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Numerical Study on the Boundary Value Problem by Using a Shooting Method

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Abstract: In the present paper, a shooting method for the numerical solution of nonlinear two-point boundary value problems is analyzed. Dirichlet, Neumann, and Sturm- Liouville boundary conditions are considered and numerical results are obtained. Numerical examples to illustrate the method are presented to verify the effectiveness of the proposed derivations. The solutions are obtained by the proposed method have been compared with the analytical solution available in the literature and the numerical simulation is guarantee the desired accuracy. Finally the results have been shown in graphically.

Keywords: Boundary Value Problem, Shooting Method, Numerical Simulation and MATLAB Programming

1. Introduction

There are many linear and nonlinear problems in science and engineering, namely second order differential equations with various types of boundary conditions, are solved either analytically or numerically. Two-point boundary value problems occur in a wide variety of problem such as modeling of chemical reactions, the boundary layer theory in fluid mechanics and heat power transmission. The wide applicability of boundary value problems in engineering and sciences calls for faster and accurate numerical methods. Many authors have attempted to obtain higher accuracy rapidly by using a numerous methods. The shooting method to compute eigen-values of fourth-order two-point boundary value problems studied by D. J. Jones [1].Wang et al [2] investigated application of the shooting method to second order multi point integral boundary value problems. Kwong and Wong [3] have studied the shooting method and non-homogeneous multipoint BVPs of second-order ODE. Abd-Elhameed et al [4] have investigated a new wavelet collection method for solving second-order multipoint boundary value problems using Chebyshev polynomials of the third and fourth kinds. See [5] studied nonlinear two point boundary value problem using two step direct method. Meade et al [6] discussed about the shooting technique for the solution two- point boundary value problems. Rahman et al [7] have studied numerical Solutions for Second Order Boundary Value Problems using Galerkin Method. Fatullayev et al [8] investigated numerical solution of a boundary value problem for a second order Fuzzy differential equation. Granas et al [9] investigated the shooting method for the numerical solution of a class of nonlinear boundary value problems. Cole and Adeboye [10] studied an alternative approach to solutions of nonlinear two point boundary value problems.

Qiao and Li [11] analyzed two kinds of important numerical methods forcalculating periodic solutions. TrungHieu [12] studied remarks on the shooting method for nonlinear two-point boundary value problem. Russell and Shampine [13] discussed numerical methods for singular boundary value problem. Sharma et al [14] studied numerical solution of two point boundary value problems using Galerkin-Finite element method. Hence the main objective of the present study is to solve nonlinear two point boundary value problems (BVP) by using simple and efficient shooting method. This well-known technique is an iterative algorithm which attempts to identify appropriate initial conditions for a related initial value problem that provides the solution to the original boundary value problem.

2. Mathematical Formulation

For a general boundary value problem for a second-order ordinary differential equation, the simple shooting method is stated as follows:

Let, $x''(t) = f(t, x(t), x'(t)), \quad t \in [a,b]$

$$x(a) = \alpha, \; x(b) = \beta \tag{2.1}$$

be the BVP in question and let x(t,s)denote the solution of the IVP

$$x''(t) = f(t, x(t), x'(t)), \; t \in [a,b] \quad x(a) = \alpha, \; x'(a) = s \tag{2.2}$$

where s is a parameter that can be varied. The IVP (2.2) is solved with different values of s with, e.g., Rung Kutta-4 method until the boundary condition on the right side $x(b) = \beta$ becomes fulfilled. As mentioned above, the solution $x(t,s)$ of (2.2) depends on the parameters. Let us define a function

$$F(s) := x(b,s) - \beta$$

If the BVP (2.1) has a solution, then the function F(s) has a root, which is just the value of the slope $x'(a)$ giving the solution x(t)of the BVP in question. The zeros of F(s) can be found with, e.g., Newton's method.

Newton's method is probably the best known method for finding numerical approximations to the zeroes of a real-valued function. The idea of the method is to use the first few terms of the Taylor series of a function F(s) in the vicinity of a suspected root, i.e.,

$$F(s_n + h) = F(s_n) + F'(s_n)h + Q(h^2)$$

Where s_n is the $n$thapproximation of the root. Now if one inserts $h = s - s_n$, one obtains

$$F(s) = F(s_n) + F'(s_n)(s - s_n)$$

As the next approximation $S_{n+1}$to the root we choose the zero of this function, i.e,

$$F(s_{n+1}) = F(s_n) + F'(s_n)(s_{n+1} - s_n) = 0$$
$$\Rightarrow s_{n+1} = s_n - \frac{F(s_n)}{F'(s_n)} \tag{2.3}$$

The derivative $F'(s_n)$ can be calculated using the forward difference formula

$$F'(s_n) = \frac{F(s_n + \delta s) - F(s_n)}{\delta s}$$

where δs is small. Notice that this procedure can be unstable near a horizontal asymptote.

3. Method of Solution Technique

Consider the boundary value problem for the second-order differential equation of the form

$$y'' = p(x)y' + q(x)y + r(x),$$
$$a \le x \le b, y(a) = \alpha\,, y(b) = \beta \tag{3.1}$$

Then the two initial value problems is given by

$$y'' = p(x)y' + q(x)y + r(x),$$
$$a \le x \le b, y(a) = \alpha, y'(a) = 0 \tag{3.2}$$
$$y'' = p(x)y' + q(x)y + r(x),$$
$$a \le x \le b, y(a) = 0, y'(a) = 1 \tag{3.3}$$

Then if $y_1(x)$ is the solution to (3.2) and $y_2(x)$ is the solution to equation (3.3) the solution to equation (3.1) is

$$y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2(x), y_2(b) \ne 0 \tag{3.4}$$

For the nonlinear case, the technique remains the same as that used to obtain a solution to equation (3.1) except that a sequence of initial value problems of the form;

$$y'' = f(x, y, y'), a \le x \le b, y(a) = \alpha, y'(a) = t_k \tag{3.5}$$

where $t_k$are real number are now required.

Let $y(x, t_k)$ be solution of the initial value problem(3.5).

We want to have a sequence $\{s_k\}$so that

$$\lim_{k\to\infty} y(x, t_k) = y(x) \tag{3.6}$$

One of the choices for $s_0$is;

$$s_0 = y'(a) = \frac{y(b) - y(a)}{b - a} = \frac{\beta - \alpha}{b - a}$$

Choosing the parameter $t_k$for $k \ge 1$ to satisfy (3.6) is not easy and can be complicated by the fact that;

$$y(b, t_k) - \beta = 0$$

is a nonlinear equation;

$$y'' = f\big(x, y(x,t), y'(x,t)\big), a \le x \le b, y(a,t) = \alpha,$$
$$y'(a,t) = t_k \tag{3.7}$$

The subscript k is dropped inside the functional notation for convenience differentiating equation (3.7) with respect to t and assuming that the order of differentiation of x and t is reversible gives

$$\frac{\partial}{\partial t} y''(x,t) = \frac{\partial}{\partial y} f\big(x, y(x,t), y'(x,t)\big) \frac{\partial}{\partial t} y(x,t) + \frac{\partial}{\partial y\prime} f\big(x, y(x,t), y'(x,t)\big) \frac{\partial}{\partial t} y'(x,t)$$

$$\frac{\partial}{\partial t} y(a,t) = 0, \quad \frac{\partial}{\partial t} y'(a,t) = 1$$

Simplification using $z(x,t)$ to represent $\frac{\partial}{\partial t} y(x,t)$ results in

$$z'' = \frac{\partial}{\partial y} f(x,y,y')z + \frac{\partial}{\partial y'} f(x,y,y')z', a \le x \le b, z(a) = 0, z'(a) = 1$$

$$= f_y z + f_{y'} z', a \le x \le b, z(a) = 0, z'(a) = 1$$

Finally, the Secant formula in general form is as follows:

$$t_k = t_{k-1} - \frac{g(t_{k-1})(t_{k-1} - t_k)}{g(t_{k-1}) - g(t_{k-2})}$$

We can update t_k using the information from $z(x,t)$ as follows:

$$t_k = t_{k-1} - \frac{(y(b,t_{k-1}) - \beta)}{z(b,t_{k-1})}$$

For a given accuracy ϵ, the algorithm terminated if;

$$|y(b,t_{k-1}) - \beta)| < \epsilon.$$

4. Results and Discussion

In this section, we explain five numerical examples of BVP which are available in the literature. The computations programming language, associated with the examples, are performed by **MATLAB** [15, 16, 17].

Example 1. Consider the boundary value problem;

$$y'' = \frac{1}{8}(32 + 2x^3 - yy'),\ 1 \le x \le 3,$$

$$y(1) = 17, y(3) = \frac{43}{3}$$

Solution: Let $f(x,y,y') = \frac{1}{8}((32 + 2x^3 - yy')$

$$f_y = -\frac{1}{8}y',\ f_{y'} = -\frac{1}{8}y$$

Solve a system of two second- order initial value problems:

$$y'' = \frac{1}{8}(32 + 2x^3 - yy'),\ 1 \le x \le 3, y(1) = 17, y'(1) = s_k$$

$$z'' = f_y z + f_{y'} z' = -\frac{1}{8}(y'z + yz'), 1 \le x \le\le 3, z(1) = 0, z'(1) = 1$$

Let $u_1 = y,\ u_2 = y',\ u_3 = z,\ u_4 = z'$.

Solve a system of four first-order initial value problems:

$$\begin{cases} u_1' = u_2 \\ u_2' = \frac{1}{8}(32 + 2x^3 - u_1 u_2) \\ u_3' = u_4 \\ u_4' = -\frac{1}{8}(u_2 u_3 + u_1 u_4) \end{cases}; \quad \begin{bmatrix} u_1(1) = 17 \\ u_2(1) = s_k \\ u_3(1) = 0 \\ u_4(1) = 1 \end{bmatrix}$$

Using MATLAB function ode45.m

Example 2.Consider the boundary value problem

$$y'' = -(y')^2 - y + \ln(x), 1 \le x \le 2, y(1) = 0, y(2) = \ln(2)$$

Solution: Let

$$f(x,y,y') = -(y')^2 - y + \ln(x),$$

$$f_y = -1, f_{y'} = -2y'$$

Solve a system of two second-order initial value problems:

$$y'' = -(y')^2 - y + \ln(x), 1 \le x \le 2, y(1) = 0, y'(1) = s_k$$

$$z'' = f_y z + f_{y'} z' = -z - 2y'z', 1 \le x \le 2, z(1) = 0, z'(1) = 1$$

Let $u_1 = y, u_2 = y', u_3 = z,\ u_4 = z'$.

Solve a system of four first-order initial value problems:

$$\begin{cases} u_1' = u_2 \\ u_2' = -(u_2)^2 - u_1 + \ln(x) \\ u_3' = u_4 \\ u_4' = -u_3 - 2u_2 u_4 \end{cases}; \quad \begin{bmatrix} u_1(1) = 0 \\ u_2(1) = s_k \\ u_3(1) = 0 \\ u_4(1) = 1 \end{bmatrix}$$

Using MATLAB function ode45.m

Example 3. Consider the boundary value problem

$$y'' = \frac{3}{2}y^2, 0 \le x \le 1, y(0) = 4, y(1) = 1$$

Solution: Let $f(x,y,y') = \frac{3}{2}y^2$

$$f_y = 3y, f_{y'} = 0$$

Solve a system of two second-order initial value problem

$$y'' = \frac{3}{2}y^2, 0 \le x \le 1, y(0) = 4, y'(0) = s_k$$

$$z'' = f_y z + f_{y'} z' = 3yz, 0 \le x \le 1, z(0) = 0, z'(0) = 1$$

Let $u_1 = y, u_2 = y', u_3 = z, u_4 = z'$.

Solve a system of four first-order initial value problem

$$\begin{cases} u_1' = u_2 \\ u_2' = \frac{3}{2}u_1^2 \\ u_3' = u_4 \\ u_4' = 3u_1 u_3 \end{cases}; \quad \begin{bmatrix} u_1(0) = 4 \\ u_2(0) = s_k \\ u_3(0) = 0 \\ u_4(0) = 1 \end{bmatrix}$$

Using MATLAB function ode45.m

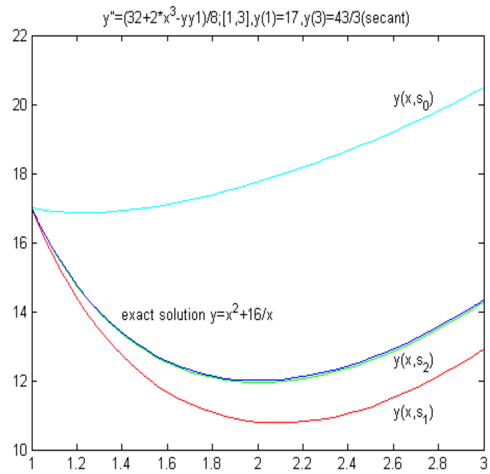

Figure 1. *Plot of exact and approximate solution.*

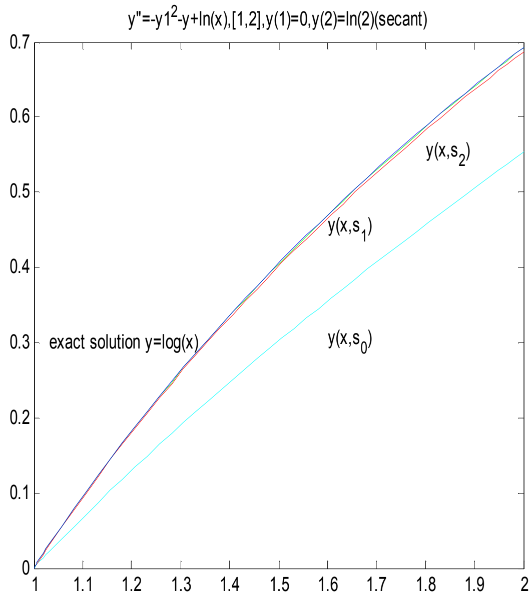

Figure 2. *Plot of exact and approximate solution.*

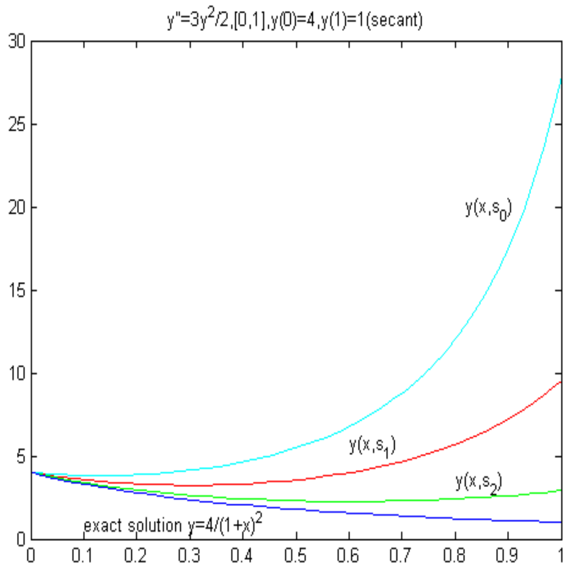

Figure 3. *Plot of exact and approximate solution.*

Figure 1, Figure 2 and Figure 3 shows that the approximate solution of Example 1, Example 2 and Example 3 is at the third iteration (shot), because the solution is almost coincident with the exact solution among the three shots.

Example 4.Consider the boundary value problem

$$y'' = y^3 - yy', 1 \le x \le 2, y(1) = \tfrac{1}{2}, y(2) = \tfrac{1}{3}$$

Solution: Let

$$f(x, y, y') = y^3 - yy'$$

$$f_y = 3y^2 - y', \; f_{y'} = -y$$

Solve a system of two second-order initial value problems:

$$y'' = y^3 - yy', 1 \le x \le 2, y(1) = \tfrac{1}{2}, y'(1) = s_k$$

$$z'' = f_y z + f_{y'} z' = (3y^2 - y')z - yz', 1 \le x \le 2, z(1) = 0, z'(1) = 1$$

Let $u_1 = y$, $u_2 = y'$, $u_3 = z$, $u_4 = z'$.
Solve a system of four first-order initial value problems:

$$\begin{cases} u_1' = u_2 \\ u_2' = (u_1)^3 - u_1u_2 \\ u_3' = u_4 \\ u_4' = 3(u_1)^2u_3 - u_2u_3 - u_1u_4 \end{cases} ; \begin{bmatrix} u_1(1) = \frac{1}{2} \\ u_2(1) = s_k \\ u_3(1) = 0 \\ u_4(1) = 1 \end{bmatrix}$$

Using MATLAB function ode45.m

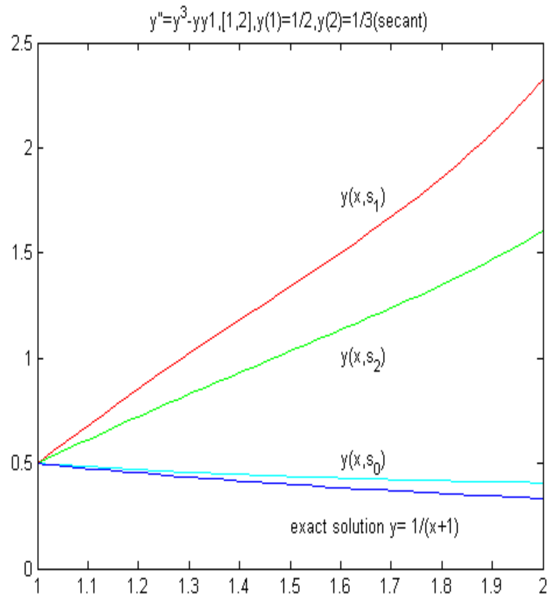

Figure 4. *Plot of exact and approximate solution.*

Example 5.Consider the boundary value problem

$$y'' = y' + 2(y - \ln x)^3 - \tfrac{1}{x}, 2 \le x \le 3,$$

$$y(2) = \tfrac{1}{2} + \ln 2, y(3) = \tfrac{1}{3} + \ln 3$$

Solution: Let

$$f(x, y, y') = y' + 2(y - \ln x)^3 - \tfrac{1}{x}$$

$$f_y = 6(y - \ln x)^2, f_{y'} = 1$$

Solve a system of two second-order initial value problems:

$$y'' = y' + 2(y - \ln x)^3 - \tfrac{1}{x}, 2 \le x \le 3, y(2) = \tfrac{1}{2} + \ln 2, y'(2) = s_k$$

$$z'' = f_y z + f_{y'} z' = 6(y - \ln x)^2 z + z', 2 \le x \le 3, z(2) = 0, z'(2) = 1$$

Let $u_1 = y$, $u_2 = y'$, $u_3 = z$, $u_4 = z'$.
Solve a system of four first-order initial value problems:

$$\begin{cases} u_1' = u_2 \\ u_2' = u_2 + 2(u_1 - \ln(x))^3 - \frac{1}{x} \\ u_3' = u_4 \\ u_4' = 6u_3(u_1 - \ln(x))^2 + u_4 \end{cases}; \quad \begin{bmatrix} u_1(2) = \frac{1}{2} + \ln(2) \\ u_2(2) = s_k \\ u_3(2) = 0 \\ u_4(2) = 1 \end{bmatrix}$$

Using MATLAB function ode45.m

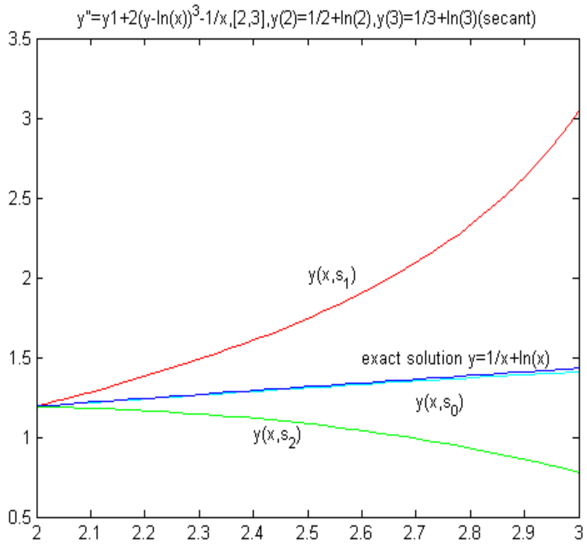

Figure 5. *Plot of exact and approximate solution.*

Figure 4 and Figure 5 shows that the approximate solution of Example 4 and Example 5 is at the first iteration, because this solution is so close to the exact solution among the three shoots.

5. Conclusion

We have developed a Shooting method to solve non-linear two point boundary value problem analytically. The given problems were tested using three iterations of shooting method. In each figure, we represent the comparison between the exact solution and each iteration, which are made in order to solve these problems. The numerical results obtained by the proposed method are in good agreement with the exact solutions.

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