will be well ordered *ordering* and set after ordering *ordered*. The set x_1 is called *well ordering relation*.

Each n -member ordering set has $n!$ ordered sets. Each denumerable ordering set has the nondenumerable set of ordered sets. The choice axiom states that any nondenumerable set can be well ordered.

Axiom 10. $\exists x_1\ x_1 \text{We}\, x_0$ *(for all* x_0*).*

It will be very useful the next definition.

Definition. A predicate $\mathrm{WO}\,x_0$ *is true, if* x_0 *well orders some set:*

$\mathrm{WO}\,x_0 \leftrightarrow \exists x_1\, x_0 \mathrm{We}\, x_1$

The well known definition of a well ordered set states that the set is an ordered pair, $\langle$ a set, its well ordering relation $\rangle$. But the first component of the pair is superfluous because one ordering set has many ordered sets but one ordered set has only one ordering set.

Theorem. Each ordered set with well ordering relation x_0 *has the unique ordering set* x_1: $WOx_0 \to \exists_1 x_1\, x_0 \mathrm{We}\, x_1$.

Proof. The set x_1 is the union of domain and range of x_0. □

So we can exclude the first component of the pair and name the well ordering relation *well ordered set*. This is very useful: to put some definitions, to formulate some theorems, and to prove some theorems. But we will use the ordered pair $\langle$ a set, a semantics of ordering relation $\rangle$. For example, $\langle x_0,<\rangle$ means that $x_1 < x_2$ but $\langle x_0,>\rangle$ means that $x_1 > x_2$ in $\langle x_1,x_2\rangle$, $x_1 \in x_0$, $x_2 \in x_0$. And $\langle x_0,\subset\rangle$ means that $x_1 \subset x_2$ in $\langle x_1,x_2\rangle$.

The next definition finds ordering set for given ordering relation.

Definition. Let the map OS take each ordering relation to ordering set:

$$\mathrm{WO}\,x_1 \to (x_0 \in \mathrm{OS}(x_1) \leftrightarrow \exists x_2\, \langle x_0,x_2\rangle \in x_1 \vee \langle x_2,x_0\rangle \in x_1)$$

3.14. Cantor Ordinals

A Cantor ordinal is a collection of all similar ordered sets.

Definition. Two sets are similar ordered (≈), if

$$x_1 \approx x_2 \leftrightarrow |\,x_1\,| = 0 \wedge |\,x_2\,| = 0 \vee |\,x_1\,| = 1 \wedge$$

$$\wedge\, |\,x_2\,| = 1 \vee \mathrm{WO}\,x_1 \wedge \mathrm{WO}\,x_2 \wedge \exists x_0\, x_0 \subseteq \mathrm{OS}(x_1) \times \mathrm{OS}(x_2) \wedge \mathrm{Fnc}_1(x_0)$$

$$\wedge \forall x_3,x_4\, \langle x_3,x_4\rangle \in x_1 \to$$

$$\to \exists x_5,x_6\, \langle x_5,x_6\rangle \in x_2 \wedge \langle x_3,x_5\rangle \in x_0 \wedge \langle x_4,x_6\rangle \in x_0$$

where $\mathrm{Fnc}_1(x_3)$ means that x_3 is a one-one function:

$$\mathrm{Fnc}_1(x_1) \leftrightarrow (\exists x_2,x_3\, x_1 \subseteq x_2 \times x_3) \wedge \forall x_2,x_3,x_4$$

$$((\langle x_2,x_3\rangle \in x_1 \wedge \langle x_2,x_4\rangle \in x_1 \to x_3 = x_4)$$
$$\wedge\, ((\langle x_2,x_3\rangle \in x_1 \wedge \langle x_4,x_3\rangle \in x_1 \to x_2 = x_4)$$

The first two operands of disjunctions in right part of the first definition state that all 0-member sets are similar ordered and all 1-member sets are similar ordered. The third operand of the disjunction defines the other similar ordered sets . This operand has five operands of conjunctions. The first two operands demand that x_1 and x_2 must be well ordered. The next two operands state that there is one-one function for all members of sets ordering by x_1 and x_2. The last operand (in the last line) states that mapping of the ordering sets keeps their order.

Cantor ordinals are not sets in T_0 but they are sets in T_1. Members of Cantor ordinals are constructed from sets of T_0. So we must define Cantors ordinals in T_0 but the definition is well only in T_1.

Definition. The predicate COrd defines Cantor ordinals,

$$\mathrm{COrd}\,x_0 \leftrightarrow \forall x_1,x_2\, x_1 \in x_0 \wedge x_2 \in x_0 \to x_1 \approx x_2$$

We can use one of similar sets as a representative of a Cantor ordinal. We will call these representatives briefly *ordinals*.

3.15. Quasi-Ordinals

Quasi-ordinals will be used to construct ordinals and cardinals.

We denote natural number 0 by $\dot{0}$. We denote $\{\dot{0}\}$ by $\dot{1}$, $\{\dot{0},\{\dot{0}\}\}=\{\dot{0},\dot{1}\}$ by $\dot{2}$, $\{\dot{0},\{\dot{0}\},\{\dot{0},\{\dot{0}\}\}\}=\{\dot{0},\dot{1},\dot{2}\}$ by $\dot{3}$ and so on. We call these sets *quasi-ordinals* and denote their collection[8] by $\dot{\alpha}$.

Definition (non-predicative).

$$x \in \dot{\alpha} \leftrightarrow x = \dot{0} \vee x = \dot{\alpha}$$

We have $\dot{\alpha} = \{\dot{0}\}$ at step 0, $\dot{\alpha} = \{\dot{0},\{\dot{0}\}\} = \{\dot{0},\dot{1}\}$ at step 1, and so on. In particular, we get the set $\{\dot{0},\dot{1},...,\dot{n}\}$ at step n. So every finite member can be got and there is a set containing all finite members. This set we denote by $\dot{\omega}$. Then we have set $\{\dot{0},\dot{1},...,\dot{\omega}\}$, which we denote by $\omega\dot{+}1$. And then we have set $\{\dot{0},\dot{1},...,\dot{\omega},\omega\dot{+}1\}$, which we denote by $\omega\dot{+}2$. And so on.

This construction of $\dot{\alpha}$ is ended, if the next step generates the new collection such that is not a set (in T_1).

We can put the order $\langle\dot{\alpha},\le\rangle$.

Definition. The predicate $\le_{\dot{\alpha}}$ *puts the natural linear order,*

$$x_1 \le_{\dot{\alpha}} x_2 \leftrightarrow x_1 \in \dot{\alpha} \wedge x_2 \in \dot{\alpha} \wedge (x_1 = x_2 \vee x_1 \subset x_2)$$

---

[8] Except $\dot{0}$ and $\dot{1}$ all quasi-ordinals are sets in T_1, they are not sets in T_0 (and $\dot{\alpha}$ is the set only in T_2). We can construct ordinals and cardinals without quasi-ordinals (using only sets of T_0) but such construction is very complex. Results, that use quasi-ordinals and results that use only sets of T_0, are the same.

Hear $x_1 \subset x_2$ points out that there is a well ordering relation with members $\langle x_1, x_2 \rangle$. We will call the well ordering relation *natural*.

We will use the next definition to construct cardinals.

Definition The predicate min defines a minimal quasi-ordinal into some set of quasi-ordinals,

$$x_1 \min x_2 \leftrightarrow x_1 \in x_2 \wedge \forall x_3\; x_3 \in x_2 \to x_3 \in \dot{\alpha} \wedge x_3 \leq_{\dot{\alpha}} x_1$$

3.16. Ordinals

We denote finite ordinals by natural numbers. And we use quasi-ordinals to construct the collection of all ordinals.

The first two ordinals 0 and 1 are $\dot{0}$ and $\dot{1}$ because they are representatives of 0-member and 1-member sets, which are well ordered. But the other ordinals are not quasi-ordinals because the other quasi-ordinals are not well ordered. So the next two ordinals 2 and 3 are $\{\langle \dot{0}, \dot{1} \rangle\}$ and $\{\langle \dot{0}, \dot{1} \rangle, \langle \dot{0}, \dot{2} \rangle, \langle \dot{1}, \dot{2} \rangle\}$. The ordinal 2 has quasi-ordinal $\dot{2}$ as the next to the all components in the ordinal. And the ordinal 3 has quasi-ordinal $\dot{3}$ as the next to the all components in the ordinal. The ordinal ω is $\{\langle \dot{0}, \dot{1} \rangle, \langle \dot{0}, \dot{2} \rangle, ..., \langle \dot{1}, \dot{2} \rangle, \langle \dot{1}, \dot{3} \rangle, ...\}$ and it has the quasi-ordinal $\dot{\omega}$ as the next to all components of the ordinal. So every ordinal has the quasi-ordinal as the next to all components of the ordinal. This means there is a one-one correspondence between ordinals and quasi-ordinals. Cantor used the correspondence implicitly.

Definition. The collection of all ordinals is Ord,

$$x_0 \in \mathrm{Ord} \leftrightarrow x_0 = \dot{0} \vee x_0 = \dot{1}$$

$$\vee (\exists x_1\; \mathrm{COrd}\, x_1 \wedge x_0 \in x_1) \wedge \forall x_1, x_2\; \langle x_1, x_2 \rangle \in x_0$$

$$\to x_1 \in \dot{\alpha} \wedge x_2 \in \dot{\alpha} \wedge x_1 <_{\dot{\alpha}} x_2$$

There are three operands of the disjunction. The first two operands state that $\dot{0}$ and $\dot{1}$ are ordinals. The third operand states that ordinals are members of some Cantor ordinal and members of ordinals are ordered pairs such that both components are quasi-ordinals and the first component is less than the second component.

We can put the order for ordinals.

Definition. The predicate $\leq_o$ puts the natural linear order for ordinals,

$$x_1 \leq_o x_2 \leftrightarrow x_1 \in \mathrm{Ord} \wedge x_2 \in \mathrm{Ord} \wedge (x_1 = x_2 \vee x_1 \subset x_2)$$

We have used ordinals as subscripts.

2.17. Cantor Cardinals and Cardinal Representatives

A Cantor cardinal is a collection of all isomorphic sets.

Definition. Two sets are isomorphic ($\cong$), if there is a one-one correspondence between them,

$$x_1 \cong x_2 \leftrightarrow \exists x_0\; x_0 \subseteq x_1 \times x_2 \wedge \mathrm{Fnc}_1(x_0)$$

where $\mathrm{Fnc}_1(x_0)$ means that x_0 is a one-one function.

Cantor cardinals are not sets in T_0 but they are sets in T_1. Because all members of a Cantor cardinal belong to T_0, we define Cantor cardinals in T_0.

Definition. The predicate CCar *defines Cantor cardinals,*

$$\mathrm{CCar}\, x_0 \leftrightarrow \forall x_1, x_2\; x_1 \in x_0 \wedge x_2 \in x_0 \to x_1 \cong x_2$$

We can use one of isomorphic sets as a representative of a Cantor cardinal. We will call these representatives briefly *cardinals*.

All ordinals are members of Cantor cardinals but there are Cantor ordinals without ordinals. For example, ordinals with two members do not exist, hence the Cantor cardinal of two member sets does not contain ordinals. But quasi-ordinals are members of every Cantor cardinals. We use a minimal quasi-ordinal of a Cantor cardinal as a cardinal[9].

Definition. Collection Car *contains all cardinals,*

$$x_0 \in \mathrm{Car} \leftrightarrow (\exists x_1 \mathrm{CCar}\, x_1 \wedge x_0 \in x_1) \wedge x_0 \in \dot{\alpha}$$

$$\wedge \exists x_1\; (\forall x_2\; x_2 \in x_1 \leftrightarrow x_2 \cong x_0 \wedge x_2 \in \dot{\alpha}) \wedge x_0 \min x_1$$

We state (in the first line) that cardinal x_0 is a member of a Cantor cardinal and an quasi-ordinal. We create (in the othert lines) set x_1 of all quasi-ordinals which are isomorphic to x_0 and we demand that x_0 is minimal into the set.

A finite cardinal with n members is equal to quasi-ordinal $\dot{n}$. The cardinals are not natural numbers but we will denote them by the numbers: 0, 1,... . This denotation is generally accepted.

We can put the natural order for infinite cardinals. This means that all infinite cardinals can be numbered by ordinals α by beginning with 0. It is generally accepted to denote infinite cardinals by $\aleph_\alpha$.

For any set x, $|x|$ is a cardinal which is isomorphic to x.

3.18. Absent Existence Axioms

The other existence axioms are absent because the external theory T_1 has the definition of sets of T_0 and by definition all introduced above objects are sets. This means they exist. In particular, a direct product of sets is a set and a power set is a set, too. So we need not give axioms of existence of these sets.

[9] The other way is to use quasi-ordinals with minimal type (D. Scott, 1965, [25]).

It is natural that some definitions of objects of T_0 present in T_1. In particular the definition of variable domain of T_0 can be given only in T_1. And some theorems that cannot be proved in T_0 will be proved in T_1.

3.19. Applied Mathematics

The theory T_0 can satisfy almost all needs of applied mathematics. As an example we shall consider Hilbert space.

Hilbert space is infinite-dimensional. Hence this space uses an infinite number of direct product operations $\times$.

The operation $\times$ is used only finite times in the theory. Nevertheless Hilbert space exists in the theory. We can use the infinite sequence of real numbers as a point in Hilbert space instead of an infinite number of the operations. Because real numbers belong to the class $P(N^3)$, a member of the sequence belongs to the class $N \times P(N^3)$ (members of sequences are numbered) and the sequence belongs to the class $P(N \times P(N^3))$. Hence Hilbert space belongs to the class $P^2(N \times P(N^3))$.

Similarly we can construct Hilbert vector space over the field of real or complex numbers.

4. Set Theory T_1

4.1. Signature, Axioms, and notation of Variables

We inherit the signature and axioms of the theory from theory T_0. Hence theory T_1 is an extension of theory T_0.

We use the new notation of variables of the theory: $x_0^1, x_1^1, x_2^1, \ldots$, because the domain of these variables is not the domain of variables of T_0. Accordingly we must change the notation of variables in the axioms of the theory T_0.

The domain of the new variables is the universe U_2. This universe is not defined in the theory. But universe U_1 of all sets of T_0 is defined.

Before we shall define what collections of sets of T_0 are sets of T_0 and what collections are not sets of T_0.

4.2. Definition of Sets of T_0

This definition can be presented only in T_1.

Definition (non-predicative). A set of T_0 *is:* N *and its members, a direct product of sets of* T_0 *and its members, a power set of a set of* T_0 *and its members:*

$$\begin{aligned} &\mathrm{set}_0 x_0^1 \leftrightarrow x_0^1 = N \lor x_0^1 \in N \\ &\lor \exists x_1^1 \exists x_2^1\, \mathrm{set}_0 x_1^1 \land \mathrm{set}_0 x_2^1 \\ &\land (x_0^1 = x_1^1 \times x_2^1 \lor x_0^1 \in x_1^1 \times x_2^1 \lor x_0^1 = P(x_1^1) \\ &\lor x_0^1 \in P(x_1^1)) \end{aligned}$$

where set_0 is a set of T_0.

By definition, all these objects exist.

4.3. Completeness of Classification in T_0

The definition of set_0 specifies what sets are of T_0 and what sets are not of T_0. Hence now we can prove that the classification of sets of T_0 covers all the sets.

Theorem. Any set of T_0 *is a member of a class of* T_0*:*

$$\mathrm{set}_0 x_0^1 \to \exists_1 x_1^1\, \mathrm{set}_0 x_1^1 \land \mathrm{class}_0 x_1^1 \land x_0^1 \in x_1^1$$

where class_0 is a class of T_0.

In the other words, any set of T_0 has a type because any class is numbered by a finite ordinal and any member of a class has type which equals the ordinal number of the class.

Proof. We use the induction rule. Let class N and its members exist at the step 0. The class N is a member of the class $P(N)$, members of N are members of the class N. So the theorem is true at this step.

We assume that only members of classes exist at a step n. And we shall prove that only members of classes are generated at the step $n+1$.

By definition, the sets generated at the step $n+1$ are only:

- any direct product of sets that are generated at the previous steps,
- any member of the direct products,
- any power set of sets that are generated at the previous steps,
- any member of the power set.

Let:

- sets x_1^1 and x_2^1 (but not atoms) be generated at previous steps,
- these sets be members of classes x_3^1 and x_4^1 respectively,
- the sets $x_1^1 \times x_2^1$ and $P(x_1^1)$ be generated at the step $n+1$.

The set $x_1^1 \times x_2^1$ is a subset of the class $x_3^1 \times x_4^1$ and a member of the class $P(x_3^1 \times x_4^1)$. Members of the set are members of the class $x_3^1 \times x_4^1$. Then the theorem is true for this set and its members.

The set $P(x_1^1)$ is a subset of the class $P(x_3^1)$ and a member of the class $P^2(x_3^1)$. Members of the set are members of the class $P(x_3^1)$. The theorem is true for this set and its members, too. This completes the proof. □

Remark. The given theorem illustrates that some properties of sets of theory T_0 unprovable in the theory can be proved in a more powerful theory.

4.4. Non-Transitivity and Foundation (Regularity) in T_0

We use the completeness of set classification in T_0 to prove non-transitivity and foundation of sets of T_0.

Theorem (of non-transitivity). The transitivity relation is not realized in the theory T_0:

$$x_1^1 \in x_2^1 \wedge x_2^1 \in x_3^1 \wedge \mathrm{set}_0 x_3^1 \to x_1^1 \notin x_3^1$$

In other words, set x_3^1 of T_0 does not contain both set x_2^1 and its member x_1^1.

Proof. Set x_3 does not contain both set x_2^1 and its member x_1^1 because x_2^1 and x_1^1 have different types. □

Corollary. Every set of T_0 does not contain itself: $\mathrm{set}_0 x^1 \to x^1 \notin x^1$.

Proof. A set and its members have different types. □

Theorem (of foundation). Every nonempty set x_1^1 of T_0 contains a set x_2^1 such that x_1^1 and x_2^1 do not contain common members:

$$\mathrm{set}_0 x_1^1 \wedge (\exists x_0^1\, x_0^1 \in x_1^1) \to \exists x_2^1\, x_2^1 \in x_1^1 \wedge \forall x_0^1\, x_0^1 \in x_2^1 \to x_0^1 \notin x_1^1$$

Proof. By non-transitivity theorem every nonempty set x_1^1 of T_0 does not contain a set x_2^1 and its member. □

4.5. Construction of Sets of Theory T_1

First we shall construct the set of all sets of T_0.

Definition. The set of all sets of T_0 is universe U_1:

$$x_0^1 \in U_1 \leftrightarrow \mathrm{set}_0 x_0^1$$

Now, using universe U_1 we can construct all other sets.

Definition. A set $x_1^1 \times x_2^1$ is a direct product of sets x_1^1 and x_2^1, if x_1^1 and x_2^1 contain ordered pairs which first components are members of x_1^1 and second components are members of x_2^1:

$$\neg\mathrm{atom}\, x_1^1 \wedge \neg\mathrm{atom}\, x_2^1 \to (x_0^1 \in x_1^1 \times x_2^1$$

$$\leftrightarrow \exists x_3^1 \exists x_4^1\, x_3^1 \in x_1^1 \wedge x_4^1 \in x_2^1 \wedge x_0^1 = \langle x_3^1, x_4^1 \rangle)$$

Definition. A set x_1^1 is a subset of a set x_2^1, if both sets are not atoms and all members of x_1^1 are members of x_2^1:

$$x_1^1 \subseteq x_2^1 \leftrightarrow \neg\mathrm{atom}\, x_1^1 \wedge \neg\mathrm{atom}\, x_2^1 \wedge \forall x_3^1\, x_3^1 \in x_1^1 \to x_3^1 \in x_2^1$$

If $x_1^1 \neq x_2^1$ then $x_1^1 \subset x_2^1$.

Definition. The set of all subsets of a set x_1^1 is the power set $P(x_1^1)$:

$$\neg\mathrm{atom}\, x_1^1 \to (x_0^1 \in P(x_1^1) \leftrightarrow x_0^1 \subseteq x_1^1)$$

4.6. Classes of Sets

The natural classification of sets differs from the natural classification of sets of T_0.

Theorem. Sets $U_1 \times U_1$ and U_1 have common members.

Proof. If x_1^1 is a set of T_0 then set $\langle x_1^1, x_1^1 \rangle$ is a member of U_1 and a member of $U_1 \times U_1$, □

Theorem. Sets $P(U_1)$ and U_1 have common members.

Proof. If x_1^1 is a set of T_0 then set $\{x_1^1\}$ is a member of U_1 and a member of $P(U_1)$. □

But natural classification of sets exists, if we construct classes by partial operations. These classes are disjoint and cover all sets.

Definition. A partial direct product $x_1^1 \hat{\times} x_2^1$ contains only members which are not members of U_1 : $x_0^1 \in x_1^1 \hat{\times} x_2^1 \leftrightarrow x_0^1 \in x_1^1 \times x_2^1 \wedge x_0^1 \notin U_1$

Definition. A partial power set $\hat{P}(x_1^1)$ contains only members which are not members of U_1 : $x_0^1 \in \hat{P}(x_1^1) \leftrightarrow x_0^1 \in P(x_1^1) \wedge x_0^1 \notin U_1$

Definition. A class is:

- the universe U_1,
- a nonempty partial direct product of classes,
- a partial power family of a class:

$$\mathrm{class}_1 x_0^1 \leftrightarrow x_0^1 = U_1$$
$$\vee \exists x_1^1 \exists x_2^1\, \mathrm{class}_1 x_1^1 \wedge \mathrm{class}_1 x_2^1 \to x_0^1 = x_1^1 \hat{\times} x_2^1 \vee x_0^1 = \hat{P}(x_1^1)$$

Classes can be numbered by ordinals $\omega \le \alpha < \omega 2$ in the sequence of their construction:

$$U_1; U_1 \hat{\times} U_1, \hat{P}(U_1); U_1 \hat{\times} (U_1 \hat{\times} U_1), U_1 \hat{\times} \hat{P}(U_1), U_1 \hat{\times}$$

$$\hat{\times} U_1 \hat{\times} U_1, \hat{P}(U_1 \hat{\times} U_1), U_1 \hat{\times} U_1 \hat{\times} (U_1 \hat{\times} U_1),$$

$$U_1 \hat{\times} U_1 \hat{\times} \hat{P}(U_1), \hat{P}(U_1) \hat{\times} U_1, \hat{P}^2(U_1), \hat{P}(U_1) \hat{\times}$$

$$\hat{\times}(U_1 \hat{\times} U_1), \hat{P}(U_1) \hat{\times} \hat{P}(U_1); \ldots$$

The semicolons separate the classes constructed at the next step.

The classes can be numbered by ordinal numbers beginning at ω. So class U_1 has ordinal number ω, class $U_1 \hat{\times} U_1$ has ordinal number $\omega+1$, and so on. But the ordinal numbers are less than $\omega 2$.

This means that T_1 is a type theory as a corollary of the classification. Members of U_1 have all spectrum of finite types. But members of an another class have the same type which equals the ordinal number of the class minus 1. In particular, members of $U_1 \times U_1$ have type ω, U_1 and the other members of $P(U_1)$ have type $\omega+1$.

4.7. Disjointness of Classes and Completeness

The classes have no common members.

Lemma. The set $\hat{P}(x_1^1)$ *and the universe* U_1 *have no common members:* $x_0^1 \notin \hat{P}(x_1^1) \vee x_0^1 \notin U_1$.

Proof. By definition members of $\hat{P}(x_1^1)$ are not members of U_1. □

Lemma. The sets $x_1^1 \hat{\times} x_2^1$ *and* $\hat{P}(x_3^1)$ *have no common members:* $x_0^1 \notin x_1^1 \hat{\times} x_2^1 \vee x_0^1 \notin \hat{P}(x_3^1)$.

Proof. Members of $x_1^1 \hat{\times} x_2^1$ are atoms but members of $\hat{P}(x_1^1)$ are not atoms. □

Lemma. The sets $x_1^1 \hat{\times} x_2^1$ *and the universe* U_1 *have no common members:* $x_0^1 \notin x_1^1 \hat{\times} x_2^1 \vee x_0^1 \notin U_1$.

Proof. Members of $x_1^1 \hat{\times} x_2^1$ are not members of U_1. □

Theorem. Two different classes have no common members:

$$\text{class}_1 x_1^1 \wedge \text{class}_1 x_2^1 \wedge x_1^1 \neq x_2^1 \rightarrow x_0^1 \notin x_1^1 \vee x_0^1 \notin x_2^1$$

Proof. We use the symbol-by-symbol comparison of parenthesis-free records of classes. If the first compared pair of symbols has equal symbols then we pass to comparison of the next pair. By the finite descent rule there will be a pair of unequal symbols. It follows from the lemmas that sets generated by a pair of unequal symbols have no common members. □

The classification is complete. All sets excluded by the partial operations are members of the class U_1. But completeness of the classification is proved in the theory T_2.

4.8. Transitivity and Null-Sets

If we do not use partial operations then the transitivity relation can be realized.

Theorem (of transitivity). The transitivity relation can be realized in the theory T_1 *for some sets,*

$$\exists x_1^1 \exists x_2^1 \exists x_3^1 \; x_1^1 \in x_2^1 \wedge x_2^1 \in x_3^1 \rightarrow x_1^1 \in x_3^1$$

Proof. Because N is both a member and a subset of U_1, then

$$N \in U_1 \wedge U_1 \in P(U_1) \rightarrow N \in P(U_1)$$

By axiom 3 each set (not atom) has only one empty subset (null-set). In particular, the universe U_1 has only one empty subset. But U_1 contains infinite number of null-sets because U_1 contains null-sets of each class of the theory T_0. □

4.9. Foundation (Regularity)

Theorem (of foundation). Every nonempty set x_1^1 contains a set x_2^1 such that x_1^1 and x_2^1 do not contain common members:

$$(\exists x_0^1 \; x_0^1 \in x_1^1) \rightarrow \exists x_2^1 \; x_2^1 \in x_1^1 \wedge \forall x_3^1 \; x_3^1 \notin x_1^1 \vee x_3^1 \notin x_2^1$$

Proof. All types are numerated by ordinals. This means that any subset of types contains the minimal type. Hence set x_1^1 contains set x_2^1 with a minimal type. Then members of x_2^1 are not contained in x_1^1 because they have types less than the minimal type. □

5. Natural Theory T_{On}

5.1. Signature, Axioms, and Variable Notation

There are no limit to construct ordinals α because there are no limit to construct quasi-ordinals. But the collection of all quasi-ordinals $\dot{\alpha}$ exists. The collection is $\dot{\text{O}}$n. This means that On exists too, and On is next after all ordinals. We call On *up-ordinal* (see 6.6 for more details). So the theory T_{On} exists.

The signature of the theory is inherited from theory T_0. The axioms are inherited from T_0, too. Then theory T_{On} is an extension of theory T_0 and all theories T_α. Objects of T_{On} are families and some families are not sets. Hence the families are an analog of classes in MK set theory. Families, which are not sets, are up-sets. They are an analog of proper classes of MK.

The variables are X_0, X_1, X_2,... . The domain of these variables is the universe U_+ containing all families. This universe is not a family and the definition of universe belongs to the external theory.

5.2. Construction of Families

We use the universe U to construct the other families.

Definition. The union of sets of all previous theories is the universe U *:*

$$X_0 \in U \leftrightarrow \exists \alpha \operatorname{set}_\alpha X_0$$

Definition. Family $X_1 \times X_2$ *is a direct product of* X_1 *and* X_2, *if* $X_1 \times X_2$ *contains all ordered pairs which first components are members of* X_1 *and second components are members of* X_2 *:*

$$\neg \text{atom}\, X_1 \wedge \neg \text{atom}\, X_2$$

$$\rightarrow (X_0 \in X_1 \times X_2 \leftrightarrow \exists X_3, X_4\ X_3 \in X_1$$

$$\wedge X_4 \in X_2 \wedge X_0 = \langle X_3, X_4 \rangle)$$

Definition. Family X_1 *is a subfamily of* X_2, *if* X_1 *and* X_2 *are not atoms and if all members of* X_1 *are members of* X_2 *:*

$$X_1 \subseteq X_2 \leftrightarrow \neg \text{atom}\, X_1 \wedge \neg \text{atom}\, X_2$$

$$\wedge \forall X_3\ X_3 \in X_1 \rightarrow X_3 \in X_2$$

We do not use the power family operation to stop any growth of family cardinals.

5.3. Sets and Up-Sets

Sets are members of families, up-sets are not members of families.

Definition. The predicate set *defines families that are members of the other families,*

$$\text{set}\, X_0 \leftrightarrow \exists X_1\ X_0 \in X_1$$

Members of U are sets.

Definition. The predicate up-set defines families that are not members of the other families,

$$\text{up}-\text{set}\, X_0 \leftrightarrow \forall X_1\ X_0 \notin X_1$$

Below we will show that families are up-sets if they are not members of *U*.

The empty subfamily of U is not a member of U.

Definition. The empty subfamily of the universe U *is the family* $\varnothing$ *:*

$$X_1 = \varnothing \leftrightarrow X_1 \subseteq U \wedge \forall X_0\ X_0 \notin X_1$$

5.4. Classes of Families

The classification exists but classes containing up-sets are not families because otherwise up-sets are members of a family. So the well definition of classes belongs to $T_{\text{On}+1}$ but we will define classes in T_{On}.

We use the partial operations of direct product $\hat{\times}$ and of power family $\hat{P}$ to create classes.

Definition. The partial direct product $\hat{\times}$ *constructs families which are not members of* U *:*

$$X_0 \in X_1 \hat{\times} X_2 \leftrightarrow X_0 \in X_1 \times X_2 \wedge X_0 \notin U$$

Some partial direct products are empty. For example $U \hat{\times} U$ is empty. But $U \hat{\times} U$ are not members of *U* or any family. Hence $U \hat{\times} U$ is an up-set.

Definition. The partial power family $\hat{P}$ *constructs families which are not members of* U *:*

$$\neg \text{atom}\, X_0 \rightarrow (X_0 \in \hat{P}(X_1) \leftrightarrow X_0 \subseteq X_1 \wedge X_0 \notin U)$$

Definition. A class of families is the universe U, *a collection constructed from classes by the partial family power and a nonempty collection constructed from classes by the partial direct product,*

$$\begin{aligned}
&\text{class}_{\text{On}}\, X_0 \leftrightarrow X_0 = U \\
&\vee\ (\exists X_1\ \text{class}_{\text{On}}\ \wedge X_0 = \hat{P}(X_1)) \\
&\vee \exists X_1, X_2\ \text{class}_{\text{On}}\, X_1 \wedge \text{class}_{\text{On}}\, X_2 \\
&\wedge (\exists X_3\ X_3 \in X_1 \quad \times X_2) \wedge X_0 = X_1 \hat{\times} X_2
\end{aligned}$$

Classes can be well ordered in the sequence of their construction:

$$U; \hat{P}(U); U \hat{\times} \hat{P}(U), \hat{P}(U \hat{\times} U), \hat{P}(U) \hat{\times} U,$$

$$\hat{P}(U) \hat{\times} \hat{P}(U);\ \ldots$$

Here the semicolons separate sets constructed at the next step. Except U all classes contain $\hat{P}$. Classes with $\hat{P}^2$ do not exist. Only class U is a family. Hence all up-sets are not members of families.

The class sequence of parenthesis-free record is

$$U; \hat{P}U; \hat{\times} U \hat{P} U, \hat{P} \hat{\times} UU, \hat{\times} \hat{P} UU, \hat{\times} \hat{P} U \hat{P} U;\ \ldots$$

Classes, which record begins at the symbol $\hat{\times}$, contain empty up-classes. The other classes (except U and $\hat{P}U$) have records beginning at symbols $\hat{P}\hat{\times}$.

The classes U, PU, $\hat{\times} U \hat{P} U$, $\hat{\times} \hat{P} UU$, and $\hat{\times} \hat{P} U \hat{P} U$ are infinite.

The class $\hat{P} \hat{\times} UU$ is finite, it has only one member, the member is the empty up-set $\{\}^{\text{On}+2}$.

The classes can be numbered in the sequence of their construction by beginning at On. So the class U has ordinal number On, the class $\hat{P}(U)$ has ordinal number $\text{On}+1$, and so on. But all ordinal numbers are less than $\text{On}+\omega$.

This means that T_{On} is a type theory. Members of U have types that are any ordinals $\alpha < \text{On}$. Members of an other class have the same type that equals ordinal number of the class minus 1. In particular, U and the other members of $\hat{P}(U)$ have type On, members of $U\hat{\times}\hat{P}(U)$ have type On+1.

The class $\hat{\times}\hat{P}U\hat{P}U$ contains ordered pairs which components are up-sets. So up-sets cannot be members of families but can be components of ordered pairs. This means any up-set can be well ordered and the choice axiom is well in the theory.

5.5. Disjointness of Classes

The classes have no common members.

Lemma. The set $\hat{P}(X)$ and the universe U have no common members: $X_0 \notin \hat{P}(X_1) \vee X_0 \notin U$.

Proof. By definition members of $\hat{P}(X)$ are not members of U. □

Lemma. The sets $X_1 \hat{\times} X_2$ and $\hat{P}(X)$ have no common members: $X_0 \notin X_1 \hat{\times} X_2 \vee X_0 \notin \hat{P}(X_3)$.

Proof. Members of $X_1 \hat{\times} X_2$ are atoms but members of $\hat{P}(X_1)$ are not atoms. □

Lemma. The sets $X_1 \hat{\times} X_2$ and the universe U have no common members: $X_0 \notin X_1 \hat{\times} X_2 \vee X_0 \notin U$.

Proof. Members of $X_1 \hat{\times} X_2$ are not members of U_1. □

Theorem. Two different classes have no common members: $\mathrm{class}_{\mathrm{On}}\, X_1 \wedge \mathrm{class}_{\mathrm{On}}\, X_2 \wedge X_1 \neq X_2 \to$

$$\to X_0 \notin X_1 \vee X_0 \notin X_2 .$$

Proof. We use the symbol-by-symbol comparison of parenthesis-free records of classes. If the first compared pair of symbols has equal symbols we pass to comparison of the next pair. By the finite descent rule there will be a pair of unequal symbols. It follows from the lemmas that sets generated by a pair of unequal symbols have no common members. □

The completeness of classification will be proved in the theory $T_{\mathrm{On}+1}$.

5.6. Cantor Up-Ordinals and Up-Ordinal Representatives

A Cantor up-ordinal is a collection of all similar ordered up-sets, its representative is up-ordinal. We have numerated classes by up-ordinals but have not defined the up-ordinals.

Before we must define the up-quasi-ordinal $\dot{\mathrm{O}}\mathrm{n}$ and the predicate $\leq_{\dot{\mathrm{O}}\mathrm{n}}$.

Definition. The up-quasi-ordinal $\dot{\mathrm{O}}n$ is

$$X \in \dot{\mathrm{O}}\mathrm{n} \leftrightarrow \exists \alpha\, X \in \dot{\alpha}_\alpha$$

where $\dot{\alpha}_\alpha$ is a quasi-ordinal defined in a theory T_α.

The predicate $\leq_{\dot{\mathrm{O}}\mathrm{n}}$ is

$$X_1 \leq_{\dot{\mathrm{O}}\mathrm{n}} X_2 \leftrightarrow X_1 \in \dot{\mathrm{O}}\mathrm{n} \wedge X_2 \in \dot{\mathrm{O}}\mathrm{n}$$
$$\wedge (X_1 = X_2 \vee X_1 \subseteq X_2)$$

Definition. The up-ordinal On is

$$X_0 \in \mathrm{Ord} \leftrightarrow \forall X_1, X_2\, \langle X_1, X_2 \rangle \in X_0$$
$$\to X_1 \in \dot{\mathrm{O}}\mathrm{n} \wedge X_2 \in \dot{\mathrm{O}}\mathrm{n} \wedge X_1 \leq_{\dot{\mathrm{O}}\mathrm{n}} X_2$$

Next we construct $\mathrm{On} \dot{+} 1$: $\mathrm{On} \dot{+} 1 = \dot{\mathrm{O}}\mathrm{n} \cup \{\dot{\mathrm{O}}\mathrm{n}\}$, and construct On+1 by using $\mathrm{On} \dot{+} 1$. (by analogy with construction of On). The up-ordinal On+2 is constructed by using up-quasi-ordinal $\mathrm{On} \dot{+} 2$:

$$\mathrm{On} \dot{+} 2 = \mathrm{On} \dot{+} 1 \cup \{\mathrm{On} \dot{+} 1\}$$

And so on.

Now we have all up-ordinals less than Om+ ω.

The other up-ordinals exist but they are unneeded.

5.7. Cantor Up-Cardinal and Up-Cardinal Representative

We call $\aleph_{\mathrm{On}}$ and all next alephs *up-cardinals*. The up-cardinal $\aleph_{\mathrm{On}}$ is the next to all cardinals.

The universe U has up-cardinal $\aleph_{\mathrm{On}}$. All subsets of U cannot have up-cardinal greater than $\aleph_{\mathrm{On}}$. And a result of direct product cannot have up-cardinal greater than $\aleph_{\mathrm{On}}$ because $|X_1 \times X_2| = \max(|X_1|, |X_2|)$. This means that only one up-cardinal exists and only one Cantor up-cardinal exists.

By definition $\aleph_{\mathrm{On}}$ is $\dot{\mathrm{O}}$n. And $|\mathrm{On}| = \dot{\mathrm{O}}$n. Hence $\aleph_{\mathrm{On}} = |\mathrm{On}|$. This means that up-cardinal $\aleph_{\mathrm{On}}$ is regular. More that, the up-cardinal is strongly inaccessible.

There are empty up-sets but no other finite up-sets. Some infinite up-sets can have cardinals. Really, the up-set of all empty subsets of U_α has up-cardinal $\aleph_{\mathrm{On}}$ but some up-sets, which are subfamily of the up-set and cofinal[10] in the up-set, can have cardinals.

5.8. Transitivity and Foundation

If we do not use partial operations then the transitivity relation can be realized.

Theorem (of transitivity). The transitivity relation can be realized in the theory T_{On} for some families:

$$\exists X_1 \exists X_2 \exists X_3\; X_1 \in X_2 \wedge X_2 \in X_3 \to X_1 \in X_3$$

Proof. Because N is both a member and a subset of U then $N \in U \wedge U \in P(U) \to N \in P(U)$. □

Any family has a foundation but the theorem of foundation can be proved only in the external theory (see 6.4)

[10] A subfamily X_0 of a well ordered family X_1 is cofinal in X_1 if $\sup X_0 = \sup X_1$. And the up-set can be well ordered.

5.9. Admissible Wfs and Russel Paradox

A wf is *admissible*, if it constructs families. But no wf constructs atoms.

Definition. A wf is admissible (adm), if it constructs a family (not atom),

$$\mathrm{adm}\,\varphi \leftrightarrow \exists X_0 \neg \mathrm{atom}\, X_0 \\ \wedge \forall X_1,...,X_n\; X_0 = \{\langle X_1,...,X_n\rangle \mid \varphi(X_0, X_1,...,X_n)\}$$

where φ is a wf and X_0 can be fictitious in φ.

The wf $X \notin X$ is well known as Russel paradox. This wf generates the universe U_+ but the universe is not a family. So the wf is not admissible. The wf $X \in X$ is the negation of $X \notin X$ and generates the complement to U^+. This complement is the empty collection and is not a family, too.

6. External Theory $T_{\mathrm{On}+1}$

6.1. Signature, Axioms, and Notation of Variables

We inherit the signature and the axioms of the theory from the theory T_0. Hence $T_{\mathrm{On}+1}$ is an extension of T_0. And $T_{\mathrm{On}+1}$ is external to T_{On}. We call objects of $T_{\mathrm{On}+1}$ *up-families* if they are not families.

We use the next notation of variables: X_0^+, X_1^+, X_2^+ .... The range of these variables is an universe and the definition of the universe does not exist in the theory. But the definition of the universe U_+ containing all families of T_{On} exists. Before giving the definition of U_+ we must formalize the definition of families and set some properties of families.

6.2. Definition of Families

This definition belongs $T_{\mathrm{On}+1}$ but we presemt it here.

Definition (non-predicative). A family of T_{On} is: U and its members, a direct product of families and its members, a subfamily of a family:

$$\mathrm{set}_{\mathrm{On}}\, X_0^+ \leftrightarrow X_0^+ = U \vee X_0^+ \in U \\ \vee \exists X_1^+ \exists X_2^+\, \mathrm{set}_{\mathrm{On}}\, X_1^+ \wedge \mathrm{set}_{\mathrm{On}}\, X_2^+ \\ \wedge (X_0^+ = X_1^+ \times X_2^+ \vee X_0^+ \in X_1^+ \times X_2^+ \vee X_0^+ \subseteq X_1^+)$$

By definition all these objects exist.

6.3. Completeness of Classification

The definition of the predicate $\mathrm{set}_{\mathrm{On}}$ specifies what collections are families and what collections are not families. Hence now we can give the more precise definition of the class $\hat{P}$ and can prove that the family classification covers all families.

Definition. The collection of subsets of U, such that the subsets are not members of U, is the class $\hat{P}$:

$$X_0^+ \in \hat{P} \leftrightarrow X_0^+ \subseteq U \wedge X_0^+ \notin U$$

Theorem. The family classification covers all families:

$$\mathrm{set}_{\mathrm{On}}\, X_0^+ \to \exists X_1^+ \mathrm{set}_{\mathrm{On}}\, X_1^+ \wedge \mathrm{class}_{\mathrm{On}}\, X_1^+ \wedge X_0^+ \in X_1^+$$

in the other words, any set of T_{On} has a type because any class is numbered by an up-ordinal and any member of an up-class (except members of U) has type which equals the up-ordinal number of the class.

Proof. We use the induction rule. Let class U and its members exist at the step 0. The theorem is true at this step because U (and every class) is a member of an other class. And all members of U are members of the class U.

We shall prove that only members of classes exist at step $n+1$, if only members of classes exist at step n.

By definition, families generated at step $n+1$ are only:

- any direct product of families, which is generated at the previous steps, and its members,
- any power family[11], which is generated at the previous steps, and its members.

Let:

- families X_1^+ and X_2^+ (but not atoms) be generated at previous steps,
- these families be members of classes X_3^+ and X_4^+ respectively,
- the families $X_1^+ \times X_2^+$ and $P(X_1^+)$ be generated at the step $n+1$.

The family $X_1^+ \times X_2^+$ is a subfamily of the class $X_3^+ \times X_4^+$ and a member of the class $P(X_3^+ \times X_4^+)$. Members of the family are members of the class $X_3^+ \times X_4^+$. Then the theorem is true for this family and its members.

The set $P(X_1^+)$ is a subfamily of the class $P(X_3^+)$ and a member of the class $P^2(X_3^+)$. Members of the family are members of the class $P(X_3^+)$. The theorem is true for this family and its members, too. This completes the proof. □

6.4. Foundation (Regularity) of Families

We use the completeness of family classification to prove the theorem of foundation.

Theorem (of foundation). Each nonempty family X_1^+ contains a family X_2^+ such that X_1^+ and X_2^+ do not contain common members:

[11] Power families exist in $T_{\mathrm{On}+1}$.

$$\mathrm{set_{On}}\, X_1^+ \wedge (\exists X_0^+\ X_0^+ \in X_1^+)$$
$$\rightarrow \exists X_2^+\ X_2^+ \in X_1^+ \wedge \forall X_0^+\ X_0^+ \notin X_2^+ \vee X_0^+ \notin X_1^+$$

Proof. All types are numerated by ordinals and up-ordinals. This means that any subset of types contains the minimal type. Hence set X_1^+ contains set X_2^+ with a minimal type. Then members of X_2^+ is not contained in X_1^+ because they have types less than the minimal type.

6.5. Construction of Up-Families

We begin to construct up-families at the universe U_+.

Definition. The up-family containing all families is the universe U_+*:*

$$X_0^+ \in U_+ \leftrightarrow \mathrm{set_{On}}\, X_0^+$$

Definition. An up-family $X_1^+ \times X_2^+$ is a direct product of X_1^+ and X_2^+, if $X_1^+ \times X_2^+$ contains all ordered pairs which first components are members of X_1^+ and second components are members of X_2^+:

$$\neg\mathrm{atom}\, X_1^+ \wedge \neg\mathrm{atom}\, X_2^+ \rightarrow (X_0^+ \in X_1^+ \times X_2^+$$

$$\leftrightarrow \exists X_3^+, X_4^+\ X_3^+ \in X_1^+ \wedge X_4^+ \in X_2^+ \wedge X_0^+ = \langle X_3^+, X_4^+ \rangle)$$

Definition. Up-family X_1^+ is a sub-up-family of up-family X_2^+, if these up-families are not atoms and if all members of the first up-family present in the second up-family:

$$X_1^+ \subseteq X_2^+ \leftrightarrow \neg\mathrm{atom}\, X_1^+ \wedge \neg\mathrm{atom}\, X_2^+ \wedge$$

$$\wedge \forall X_3^+\ X_3^+ \in X_1^+ \rightarrow X_3^+ \in X_2^+$$

Now we finish to construct up-families.

The statement that other up-families do not exist is impossible to formalize in the external theory. But this is unnecessary because the external theory is used only to research characteristics of up-sets and to prove those theorems that cannot be proved in T_{On}.

7. Conclusion

We have constructed stairs of set theories from the simple set theory for applied mathematics and to the set theory that is more strong than existing set theories. We have constructed external theories for these two theories and use former theories to prove some theorems that cannnot be proved in the latter theories, in particular, existence of natural classification of sets such that every set belongs to a class and classes are disjoint. All set theories have the same axiomatics and every new theory of the stairs is an extension of previouse theories. We have constructed the new base of ordinals and cardinals but we do not use the base to construct large cardinals. But we believe that the base will allow to costruct all existing and some new large cardinals.

References

[1] G. Cantor, 1874, Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen, *J. reine and angew. Math.* 77, 258-262.

[2] B. Russell, 1903, *The Principles of Mathematics*, Cambridge University Press.

[3] B. Russell, 1908, Mathematical logic as based on the theory of types, *Amer. J. of Math.* 30, 222-262.

[4] E. Zermelo, 1908, Untersuchungen über die Grundlagen der Mengenlehre, *Math. Ann.*, 65, 261-281.

[5] K. Gödel, 1931, Über formal untentschedibare Saetze der Principia Mathematica und verwandter Systeme I, Monatsh *Math. Phys.*, 38, 349-360.

[6] A. Tarski, 1931, Sur les ensembles definissables de nombres reels I, *Fund. Math.*, 17, 210-239.

[7] W. Quine, 1937, New foundations for mathematical logic, *Am. Math. Monthly*, 44, 70-80.

[8] W. Quine, 1940, *Mathematical logic*, Cambridge Mass.

[9] A. Church, 1940, A formulation of the simple theory of types, *J. of Symb. Log.* 5(2), 56-68.

[10] Hao Wang, 1954, The formalization of mathematics, *J. of Symb. Log.*, 19, 241-266.

[11] A. Fraenkel, 1922, Zu den Grundlagen der Cantor-Zermeloschen Mengenlehre, *Math. Annalen*, 86, 230-237.

[12] J. Neumann, 1925-1928, Eine Axiomatizierung der Mengenlehre, *J. für Math.*, 154, 219-240 (1925); Die Axiomatizierung der Mengenlehre, *Math. Z.*, 27, 669-752 (1928).

[13] R.M. Robinson, 1937, The Theory of the Classes. A modification of von Neumann's system, *J. of Symb. Log.*, 2, 69-72.

[14] P. Bernays, 1937-1954, A system of axiomatic set theory, *J. of Symb. Logic*, I: 2, 65-77 (1937), II: 6, 1-17 (1941), (1942), III: 7, 65-89 (1942), IV: 7, 133-145 (1942), V: 8, 89-106 (1943), VI: 13, 65-79 (1948), VII: 19, 81-96 (1954).

[15] K. Gödel, 1940, *The consistency of the axiom of choice and of the generalized continuum hypothesis with axioms of set theory*, Princeton.

[16] J. Kelley, 1955, *General topology*, Van Nostrand.

[17] A. Morse, 1965, *A teory of sets*, Acadimiv Press.

[18] G. Boolos, 1971, The iterative conception of set, *J. of phil.* 68 215-231.

[19] P. Maddy, 1988, Believing the Axioms, *J. of Symb. Log.* I: 53(2), 481-511; II: 53(3), 736-764.

[20] W. Woodin, 2001, The continuum hypothesis, part II, *Notices of Amer. Math. Soc.* 48(7), 681-690.

[21] A. Kanamori, 2003, *The Higher Infinite: Large Cardinals in Set Theory from their beginnings* (2nd ed.), Springer.

[22] J.R. Steel, 2000, Mathematics needs new axioms, *Bull. Symb. Log.*, 6 422-433.

[23] M. Foreman, A. Kanamory, ed., 2010, *Handbook of set theory*, v. 1-3, Springer.

[24] E. Mendelson, 2013, *Introduction to mathematical logic*, 5th edition, Chapman and Hall.

[25] D. Scott, 1965, Logic with denumerably long formulas and finite strings of quantifiers, in *Proceedings of the 1963 International Symposium, Theory of Models, Berkeley*, 329-341.