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New fixed point theorems of condensing mappings satisfying the interior condition in Banach spaces

Shujia Chen, Shaoyuan Xu*

Department of Mathematics and Statistics, Hanshan Normal University, Chaozhou, China
* corresponding author

Email address:
xushaoyuan@126.com (Shaoyuan Xu), 1076429058@qq.com (Shujia Chen)

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Abstract: In this paper, based on a basic result on condensing mappings satisfying the interior condition, some new fixed point theorems of the condensing mappings of this kind are obtained. As a result, the famous Petryshyn's theorem and some results of the Reference [8] are extended to the condensing mappings satisfying the interior condition.

Keywords: Condensing Mappings, Interior Condition, Fixed Point, Banach Space

1. Introduction and Preliminaries

As is well known, the condensing mappings are a class of important nonlinear operators, existing extensively in the nonlinear differential and integral equations. So, the fixed point theorems of the condensing mappings play a central role in the study of existence of the solutions of these equations (see [1-8]).

Let X be a real Banach space and D an open subset of X with $\theta \in D$ where θ denotes the zero element of X. Denote by $\overline{D}$ and ∂D the closure and boundary of D respectively. It is well known that if D is bounded and if $A:\overline{D} \to X$ is a condensing mapping, then we have had some well-known theorems as follows (see [1-5, 8]).

Theorem 1.1 Suppose that one of the following conditions is satisfied:

(i) (Leray − Schauder) $Ax \neq \lambda x, \forall x \in \partial D$;

(ii) (Roth) $\|A(x)\| \leq \|x\|, \forall x \in \partial\Omega$;

(iii) (Petryshyn) $\|A(x)\| \leq \|A(x)-x\|, \forall x \in \partial\Omega$;

(iv) (Altman) $\|A(x)-x\|^2 \geq \|A(x)\|^2 - \|x\|^2, \forall x \in \partial\Omega$.

Then A has at least one fixed point in $\overline{D}$.

Recently, Antonio and Morales [7] introduce a new condition which resembles the Leray-Schauder boundary condition mentioned above. It is called the Interior Condition defined as follows.

Definition 1.1 [7] We say that a mapping $A:\overline{D} \to X$ satisfies the Interior Condition if there exists $\delta > 0$ such that

$$Ax \neq \lambda x, \forall x \in D^*, \lambda > 1, Ax \notin \overline{D},$$

where

$$D^* = \{x \in D : dist(x, \partial x) < \delta\}. \qquad (1.1)$$

Recall that the set-measure of non-compactness of a bounded subset E of X is defined as (see [2-4])

$\gamma[E] = \inf\{\ d > 0 : E$ can be covered by finitely many sets of diameter $\leq d\ \}$:

A mapping $A: D \subset X \to X$ is called a condensing mapping (see [2-4]), if A is continuous and $\gamma[A[E]] < \gamma[E]$ for all bounded subset E of D where $\gamma[E] > 0$.

Definition 1.2 [7] Let X be a real Banach space and D an bounded open subset of X with $\theta \in D$. We say that D is strictly star-shaped with respect to the origin if the following condition is satisfied:

$$\{tx : t > 0\} \cap \partial G = \{x\}, \forall x \in \partial G. \qquad (1.2)$$

From now on, we will assume without loss of generality, that the star-shaped assumption will always be considered with respect to the origin, unless the contrary is mentioned.

In this paper, based on a basic result on condensing mappings satisfying the interior condition, some new fixed point theorems of the condensing mappings of this kind are obtained. As a result, the famous Petryshyn' s theorem and some results of the Reference [8] are extended to the condensing mappings satisfying the interior condition.

2. Main Results

Lemma 2.1[7] Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. If $A:\overline{D}\to X$ is a condensing mapping satisfying the Condition (I-C): there exists $\delta>0$ such that

$$Ax \neq \lambda x, \forall x \in D^*, \lambda > 1, Ax \notin \overline{D}, \quad \text{(I-C)}$$

where

$$D^* = \{x \in D : dist(x, \partial G) < \delta\}$$

for some $\delta>0$, then A has at least one fixed point in $\overline{D}$.

We are now in the position to apply the above basic result to derive some new fixed point theorems for the condensing mappings satisfying the Condition (I-C) which extend many well-known results to the case of the mappings satisfying the interior condition.

Theorem 2.1 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A:\overline{D}\to X$ be a condensing mapping. Suppose also that there exist $1 \le \alpha < \beta$ or $1 < \alpha \le \beta$ such that

$$\|Ax - x\|^{\alpha}\|x\|^{\beta} \ge \|Ax\|^{\alpha}\|Ax + x\|^{\beta} - \|Ax\|^{\alpha}\|x\|^{\beta};$$

$$\forall x \in D^* \text{ and } Ax \notin \overline{D}, \quad (2.1)$$

where

$$D^* = \{x \in D : dist(x, \partial G) < \delta\}$$

for some $\delta>0$, then A has at least one fixed point in $\overline{D}$.

Proof If the operator A has a fixed point on ∂D, then A has at least one fixed point on D. Now suppose that A has no fixed points on ∂D. Next we shall prove that the condition (I-C) is satisfied.

Suppose it is not true. Then there exist $x_0 \in \partial D$ and $\mu_0 \ge 1$ such that $Ax_0 = \mu_0 x_0$. It is easy to see that $\mu_0 > 1$. Now, consider the function defined by

$$f(t) = (t-1)^{\alpha} - t^{\alpha}(t+1)^{\beta} + t^{\alpha}, \forall t \ge 1.$$

Since

$f'(t) = \alpha(t-1)^{\alpha-1} - \alpha t^{\alpha-1}(t+1)^{\beta} + \alpha t^{\alpha-1} - \beta t^{\alpha}(t+1)^{\beta-1}$ < 0 by formal differentiation, $f(t)$ is a strictly decreasing function in $[1,\infty)$. And so $f(t) < f(1)$ for $t>1$, i.e., $(t-1)^{\alpha} < t^{\alpha}(t+1)^{\beta} - t^{\alpha}$, for any $t>1$. Consequently, noting that $\|x_0\| \neq 0, \mu_0 > 1$, we have

$$\|Ax_0 - x_0\|^{\alpha}\|x_0\|^{\beta}$$

$$= \|\mu_0 x_0 - x_0\|^{\alpha}\|x_0\|^{\beta}$$

$$= (\mu_0 - 1)^{\alpha}\|x_0\|^{\alpha+\beta}$$

$$< \left[\mu_0{}^{\alpha}(\mu_0+1)^{\beta} - \mu_0{}^{\alpha}\right]\|x_{\alpha}\|^{\alpha+\beta}$$

$$= \|Ax_0\|^{\alpha}\|Ax_0 + x_0\|^{\beta} - \|Ax_0\|^{\alpha}\|x_0\|^{\beta},$$

which is a contradiction to (2.1), and so the condition (I-C) is satisfied. Therefore, it follows from Lemma 2.1 that the conclusion of Theorem 2.1 holds.

From Theorem 2.1, we can easily get the subsequent four corollaries.

Corollary 2.1 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A:\overline{D}\to X$ be a condensing mapping. Suppose also that there exists $\alpha>1$, such that

$$(\|Ax - x\|\|x\|)^{\alpha} \ge (\|Ax\|\|Ax + x\|)^{\alpha} - (\|Ax\|\|x\|)^{\alpha},$$

$$\forall x \in D^* \text{ and } Ax \notin \overline{D},$$

where

$$D^* = \{x \in D : dist(x, \partial G) < \delta\}$$

for some $\delta>0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\alpha = \beta$. It follows from Theorem 2.1 that the conclusion of Corollary 2.1 holds true.

Corollary 2.2 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A:\overline{D}\to X$ be a condensing mapping. Suppose also that A satisfies the following condition

$$\sqrt{(\|Ax - x\|\|x\|)^3} \ge \sqrt{(\|Ax\|\|Ax + x\|)^3} - \sqrt{(\|Ax\|\|x\|)^3};$$

$$\forall x \in D^* \text{ and } Ax \notin \overline{D},$$

where

$$D^* = \{x \in D : dist(x, \partial G) < \delta\}$$

for some $\delta>0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\alpha = \beta = \frac{3}{2}$. It follows from Theorem 2.1 that the conclusion of Corollary 2.2 holds true.

Corollary 2.3 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A: \overline{D} \to X$ be a condensing mapping. Suppose also that A satisfies the following condition

$$(\|Ax - x\|\|x\|)^2 \geq (\|Ax\|\|Ax + x\|)^2 - (\|Ax\|\|x\|)^2;$$

$$\forall x \in D^* \text{ and } Ax \notin \overline{D};$$

where

$$D^* = \{x \in D : dist(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\alpha = \beta = 2$. It follows from Theorem 2.1 that the conclusion of Corollary 2.3 holds true.

Corollary 2.4 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A: \overline{D} \to X$ be a condensing mapping. Suppose also that there exist $\beta > 1$, such that

$$\|Ax - x\|\|x\|^\beta \geq \|Ax\|\|Ax + x\|^\beta - \|Ax\|\|x\|^\beta,$$

$$\forall x \in D^* \text{ and } Ax \notin \overline{D},$$

where

$$D^* = \{x \in D : dist(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\alpha = 1$. It follows from Theorem 2.1 that the conclusion of Corollary 2.4 holds true.

Theorem 2.2 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A: \overline{D} \to X$ be a condensing mapping. Suppose also that there exist $\alpha \geq 1, \beta \geq 0$, such that

$$\|Ax + x\|^{\alpha+\beta} \leq \|Ax\|^\beta \|Ax - x\|^\alpha + \|x\|^{\alpha+\beta};$$

$$\forall x \in D^* \text{ and } Ax \notin \overline{D}, \tag{2.2}$$

where

$$D^* = \{x \in D : dist(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof If the operator A has a fixed point on ∂D, then A has at least one fixed point on D. Now suppose that A has no fixed points on ∂D. Next we shall prove that the condition (I-C) is satisfied.

Suppose it is not true. Then there exist $x_0 \in \partial D$ and $\mu_0 \geq 1$ such that $Ax_0 = \mu_0 x_0$. It is easy to see that $\mu_0 > 1$. Now, consider the function defined by

$$f(t) = (t+1)^{\alpha+\beta} - t^\beta (t-1)^\alpha - 1, \forall t \geq 1.$$

Since
$f'(t) = \alpha\left[(t+1)^{\alpha+\beta-1} - t^\beta (t-1)^\alpha\right] + \beta\left[(t+1)^{\alpha+\beta-1} - t^{\beta-1}(t-1)^\alpha\right]$
> 0 by formal differentiation, $f(t)$ is a strictly decreasing function in $[1, \infty)$. And so $f(t) > f(1)$ for $t > 1$, Thus, $(t+1)^{\alpha+\beta} > t^\beta (t-1)^\alpha + 1$, for any $t > 1$. Consequently, noting that $\|x_0\| \neq 0, \mu_0 > 1$, we have

$$\begin{aligned}
&\|Ax_0 + x_0\|^{\alpha+\beta} \\
&= \|\mu_0 x_0 + x_0\|^{\alpha+\beta} \\
&= (\mu_0 + 1)^{\alpha+\beta} \|x_0\|^{\alpha+\beta} \\
&> \left[\mu_0{}^\beta (\mu_0 - 1)^\alpha + 1\right] \|x_0\|^{\alpha+\beta} \\
&= \|\mu_0 x_0\|^\beta \|\mu_0 x_0 - x_0\|^\alpha + \|x_0\|^{\alpha+\beta} \\
&= \|Ax_0\|^\beta \|Ax_0 - x_0\|^\alpha + \|x_0\|^{\alpha+\beta},
\end{aligned}$$

which is a contradiction to (2.2), and so the condition (I-C) is satisfied. Therefore, it follows from Lemma 2.1 that the conclusion of Theorem 2.2holds.

From Theorem 2.2, we can easily obtain the subsequent five corollaries.

Corollary 2.5 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A: \overline{D} \to X$ be a condensing mapping. Suppose also that there exists $\alpha > 1$ such that

$$\|Ax + x\|^{2\alpha} \leq \|Ax\|^\alpha \|Ax - x\|^\alpha + \|x\|^{2\alpha},$$

$$\forall x \in D^* \text{ and } Ax \notin \overline{D},$$

where

$$D^* = \{x \in D : dist(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\beta = \alpha > 1$. It follows from Theorem 2.2 that the conclusion of Corollary 2.5 holds true

Corollary 2.6 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A: \overline{D} \to X$ be a condensing mapping. Suppose also that A satisfies the following condition

$$\|Ax + x\|^3 \leq (\|Ax\|\|Ax - x\|)^{\frac{3}{2}} + \|x\|^3,$$

$$\forall x \in D^* \text{ and } Ax \notin \overline{D},$$

where

$$D^* = \{x \in D : dist(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\alpha = \beta = 3/2$. It follows from Theorem 2.2 that the conclusion of Corollary 2.6 holds true.

Corollary 2.7 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A : \overline{D} \to X$ be a condensing mapping. Suppose also that there exists $\alpha \geq 1$ such that

$$\|Ax + x\|^{\alpha} \leq \|Ax - x\|^{\alpha} + \|x\|^{\alpha},$$

$$\forall x \in D^* \text{ and } Ax \notin \overline{D},$$

where

$$D^* = \{x \in D : dist(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\beta = 0$. It follows from Theorem 2.2 that the conclusion of Corollary 2.7 holds true

Corollary 2.8 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A : \overline{D} \to X$ be a condensing mapping. Suppose also that A satisfies the following condition

$$\|Ax + x\|^{2} \leq \|Ax - x\|^{2} + \|x\|^{2},$$

$$\forall x \in D^* \text{ and } Ax \notin \overline{D},$$

where

$$D^* = \{x \in D : dist(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\alpha = 1, \beta = 1$. It follows from Theorem 2.2 that the conclusion of Corollary 2.8 holds true.

Remark 1 We can see that the formula which satisfies the condition of Corollary 2.8 above, must also satisfies the famous Altman' s theorem, namely,

$$\|Ax - x\|^{2} \geq \|Ax\|^{2} + \|x\|^{2}, \forall x \in D^* \text{ and } Ax \notin \overline{D}$$

Therefore, Corollary 2.8 and Theorem 2.2 are the useful supplements of the famous Altman' s theorem.

Corollary 2.9 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A : \overline{D} \to X$ be a condensing mapping. Suppose also that A satisfies the following condition

$$\|Ax + x\|^{2} \leq \|Ax - x\|^{\frac{3}{2}} \|Ax\|^{\frac{1}{2}} + \|x\|^{2},$$

$$\forall x \in D^* \text{ and } Ax \notin \overline{D},$$

where

$$D^* = \{x \in D : dist(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\alpha = \frac{3}{2}, \beta = \frac{1}{2}$. It follows from Theorem 2.2 that the conclusion of Corollary 2.9 holds true.

Theorem 2.3 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A : \overline{D} \to X$ be a condensing mapping. Suppose also that there exist $\alpha \geq 1, \beta \geq 0$, such that

$$\|Ax\|^{\alpha} \|Ax + x\|^{\beta} \leq \|Ax\|^{\beta} \|Ax - x\|^{\alpha},$$

$$\forall x \in D^* \text{ and } Ax \notin \overline{D}; \tag{2.3}$$

where

$$D^* = \{x \in D : dist(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof If the operator A has a fixed point on ∂D, then A has at least one fixed point on D. Now suppose that A has no fixed points on ∂D .Next we shall prove that the condition (I-C) is satisfied.

Suppose it is not true. Then there exist $x_0 \in \partial D$ and $\mu_0 \geq 1$ such that $Ax_0 = \mu_0 x_0$. It is easy to see that $\mu_0 > 1$. Now, consider the function defined by

$$f(t) = t^{\alpha}(t+1)^{\beta} - t^{\beta}(t-1)^{\alpha}, \forall t \geq 1.$$

Since

$f'(t) = \alpha\left[t^{\alpha-1}(t+1)^{\beta} - (t-1)^{\alpha-1}t^{\beta}\right] + \beta\left[t^{\alpha}(t+1)^{\beta-1} - t^{\beta-1}(t-1)^{\alpha}\right]$ > 0 by formal differentiation, $f(t)$ is a strictly decreasing function in $[1, \infty)$. And so $f(t) > f(1)$ for $t > 1$, Thus, $t^{\alpha}(t+1)^{\beta} > t^{\beta}(t+1)^{\alpha}$, for any $t > 1$. Consequently, noting that $\|x_0\| \neq 0, \mu_0 > 1$, we have

$$\|Ax_0\|^{\alpha} \|Ax_0 + x_0\|^{\beta}$$

$$= \|\mu_0 x_0\|^{\alpha} \|\mu_0 x_0 + x_0\|^{\beta}$$

$$= \mu_0{}^{\alpha} (\mu_0 + 1)^{\beta} \|x_0\|^{\alpha+\beta}$$

$$> \mu_0{}^{\beta} (\mu_0 - 1)^{\alpha} \|x_0\|^{\alpha+\beta}$$

$$=\|u_0x_0\|^{\beta}\|\mu_0x_0-x_0\|^{\alpha}$$

$$=\|Ax_0\|^{\beta}\|Ax_0-x_0\|^{\alpha},$$

which is a contradiction to (2.3), and so the condition (I-C) is satisfied. Therefore, it follows from Lemma 2.1 that the conclusion of Theorem 2.3 holds.

From Theorem 2.3 we can easily get the subsequent such a corollary.

Corollary 2.10 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta\in D$. Let $A:\overline{D}\to X$ be a condensing mapping. Suppose also that A satisfies the following condition

$$\|Ax\|\le\|Ax-x\|,\forall x\in D^* \text{ and } Ax\notin\overline{D};$$

where

$$D^*=\{x\in D: dist(x,\partial G)<\delta\}$$

for some $\delta>0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\alpha=1,\beta=0$. It follows from Theorem 2.3 that the conclusion of Corollary 2.10 holds true.

Remark 2 Corollary 2.10 is the famous Petryshyn' s theorem, so the Theorem 2.3 extends the famous Petryshyn' s theorem to the case of condensing mappings satisfying the interior condition.

Theorem 2.4 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta\in D$. Let $A:\overline{D}\to X$ be a condensing mapping. Suppose also that there exist $\alpha\ge1,\beta\ge0$, such that

$$\|Ax-x\|^{\alpha}\|x\|^{\alpha+\beta}\ge\|Ax\|^{\alpha}\|Ax+x\|^{\alpha+\beta}-\|x\|^{2\alpha+\beta},$$

$$\forall x\in D^* \text{ and } Ax\notin\overline{D}, \tag{2.4}$$

where

$$D^*=\{x\in D: dist(x,\partial G)<\delta\}$$

for some $\delta>0$, then A has at least one fixed point in $\overline{D}$.

Proof If the operator A has a fixed point on ∂D, then A has at least one fixed point on D. Now suppose that A has no fixed points on ∂D .Next we shall prove that the condition (I-C) is satisfied.

Suppose it is not true. Then there exist $x_0\in\partial D$ and $\mu_0\ge1$ such that $Ax_0=\mu_0x_0$. It is easy to see that $\mu_0>1$. Now, consider the function defined by

$$f(t)=(t-1)^{\alpha}-t^{\alpha}(t+1)^{\alpha+\beta}+1,\forall t\ge1.$$

Since

$$f'(t)=\alpha\left[(t-1)^{\alpha-1}-t^{\alpha}(t+1)^{\alpha+\beta}-t^{\alpha}(t+1)^{\alpha+\beta-1}\right]-\beta\left[t^{\alpha}(t+1)^{\alpha+\beta-1}\right]$$

<0 by formal differentiation, $f(t)$ is a strictly decreasing function in $[1,\infty)$. And so $f(t)<f(1)$ for $t>1$, Thus, $(t-1)^{\alpha}<t^{\alpha}(t+1)^{\alpha+\beta}-1$, for any $t>1$. Consequently, noting that $\|x_0\|\ne0,\mu_0>1$, we have

$$\|x_0\|^{\alpha+\beta}\|\mu_0x_0-x_0\|^{\alpha}$$

$$=\|Ax_0-x_0\|^{\alpha}\|x_0\|^{\alpha+\beta}$$

$$=(\mu_0-1)^{\alpha}\|x_0\|^{2\alpha+\beta}$$

$$<\left[\mu_0{}^{\alpha}(\mu_0+1)^{\alpha+\beta}-1\right]\|x_0\|^{2\alpha+\beta}$$

$$=\|u_0x_0\|^{\alpha}\|\mu_0x_0+x_0\|^{\alpha+\beta}-\|x_0\|^{2\alpha+\beta}$$

$$=\|Ax_0\|^{\alpha}\|Ax_0+x_0\|^{\alpha+\beta}-\|x_0\|^{2\alpha+\beta},$$

which is a contradiction to (2.4), and so the condition (I-C) is satisfied. Therefore, it follows from Lemma 2.1 that the conclusion of Theorem 2.4 holds.

From Theorem 2.4, we can easily obtain the subsequent three corollaries.

Corollary 2.11 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta\in D$. Let $A:\overline{D}\to X$ be a condensing mapping. Suppose also that there exists $\alpha\ge1$ such that

$$\|x\|^{\alpha}\|Ax-x\|^{\alpha}\ge\|Ax\|^{\alpha}\|Ax+x\|^{\alpha}-\|x\|^{2\alpha},$$

$$\forall x\in D^* \text{ and } Ax\notin\overline{D},$$

where

$$D^*=\{x\in D: dist(x,\partial G)<\delta\}$$

for some $\delta>0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\beta=0$. It follows from Theorem 2.4 that the conclusion of Corollary 2.11 holds true.

Corollary 2.12 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta\in D$. Let $A:\overline{D}\to X$ be a condensing mapping. Suppose also that A satisfies

$$\|x\|\|Ax-x\|\ge\|Ax\|\|Ax+x\|-\|x\|^{2},$$

$$\forall x\in D^* \text{ and } Ax\notin\overline{D},$$

where

$$D^*=\{x\in D: dist(x,\partial G)<\delta\}$$

for some $\delta>0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\alpha=1,\beta=0$. It follows from Theorem 2.4

that the conclusion of Corollary 2.12 holds true.

Corollary 2.13 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A:\overline{D} \to X$ be a condensing mapping. Suppose also that A satisfies

$$\|Ax - x\|\|x\|^2 \geq \|Ax\|\|Ax + x\|^2 - \|x\|^3,$$

$$\forall x \in D^* \text{ and } Ax \notin \overline{D},$$

where

$$D^* = \{x \in D : dist(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\alpha = 1, \beta = 1$. It follows from Theorem 2.4 that the conclusion of Corollary 2.13 holds true.

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