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# Enumeration of Triangles in Cayley Graphs

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**Abstract:** Significant contributions can be found on the study of the cycle structure in graphs, particularly in Cayley graphs. Determination of Hamilton cycles and triangles, the longest and shortest cycles attracts special attention. In this paper an enumeration process for the determination of number of triangles in the Cayley graph associated with a group not necessarily abelian and a symmetric subset of the group.

**Keywords:** Cayley Graphs, Fundamental Triangle, Triangle and Group

## 1. Introduction

The cycle structure of Cayley graphs associated with certain arithmetic functions were studied by Berrizbeitia and Giudici[2][3] and Dejter and Giudici [5]. Maheswari and Madhavi [7] [8] enumerated Hamilton cycles and triangles in arithmetic Cayley graphs associated with Euler totient function and quadratic residues modulo a prime. The number of triangles in the arithmetic Cayley graph associated with the divisor function is determined in [4].

In [2] Berrizbeitia and Giudici consider sequences of Cayley graphs $Cay(G_n, S_n)$ satisfying the multiplicative arithmetic property, where $G_n$ is a finite abelian group and $S_n$ is a subset of $G_n$ . The sequence of Cayley graphs $X_n = Cay(G_n, S_n)$ as the multiplicative arithmetic property (map) if for each pair of positive relatively prime integers $(m,n)$ there is a group isomorphism $\varphi_{n,m}$ from $G_{nm}$ to $G_n \times G_m$ such that $\varphi_{n,m}$ maps $S_{nm}$ onto $S_n \times S_m$. In [2], it is proved that if $X_n = Cay(G_n, S_n)$ is a sequence of Cayley graphs with the map, then that function $2kp_k(n)$ is a linear combination of multiplicative arithmetic functions, where $p_k(n)$ denotes that number of induced k-cycles of $X_n$ using this formula the $p_3(n)$ and $p_4(n)$ are obtained for the sequence $Cay(Z_n, U_n)$ in terms of prime divisors of $n$ , where $Z_n$ is the ring of integers modulo $n$ and $U_n$ is the multiplicative group of units modulo $n$ . In this paper we give a formula for determining the number of triangles in a Cayley graph $G(X,S)$ where $X$ is a finite group, not necessarily abelian and $S$ is a symmetric subset of $X$ .

## 2. Enumeration of Triangles in Cayley Graphs

Let $(X, \cdot)$ be a group. A subset $S$ of $X$ is called a symmetric subset if $s^{-1} \in S$ for all $s \in S$ . The graph $G$ with vertex set $X$ and edge set $\{(g,h) : g^{-1}h \in S \text{ or } hg^{-1} \in S\}$ is called the Cayley graph of $X$ corresponding to the symmetric sub set $S$ of $X$ .We denote this graph by $G(X, S)$ and assume that $S$ does not contain the identity element $e$ of $X$ so that $G(X, S)$ contains no loops. Clearly $G(X, S)$ is an undirected graph which is $|S|$- regular with size $|X||S|/2$ and vertex transitive.

$G(X, S)$ associated with a group $(X, \cdot)$ not necessarily abelian, and a symmetric sub set $S$ of $X$ .

Definition 2.1: Let $e$ be the identity element of the group $(X, \cdot)$ . For $a,b \in X$ , if the triad $(e,a,b)$ is a triangle in $G(X,S)$ then $(e,a,b)$ is called a fundamental triangle.

Lemma 2.2: For a given $a \in S$ the number of fundamental triangles in $G(X,S)$ is $|S \cap aS|$ .

Proof: Let $a \in S$ . For any $b \in X$ , $(e,a,b)$ is a fundamental triangle $\Leftrightarrow (e, a), (e, b)$ and $(a, b)$ are edges in $G(X,S)$ $\Leftrightarrow e^{-1}a,\ e^{-1}b$ , $a^{-1}b \in S$ $\Leftrightarrow a, b, a^{-1}b \in S$ $\Leftrightarrow a \in S$ and

$b \in S \cap aS$ .That is, for a given $a \in S$ and for each $b \in S \cap aS$ the triad $(e,a,b)$ is a fundamental triangle in $G(X,S)$ and vice versa so that the number of fundamental triangles in $G(X,S)$ is $|S \cap aS|$.

Lemma 2.3: The number of distinct fundamental triangles in $G(X,S)$ is $\frac{1}{2}\sum_{a\in S}|S \cap aS|$.

Proof: By the Lemma 2.2, for each $a \in S$ the number of fundamental triangles in $G(X,S)$ is $|S \cap aS|$ so that the total number of fundamental triangles in $G(X,S)$ is $\sum_{a\in S}|S \cap aS|$.

However the triangles $(e,a,b)$ and $(e,b,a)$ represent the same fundamental triangle since $S$ is a symmetric subset of $X$,

$$a^{-1}b \in S \iff b^{-1}a \in S .$$

So the number of distinct fundamental triangles in $G(X,S)$ is

$$\frac{1}{2}\sum_{a\in S}|S \cap aS|$$

Theorem 2.4: The number of distinct triangles in $G(X,S)$ is

$$\frac{1}{6}|X|\sum_{a\in S}|S \cap aS|$$

Proof: Let $e$ be the identity element of the group $(X,\cdot)$ and let $g$ be any vertex of $G(X,S)$. Since $G(X,S)$ is vertex transitive and regular, the number of triangles in $G(X,S)$ with $g \in X$ as one vertex is equal to the number of fundamental triangles in $G(X,S)$ namely,

$$\frac{1}{2}\sum_{a\in S}|S \cap aS|$$

and the number of triangles in $G(X,S)$ is

$$\frac{1}{2}|X|\sum_{a\in S}|S \cap aS|$$

However each triangle in $G(X,S)$ is counted thrice, namely, once by each of its three vertices so that the number of distinct triangles in $G(X,S)$ is

$$\frac{1}{6}|X|\sum_{a\in S}|S \cap aS|.$$

The following Corollary is immediate.

Corollary 2.5: The Cayley graph $G(X,S)$ has no triangles if and only if $S \cap aS = \phi$ for all $a \in S$.

Example2.6: For the Dihedral group and its symmetric subset $D_4 = \{\rho_0, \rho_1, \rho_2, \rho_3, \mu_1, \mu_2, \delta_1, \delta_2\}$ $S = \{\mu_1, \mu_2, \delta_1\}$, where

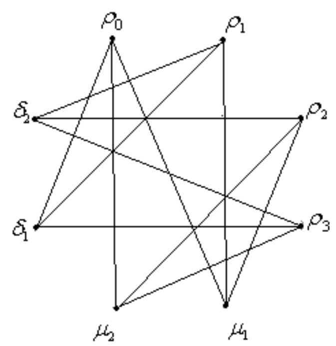
$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix},$$

$$\rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, \mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

$$\mu_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \delta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \text{ and}$$

$$\delta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

consider the Cayley graph $G(D_4, S)$



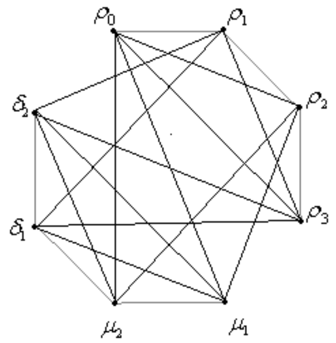

*Cayley Graph* $G(D_4,S)$

Since $\mu_1 S = \{\rho_0, \rho_2, \rho_3\}$ , $\mu_2 S = \{\rho_2, \rho_0, \rho_1\}$ and $\delta_1 S = \{\rho_1, \rho_2, \rho_0\}$ we have, $|S \cap \mu_2 S| = 0$ and $|S \cap \delta_1 S| = 0$. So the number of distinct triangles in $G(D_4,S)$ is equal to

$$\frac{1}{6}|D_4|\sum_{a\in S}|S \cap aS|$$

and $G(D_4, S)$ has no triangles .

Example 2.7: Consider the Cayley graph $G(D_4,S)$ where $S = \{\rho_1, \rho_2, \rho_3, \mu_1, \mu_2\}$.




*Cayley Graph* $G(D_4,S)$

Since $\rho_3 S=\{\rho_0, \rho_1, \rho_2, \delta_2, \delta_1\}$ , $\mu_1 S=\{\delta_2, \mu_2, \delta_1, \rho_0, \rho_2\}$ and $\mu_2 S=\{\delta_1, \mu_1, \delta_2, \rho_2, \rho_0\}$ , we have $|S\cap\rho_1 S|=2$ , $|S\cap\rho_2 S|=4$ , $|S\cap\rho_3 S|=2$ , $|S\cap\mu_1 S|=2$ and $|S\cap\mu_2 S|=2$ . So the number of distinct triangles in $G(D_4,S)$ is equal to

$$\frac{1}{6}|D_4|\sum_{a\in S}|S\cap aS|=\frac{8}{6}[2+4+2+2+2]=16.$$

## 3. Deductions

### *3.1. Enumeration of Triangles in the Euler Totient Cayley Graph*

Let $n\geq 1$ be an integer and let $(Z_n,\oplus)$ be the group of residue classes modulo $n$ with respect to the addition $\oplus$ modulo $n$ . Then the set $S=\{s:1\leq s<n$ and $s$ is relatively prime to $n\}$ is a symmetric subset of $(Z_n,\oplus)$ . The Euler totient Cayley graph $G(Z_n,\Phi)$ is the graph whose vertex set is

$$V=Z_n=\{0,1,2,...,n-1\}$$

and the edge set

$$E=\{(x,y):x,y\in V, x-y\in S \ or \ y-x\in S\}.$$

In [8] it is established that the graph $G(Z_n,\Phi)$ is $\varphi(n)$ - regular with size $n\varphi(n)/2$ , Hamiltonian, connected, bipartite and contains no triangles if $n$ is even.

For any vertex $a$ in $G(Z_n,\Phi)$,

$$a\in S\cap(1+S)\Leftrightarrow a\in S \, and \, a\in 1+S$$

$\Leftrightarrow a,a-1\in S \Leftrightarrow a$ and $a-1$ belong to $S$ , or, they constitute a pair of consecutive integers less than $n$ and relatively prime to $n$ .

Thus $|S\cap(1+S)|=\varphi^{(2)}(n)$ , the Schemmel totient function which denotes the number of pairs of consecutive positive integers less than $n$ and relatively prime to $n$ . Further using the fact that $(S,\otimes)$ is a multiplicative subgroup of order $\varphi(n)$ of the semi-group $(Z_n^{*},\otimes)$ , where $Z_n^{*}=Z_n-\{0\}$ , and $\otimes$ denotes the multiplication modulo $n$ , one can see that $|S\cap(a+S)|=|S\cap(1+S)|$ for $a\in S$ . To establish this, for $a\in S$ define the map

$$f:S\cap(1+S)\rightarrow S\cap(a+S)$$

by $f(x)=ax$ for all $x=S\cap(1+S)$.

Let $x\in S\cap(1+S)$ .Then $x\in S$ and $x\in 1+S$ so that $x=1+s$ and $ax=a(1+s) =a+as$ for some $s\in S$ .Since $(S,\otimes)$ is a group, $a, x, s\in S$ implies that $ax, as\in S$ so that $ax=a+as\in a+aS$ .

Hence the $f$ maps $S\cap(1+S)$ into $S\cap(a+S)$.

For $x_1,x_2\in S\cap(1+S)$, $f(x_1)=f(x_2)$ implies that $ax_1=ax_2$ . Since $(S,\otimes)$ is a group, this gives $x_1=x_2$ and $f$ is one- to - one.

Let $y\in S\cap(a+S)$ . Then $y\in S$ and $y=a+s$ for some $s\in S$ .Since $(S,\otimes)$ is a group, for $a,s\in S$ there exist $s_1\in S$ such that $s=as_1$ so that

$$y=a+s=a+as_1=a(1+s_1),$$

or, $1+s_1=a^{-1}y\in S$ , since $a,y\in S$ .

Also $s_1\in S$ implies that $1+s_1\in 1+S$ .That is, $1+s_1\in S\cap(1+S)$ . For this $1+s_1\in S\cap(1+S)$ and $f(1+s_1)=a(1+s_1)=y$ .This shows that $f$ is onto and hence a bijection showing that $|S\cap(1+S)|=|S\cap(a+S)|$.

So by the Theorem 2.4, the number of distinct triangles in the graph $G(Z_n,\Phi)$ is equal to

$$\frac{|Z_n|}{6}\sum_{a\in S}|S\cap(a+S)| =\frac{n}{6}\sum_{a\in S}|S\cap(1+S)|$$

$$=\frac{n}{6}\varphi^{(2)}(n)|S|$$

$$=\frac{n}{6}\varphi^{(2)}(n)\varphi(n)=\frac{n}{6}n\prod_{p|n}(1-\frac{2}{p})n\prod_{p|n}(1-\frac{1}{p})$$

$$=\frac{n^3}{6}\prod_{p|n}(1-\frac{2}{p})(1-\frac{1}{p}),$$

since $\varphi(n)=n\prod_{p|n}(1-\frac{1}{p})$ and $\varphi^{(2)}(n)=n\prod_{p|n}(1-\frac{2}{p})$ (see p.147 of [6]).

Remark 3.1.1: If $n$ is even then $2\,|\,n$ and the term corresponding to $p=2$ in the above product is zero so that $G(Z_n,\Phi)$ contains no triangles.

Example 3.1.2: Consider the Euler totient Cayley graph $G(Z_n,\Phi)$ for $n=45$. Here $n=3^2 5$. So the number of distinct triangles in the graph $G(Z_{315},\Phi)$ is

$$\frac{(45)^3}{6}(1-\frac{2}{3})(1-\frac{2}{5})(1-\frac{1}{3})(1-\frac{1}{5})=1620.$$

### *3.2. Enumeration of Triangles in the Quadratic Residue Cayley Graph*

Let $p$ be an odd prime and consider the set $Q$ of the quadratic residues modulo $p$. Let $Q^*=\{s,p-s:s\in Q\}$. Then $Q^*$ is a symmetric subset of the additive abelian group $(Z_p,\oplus)$. The quadratic residue Cayley graph is the graph whose vertex set $V=Z_p=\{0,1,2,...,p-1\}$ and the edge set

$$E=\{(x,y):x,y\in V,x-y\in Q^*\ or\ y-x\in Q^*\}.$$

For any vertex $a$ in $G(Z_p,Q^*)$, we have $a\in Q^*\cap(1+Q^*)\Leftrightarrow a\in Q^*$ and $a\in 1+Q^*$

$\Leftrightarrow a\in Q^*$ and $a-1\in Q^*$ $\Leftrightarrow a$ and $a-1$

are consecutive quadratic residues modulo $p$. Thus

$$\left|Q^*\cap(1+Q^*)\right|=Q^{(2)}(p),$$

where $Q^{(2)}(p)$ denotes the number of pairs of consecutive integers less than $p$, since $(Q^*,\otimes)$ is a subgroup of the multiplicative group $(Z_p^*,\otimes)$, one can see as in the case of Euler totient Cayley graph that

$$\left|Q^*\cap(a+Q^*)\right|=\left|Q^*\cap(1+Q^*)\right|$$

for all $a\in Q^*$.

So by the Theorem 2.4 the number of distinct triangles in the graph $G(Z_p,Q^*)$ is

$$\frac{|Z_p|}{6}\sum_{a\in Q^*}\left|Q^*\cap(a+Q^*)\right|=\frac{p}{6}\sum_{a\in Q^*}\left|Q^*\cap(1+Q^*)\right|$$

$$=\frac{p}{6}\left|Q^*\cap(1+Q^*)\right|\left|Q^*\right|=\frac{p}{6}Q^{(2)}(p)\left|Q^*\right|$$

If $p\equiv 1 \pmod 4$, then $Q^*=Q$, $|Q|=\frac{p-1}{2}$ and $Q^{(2)}(p)=\frac{1}{4}[p-4-(-1)^{(p-1)/2}]=\frac{p-5}{4}$ (see section 10.1 of [1]). So the number of distinct triangles in $G(Z_p,Q)$ is $\frac{p(p-1)}{48}(p-5)$.

If $p\equiv 3 \pmod 4$, then $Q^*=\{1,2,3,...,p-1\}$, $\left|Q^*\right|=(p-1)$ and $Q^{(2)}(p)=p-2$ so that the number of distinct triangles in $G(Z_p,Q^*)$ is $\frac{p(p-1)(p-2)}{6}$.

Example 3.2.1: Consider the quadratic residue Cayley graph $G(Z_{73},Q)$ .Here $73\equiv 1 \pmod 4$. So the number of distinct triangles in $G(Z_{73},Q)$ is equal to

$$\frac{p(p-1)(p-5)}{48}=\frac{73\times 72\times 68}{48}=7446.$$

Example 3.2.2: Consider the quadratic residue Cayley graph $G(Z_{71},Q^*)$.

Here $71\equiv 3 \pmod 4$. So the number of distinct triangles in $G(Z_{71},Q^*)$ is equal to $\frac{p(p-1)(p-2)}{6}$ $=\frac{71\times 70\times 69}{6}=57155$.

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