
Derivations of some filiform Leibniz algebras

AL-hossain Ahmad, Khiyar AL-hossain

Department of Mathematics, AL Qunfudha University College, Umm AL Qura University, City of AL Qunfudha, Kingdom of Saudi Arabia

Email address:

hossa-1427@hotmail.com (AL-hossain A.), asas.1429@hotmail.com (K. AL-hossain)

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Abstract: In this paper the two classes of filiform Leibniz algebras μ_0^n and μ_1^n in $(n + 1)$ dimensions of filiform Leibniz algebras such that $n \geq 2$ will be considered. The study includes derivations of naturally graded Leibniz algebras of first class L_n and second class W_n , be algebras whose multiplications rules are defined by the μ_0^n and μ_1^n , respectively. The algebras of derivations of naturally graded Leibniz algebras are described by linear transformations and dimensions derivations. Finally, we determine number of derivations of naturally graded Leibniz algebras.

Keywords: Leibniz Algebra, Filiform Leibniz Algebra, Characteristically Nilpotent Algebra, Graded Leibniz Algebra, Derivation

1. Introduction

In 1955 Jacobson established that, over a field of characteristic zero, any Lie algebra which has non-degenerate derivations is nilpotent [17]. In the same paper [11] he asked for the converse. This result is assumed to be the origin of the theory of characteristically nilpotent Lie algebras. An example of a nilpotent Lie algebra all of whose derivations are nilpotent (hence degenerate), answering the above question negatively, was constructed in Dixmier and Lister [6]. Lie algebras whose derivations are nilpotent endomorphisms have been called characteristically nilpotent. The result of Dixmier and Lister is assumed to be the origin of the theory of characteristically nilpotent Lie algebras. They defined a generalization of the central descending sequence and called the algebras satisfying the nullity of a power characteristically nilpotent. The (co)homology theory, representations and related problems of Leibniz algebras were studied by Loday and Pirashvili, [15], and others. Since the class of Leibniz algebras are a noncommutative generalization of the class of Lie algebras, we naturally face the problem of finding a good relationship (as well as in the case of Lie algebras [7]) for the algebra of derivations of L . The investigations in the present paper are devoted to this problem. In particular, for naturally graded complex Leibniz algebras, we describe their algebras of derivations generating the algebras of derivations of filiform Leibniz algebras. This description enables us to distinguish the characteristically nilpotent algebras in the class of filiform Leibniz algebras. In

the present paper we study the derivation algebras of low-dimensional Leibniz algebras. The outline of the paper is as follows. Section 1 is a brief introduction. Section 2 we give derivations of filiform Leibniz algebras L_n . In Section 3 we give derivations of filiform Leibniz algebras W_n .

Definition 1.1. [17] An algebra L over F a field is said to be Leibniz if the Leibniz identity:

$$[x, [y, z]] = [[x, y], z] + [[x, z], y]$$

holds for any x, y, z in L , where $[,]$ stands for the multiplication in L .

Note that if the identity $[x, x] = 0$ holds in L , then the Leibniz identity becomes the Jacobi identity. Thus, the Leibniz algebras are the "noncommutative" analog of Lie algebras. For an arbitrary algebra L , we define the sequence

$$L^1 \equiv L \quad L^{n+1} = [L^n, L].$$

Definition 1.2. [18] A Leibniz algebra L is said to be nilpotent if there exists a positive integer s in \mathbb{N} such that $L^1 \supset L^2 \supset \dots \supset L^s = 0$. The smallest integer s for which $L^s = 0$ is called the nilindex of L .

Definition 1.3. [7] An n -dimensional Leibniz algebra L is said to be filiform if $\dim L^i = n - i$, where $2 \leq i \leq n$.

Definition 1.4. [1] Given a filiform Leibniz algebra L , put $L_i = L^i / L^{i+1}$, $1 \leq i \leq n$ and $\text{gr}L = L_1 \oplus L_2 \oplus \dots \oplus L_{n-1}$. Then $[L_i, L_j] \subseteq L_{i+j}$ and we obtain the graded algebra $\text{gr}L$. If $\text{gr}L$ and L are isomorphic, denoted by $\text{gr}L = L$, we say that the algebra L is naturally graded.

Theorem 1.1. [2] Every $(n+1)$ -dimensional naturally graded complex non-Lie filiform Leibniz algebra is isomorphic to one of the following two algebras:

$$\mu_0^n: [e_0, e_0] = e_2, [e_i, e_0] = e_{i+1}, 1 \leq i \leq n-1$$

$$\mu_1^n: [e_0, e_0] = e_2, [e_i, e_0] = e_{i+1}, 2 \leq i \leq n-1$$

where e_0, e_1, \dots, e_n is a basis of the algebra L .

Corollary 1.1. [2] Every $(n+1)$ -dimensional complex non-Lie filiform Leibniz algebra is isomorphic to one of the following non-Lie filiform Leibniz algebras:

$$\mu_0^n(\beta): [e_0, e_0] = e_2, [e_i, e_0] = e_{i+1}, 1 \leq i \leq n-1,$$

$$[e_0, e_1] = \alpha_3 e_3 + \alpha_4 e_4 + \dots + \alpha_{n-1} e_{n-1} + \theta e_n$$

$$[e_i, e_1] = \alpha_3 e_{i+2} + \alpha_4 e_{i+3} + \dots + \alpha_{n+1-i} e_n, 1 \leq i \leq n-2$$

where $\alpha_3, \alpha_4, \dots, \alpha_n, \theta \in \mathbb{C}$, and

$$\mu_1^n(\beta): [e_0, e_0] = e_2, [e_i, e_0] = e_{i+1}, 2 \leq i \leq n-1,$$

$$[e_0, e_1] = \beta_3 e_3 + \beta_4 e_4 + \dots + \beta_n e_n, [e_i, e_1] = \gamma e_n,$$

$$[e_i, e_1] = \beta_3 e_{i+2} + \beta_4 e_{i+3} + \dots + \beta_{n+1-i} e_n, 2 \leq i \leq n-2,$$

where $\beta_3, \beta_4, \dots, \beta_n, \gamma \in \mathbb{C}$, and e_0, e_1, \dots, e_n is a basis of the algebra L .

Definition 1.5. [1] A linear transformation d of a Leibniz algebra L is called a derivation if for any x, y in L

$$d([x, y]) = [d(x), y] + [x, d(y)]$$

The space of all derivations of the algebra L equipped with the multiplication defined as the commutator, forms a Lie algebra which is denoted by $\text{Der}(L)$. It is clear that the operator of right multiplication R_x by an element x of the algebra L (that is $R_x(y) = [y, x]$) is also derivation. Derivations of this type are called inner derivations. Similar to the Lie algebras case the set of the inner derivation $\text{Inn}(L)$ forms an ideal of the algebra $\text{Der}(L)$.

Lemma 1.1. [17] For any $r \leq (p-1)(1-j)$, one has $F_r Z^j(L, L) = Z^j(L, L)$.

Let L_n and W_n be algebras whose multiplication rules are defined by the multiplication μ_0^n and μ_1^n , respectively.

2. Derivation of Filiform Leibniz Algebra

L_n

Proposition 2.1. The linear transformations t_1, t_2, t_3, t_4 and $d_k, 1 \leq k \leq n-2$ of L_n defined by the rules:

$$t_1(e_0) = e_0, t_1(e_i) = e_i, 1 \leq i \leq n,$$

$$t_2(e_0) = e_1, t_2(e_i) = e_i, 1 \leq i \leq n,$$

$$t_3(e_0) = e_n$$

$$t_4(e_1) = e_n$$

$$d_k(e_0) = e_{k+1}, d_k(e_i) = e_{k+i}, 1 \leq i \leq n-k.$$

form a basis of the space $\text{Der}(L_n)$.

Proof. We introduce a grading of the algebra by setting $L_n = L_1 \oplus L_2 \oplus \dots \oplus L_i$ by setting $L_1 = \text{lin}(e_0, e_1), L_i = \text{lin}(e_i)$ for $2 \leq i \leq n$. Since $Z_1(L_n, L_n) = \text{Der}(L_n)$ and grading of the algebra L_n is finite, there is a finite grading in the space $\text{der}(L_n)$. Let d in $\text{Der}(L_n)$. In this case, by Lemma 1.1 we have $d = d_0 + d_1 + \dots + d_{n-2} + d_{n-1}$, where d_i in $\text{Der}(L_n)$ and $d_i(L_j) \subseteq L_{i+j}$.

Consider the element $d_0 \in \text{Der}(L_n)$ It is clear that

$$d_0(e_i) = \begin{cases} \alpha_0 e_0 + \alpha_1 e_1, & \text{for } i = 0, \\ \beta_0 + \beta_1 \beta_1, & \text{for } i = 1, \\ \gamma_i e_i, & \text{for } 2 \leq i \leq n, \end{cases}$$

where $\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_i, 2 \leq i \leq n$, are scalars (elements of the field).

Consider the family of derivations

$$d_0([e_i, e_j]) = [d_0(e_i), e_j] + [e_i, d_0(e_j)].$$

If $j = 1$ we obtain

$$[e_i, d_0(e_1)] = [e_i, \beta_0 e_0 + \beta_1 e_1] = \beta_0 e_{i+1} = 0 \Rightarrow \beta_0 = 0$$

If $i = 1$ and $j = 0$ then

$$d_0(e_2) = [d_0(e_1), e_0] + [e_1, d_0(e_0)] \Rightarrow \gamma_2 e_2 = (\beta_1 + \alpha_0) e_2 \Rightarrow \gamma_2 = \beta_1 + \alpha_0$$

i.e., $\gamma_2 = \beta_1 + \alpha_0$

If $i = 0$ and $j = 0$ then

$$d_0(e_2) = [d_0(e_0), e_0] + [e_0, d_0(e_0)] \Rightarrow \gamma_2 e_2 = \alpha_0 e_2 + \alpha_1 e_2 + \alpha_0 e_2 \Rightarrow \gamma_2 e_2 = (2\alpha_0 + \alpha_1) e_2$$

i.e., $\gamma_2 = 2\alpha_0 + \alpha_1$

If $j = 0$ and $2 \leq i \leq n$ we obtain

$$d_0(e_{i+1}) = \gamma_i e_{i+1} + \alpha_0 e_{i+1} \Rightarrow \gamma_{i+1} = \gamma_i + \alpha_0$$

However, since $\gamma_2 = 2\alpha_0 + \alpha_1$ it follows that $\gamma_1 = \alpha_0 + \alpha$

Thus,

$$d_0(\sum_{i=0}^n \lambda_i e_i) = \lambda_0(\alpha_0 + \alpha_1 e_1) + \lambda_1(\alpha_0 e_1 + \alpha_1 e_1) + \sum_{i=2}^n \lambda_i \gamma_i e_i = \lambda_0(\alpha_0 + \alpha_1) + \alpha_0(\lambda_0 e_0 + \sum_{i=1}^n \lambda_i e_i) + \alpha_1(\lambda_0 e_1 + \sum_{i=1}^n \lambda_i e_i),$$

i.e., $d_0 = \alpha_0 t_1 + \alpha_1 t_2$

Consider the elements $d_k \in \text{Der}(L_n)$ for $1 \leq k \leq n-2$. It is clear that

$$d_k(e_i) = \begin{cases} \tau_0 e_{k+1}, & \text{for } i = 0, \\ \tau_i e_{k+i}, & \text{for } 1 \leq i \leq n-k, \end{cases}$$

where $\tau_i, 0 \leq i \leq n-k$ are scalars (elements of the field).

Consider the following property of the derivations

$$d_k([e_i, e_j]) = [d_k(e_i), e_j] + [e_i, d_k(e_j)].$$

If $j = 0$ and $i = 1$, we have

$$d_k(e_2) = \tau_1 [e_{k+1}, e_0] \Rightarrow \tau_2 e_{k+2} = \tau_1 e_{k+2} \Rightarrow \tau_2 = \tau_1.$$

If $j = 0$ and $i = 0$, we have

$$d_k(e_2) = \tau_0 [e_{k+1}, e_0] \Rightarrow \tau_2 e_{k+2} = \tau_0 e_{k+2} \Rightarrow \tau_2 = \tau_0.$$

if $j = 0$ and $1 \leq i \leq n - k$, we obtain

$$d_k(e_{i+1}) = \tau_i e_{k+i+1} \Rightarrow \tau_{i+1} e_{k+i+1} = \tau_i e_{k+i+1} \Rightarrow \tau_{i+1} = \tau_i.$$

i.e., $\tau_0 = \tau_1 = \dots = \tau_{n-k}$.

Thus,

$$d_k(\sum_{i=0}^n \lambda_i e_i) = \tau_0 (\lambda_0 e_{k+1} + \sum_{i=1}^{n-1} \lambda_i e_{k+i}).$$

Consider the elements $d_{n-1} \in \text{Der}(L_n)$. It is clear that

$$d_{n-1}(e_i) = \begin{cases} \delta_0 e_n, & \text{for } i = 0, \\ \delta_1 e_n, & \text{for } i = 1, \end{cases}$$

where δ_0 , and δ_1 are scalars (elements of the field).

Consider the following property of the derivations

$$d_{n-1}([e_i, e_j]) = [d_{n-1}(e_i), e_j] + [e_i, d_{n-1}(e_j)].$$

If $i = 0$ and $j = 0$, we have

$$d_{n-1}(e_2) = \delta_0 [e_n, e_0] + \delta_0 [e_0, e_n] \Rightarrow 0 = 0$$

If $i = 1$ and $j = 0$, we have

$$d_{n-1}(e_2) = \delta_1 [e_n, e_0] + \delta_0 [e_1, e_n] \Rightarrow 0 = 0$$

Thus,

$$d_{n-1}(\sum_{i=0}^n \lambda_i e_i) = \lambda_0 \delta_0 e_n + \lambda_1 \delta_1 e_n.$$

This proves the proposition. we note these mappings are derivations and are linearly independent.

Corollary 2.1.

$$\dim \text{Der}(L_n) = \dim Z^1(L_n, L_n) = n + 2.$$

Corollary 2.2.

$$\dim H^1(L_n, L_n) = n + 1$$

3. Derivation of Filiform Leibniz Algebra W_n

Proposition 3.1. The linear transformations t_1, t_2, t_3, t_4, t_5 and d_k of $1 \leq k \leq n - 2$ defined by the rules:

$$t_1(e_0) = e_0, t_1(e_i) = ie_i, 2 \leq i \leq n,$$

$$t_2(e_0) = e_1,$$

$$t_3(e_1) = e_1,$$

$$t_4(e_0) = e_n,$$

$$t_5(e_1) = e_n,$$

$$d_k(e_0) = e_{k+1}, d_k(e_i) = e_{k+i}, 2 \leq i \leq n - k.$$

form a basis of the space $\text{Der}(W_n)$.

Proof. We introduce a grading of the algebra $W_n = W_1 \oplus W_2 \oplus \dots \oplus W_n$ by setting $W_1 = \text{lin}(e_0, e_1)$, $W_i = \text{lin}(e_i)$ for $2 \leq i \leq n$. Since $Z_1(W_n, W_n) = \text{Der}(W_n)$ and grading of the algebra W_n is finite, there is a finite grading in the space $\text{der}(W_n)$. Let d in $\text{Der}(W_n)$. In this case, by lemma 1.1 we have

$d = d_0 + d_1 + \dots + d_{n-2} + d_{n-1}$ where $d_i \in \text{Der}(W_n)$ and $d_i(W_j) \subseteq W_{i+j}$.

Consider the element d_0 in $\text{Der}(W_n)$. It is clear that

$$d_0(e_i) = \begin{cases} \alpha_0 e_0 + \alpha_1 e_1, & \text{for } i = 0, \\ \beta_0 + \beta_1 e_1, & \text{for } i = 1, \\ \gamma_i e_i, & \text{for } 2 \leq i \leq n, \end{cases}$$

where $\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_i, 2 \leq i \leq n$, are scalars (elements of the field).

Consider the family of derivations

$$d_0([e_i, e_j]) = [d_0(e_i), e_j] + [e_i, d_0(e_j)]$$

If $j = 1$ we obtain

$$[e_i, d_0(e_1)] = [e_i, \beta_0 e_0 + \beta_1 e_1] = \beta_0 e_{i+1} = 0 \Rightarrow \beta_0 = 0.$$

If $i = 0$ and $j = 0$ then

$$d_0(e_2) = [d_0(e_0), e_0] + [e_0, d_0(e_0)] \Rightarrow \gamma_2 e_2 = 2\alpha_0 e_2$$

i.e., $\gamma_2 = 2\alpha_0$

If $j = 0$ and $2 \leq i \leq n$ we obtain

$$d_0(e_{i+1}) = \gamma_i e_{i+1} + \alpha_0 e_{i+1} \Rightarrow \gamma_{i+1} = \gamma_i + \alpha_0.$$

i.e., $\gamma_i = i\alpha_0$

Thus,

$$d_0(\sum_{i=0}^n \lambda_i e_i) = \lambda_0 (\alpha_0 e_0 + \alpha_1 e_1) + \lambda_1 \beta_1 e_1 + \alpha_0 \sum_{i=2}^n \lambda_i e_i = \alpha_0 (\lambda_0 e_0 + \sum_{i=2}^n \lambda_i e_i) + \alpha_1 \lambda_0 e_1 + \beta_1 \lambda_1 e_1$$

i.e., $d_0 = \alpha_0 t_1 + \alpha_1 t_2 + \beta_1 t_3$

Consider the elements d_k in $\text{Der}(W_n)$ for $1 \leq k \leq n - 2$. It is clear that

$$d_k(e_i) = \begin{cases} \tau_0 e_{k+1}, & \text{for } i = 0, \\ \tau_i e_{k+i}, & \text{for } 1 \leq i \leq n - k, \end{cases}$$

where $\tau_i, 0 \leq i \leq n - k$ are scalars (elements of the field).

Consider the following property of the derivations:

$$d_k([e_i, e_j]) = [d_k(e_i), e_j] + [e_i, d_k(e_j)]$$

If $j = 0$ and $i = 1$, we have

$$\tau_1 [e_{k+1}, e_0] = 0 \Rightarrow \tau_1 e_{k+2} = 0 \Rightarrow \tau_1 = 0.$$

If $j = 0$ and $i = 0$, we have

$$d_k(e_2) = \tau_0 [e_{k+1}, e_0] \Rightarrow \tau_2 e_{k+2} = \tau_0 e_{k+2} \Rightarrow \tau_2 = \tau_0.$$

if $j = 0$ and $2 \leq i \leq n - k$, we obtain

$$d_k(e_{i+1}) = \tau_i e_{k+i+1}, \Rightarrow \tau_{i+1} e_{k+i+1} = \tau_i e_{k+i+1} \Rightarrow \tau_{i+1} = \tau_i.$$

i.e., $\tau_0 = \tau_2 = \dots = \tau_{n-k}$.

Thus,

$$d_k(\sum_{i=0}^n \lambda_i e_i) = \tau_0 (\lambda_0 e_{k+1} + \sum_{i=2}^{n-k} \lambda_i e_{k+i}).$$

Consider the elements $d_{n-1} \in \text{Der}(W_n)$. It is clear that

$$d_{n-1}(e_i) = \begin{cases} \delta_0 e_n, & \text{for } i = 0, \\ \delta_1 e_n, & \text{for } i = 1, \end{cases}$$

where δ_0 , and δ_0 are scalars (elements of the field).

Consider the following property of the derivations

$$d_{n-1}([e_i, e_j]) = [d_{n-1}(e_i), e_j] + [e_i, d_{n-1}(e_j)]$$

If $i = 0$ and $j = 0$, we have

$$d_{n-1}(e_2) = \delta_0 [e_n, e_0] + \delta_0 [e_0, e_n] \Rightarrow 0 = 0$$

If $i = 1$ and $j = 0$, we have

$$\delta_1 [e_n, e_0] + \delta_0 [e_1, e_n] = 0 \Rightarrow 0 = 0$$

Thus,

$$d_{n-1}(\sum_{i=0}^n \lambda_i e_i) = \lambda_0 \delta_0 e_n + \lambda_1 \delta_1 e_n.$$

i.e, $d_{n-1} = \delta_0 t_4 + \delta_1 t_5$ This proves the proposition. We note these mappings are derivations and are linearly independent.

Corollary 3.1.

$$\dim \text{Der}(W_n) = \dim Z_1(W_n, W_n) = n + 3.$$

Corollary 3.2.

$$\dim H^1(W_n, W_n) = n + 2$$

4. Conclusion

Notations of this paper about any $(n + 1)$ dimensions of some filiform Leibniz algebras (L_n) and (W_n) .

1. notations about Derivations filiform Leibniz algebra (L_n) .

(a) The linear transformations t_1, t_2, t_3, t_4 , and dk ; $1 \leq k \leq n-2$ of L_n are a basis of the space $\text{Der}(L_n)$.

(b) We can find dimensions derivations of filiform Leibniz algebra (L_n) , by using

$$\dim \text{Der}(L_n) = n + 2.$$

(c) we can determine that the number equations of derivations for filiform Leibniz algebra L_n for any $(n + 1)$ dimensions such that $n \geq 2$ by using

$$\text{number Der}(L_n) = (n^2 + 5n + 2)/2$$

2. notations about Derivations filiform Leibniz algebra (W_n) .

(a) The linear transformations t_1, t_2, t_3, t_4, t_5 , and dk , $1 \leq k \leq n-2$, of W_n are a basis of the space $\text{Der}(W_n)$.

(b) We can find dimensions derivations of filiform Leibniz algebra (W_n) , by using

$$\dim \text{Der}(W_n) = n + 3.$$

(c) we can determine that number equations of derivations for filiform Leibniz algebra W_n for any $(n + 1)$ dimensions such that $n \geq 2$, by using

$$\text{number Der}(W_n) = \begin{cases} 3n, & \text{when } n \text{ is even,} \\ (n^2 + n + 6)/2, & \text{when } n \text{ is odd,} \end{cases}$$

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