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Weihgted Cesaro Sequence Space and Related Matrix Transformation

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Abstract: In this paper we define the weighted Cesaro sequence spaces *ces (p, q)*.We prove the space *ces(p, q)* is a complete paranorm space. In section-2 we determine its Kothe-Toeplitz dual and continuous dual. In section-3 we establish necessary and sufficient condition for a matrix A to map *ces (p, q)* to l_∞ and *ces(p, q)* to c, where l_∞ is the space of all bounded sequences and c is the space of all convergent sequences. We also get some known and unknown interesting results as corollaries.

Keywords: Sequence Space, Kothe-Toeplitz Dual, Matrix Transformation

1. Introduction

Let ω be the space of all (real or complex) sequences and let l_∞ and c respectively the Banach spaces of bounded and convergent sequence $x = (x_n)$ endowed with the norm

$$||x|| = \sup_{k\geq 1} |x_k|$$

In [8] Shiue introduce the Cesaro sequence space ces_p as

$$ces_p = \left\{x = (x_k) \in \omega : \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p < \infty\right\} \; for\ 1 < p < \infty$$

$$and\ ces_\infty = \left\{x = (x_k) \in \omega : \sup_{n\geq 1} \frac{1}{n}\sum_{k=1}^{n}|x_k|\right\} \; for\ p = \infty$$

In [4] Leibowitz studied some properties of this space and showed that it is a Banach space. Lim [9] defined this space in a different norm as

$$ces_p = \left\{x = (x_k) \in \omega : \sum_{r=0}^{\infty}\left(\frac{1}{2^r}\sum_{r}|x_k|\right)^p < \infty\right\} \; for\ 1 < p < \infty$$

and

$$ces_\infty = \left\{x = (x_k) \in \omega : \sup_{r\geq 0} \frac{1}{2^r}\, |x_k| < \infty\right\} \; for\ p = \infty$$

where $\sum_r$ *denotes* a sum over the ranges $[\,2^r, 2^{r+1})$, determined its dual spaces and characterize some matrix classes. Later in [10] Lim extended this space $ces_p\ to\ ces(p)$ for the sequence $p = (p_r)$ with $\inf p_r > 0$ and defined as

$$ces(p) = \left\{x = (x_k) \in \omega : \sum_{r=0}^{\infty}\left(\frac{1}{2^r}\sum_{r}|x_k|\right)^{p_r} < \infty\right\}.$$

For positive sequence of real numbers $(p_n), (q_n)\ and\ Q_n = q_1 + q_2 + \cdots + q_n$

Johnson and Mohapatra [11] defined the Cesaro sequence space $ces(p, q)\ as$

$$ces(p, q) = \left\{x = (x_k) \in \omega : \sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k\,|x_k|\right)^{p_r} < \infty\right\}$$

and studied some inclusion relations.

What amounts to the same thing defined by Khan and Rahman [3] as

$$ces(p, q) = \left\{x = (x_k) \in \omega : \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x_k|\right)^{p_r} < \infty\right\}$$

For $p = (p_r)$ with $\inf p_r > 0$, $Q_{2^r} = q_{2^r} + q_{2^r+1} + \cdots + q_{2^{r+1}-1}$ and $\sum_r$ denotes a sum over the ranges $[2^r, 2^{r+1})$.

They determined it's Kothe -Toeplitz dual and characterized some matrix classes.

In this paper we define the Cesaro weighted sequence space $ces(p,q)$ in the following way.

Definition. If (q_n) is a bounded sequence of positive real numbers, then for $p = (p_r)$ with $\inf p_r > 0$, the Cesaro weighted sequence space $ces(p,q)$ is defined by

$$ces(p,q) = \left\{ x = (x_k) \in \omega : \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k| \right)^{p_r} < \infty \right\}$$

where $Q_{2^r} = q_{2^r} + q_{2^r+1} + \cdots + q_{2^{r+1}-1}$ and $\sum_r$ denotes a sum over the range $2^r \le k < 2^{r+1}$.

If $q_n = 1$ for all n, then $ces(p,q)$ reduces to $ces(p)$ studied by Lim[10]. Also, if $p_n = p$ for all n and $q_n = 1$ for all n, then $ces(p,q)$ reduces to ces_p studied also Lim[9]. Obviously, $l(p) \subset ces(p) \subset ces(p,q)\, for\ p_r \ge 1$.

In their paper [1] Maji and Srivastava defined the weighted Cesaro sequence space $ces(p,q)$ with a different norms and studied on some operators and inclusion results.

The main purpose of this note is to define and investigate the weighted Cesaro sequence space $ces(p,q)$, determine its Kothe-Toeplitz dual and characterize the class of matrices $(ces(p,q), l_\infty)$ and $(ces(p,q), c)$, where l_∞ and c are respectively the spaces of bounded and convergent complex sequences. By specializing sequences (p_n) and (q_n), we get the results of Lim([9], [10]) as corollaries. Meanwhile, we also determine all continuous linear functional on $ces(p,q)$ for all $1 < p_r < \infty$.

With regard to notation, the dual space of $ces(p,q)$, i.e, the space of all continuous linear functionals on $ces(p,q)$, will be denoted by $ces^*(p,q)$. We write

$$A_r(n) = \max_r \left| \frac{a_{n,k}}{q_k} \right|$$

where for each n the maximum with respect to k in $[2^r, 2^{r+1})$.

Throughout the paper the following well-known inequality (see [6] or [7]) will be frequently used. For any $E > 0$ and any two complex numbers a and b we have

$$|ab| \le E(|a|^t E^{-t} + |b|^t) \tag{1}$$

Where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$

To begin with, we show that the space $ces(p,q)$ is paranormed by

$$g(x) = \left(\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k| \right)^{p_r} \right)^{1/M} \tag{2}$$

Provided $H = \sup_r p_r < \infty$ and $M = \max\{1, H\}$

Clearly

$$g(\theta) = 0$$

$$g(-x) = g(x),$$

where $\theta = (0,0,0,\ldots)$

Since $p_r \le M, M \ge 1$ so for any $x, y \in ces(p,q)$ we have by Minkowski's inequality

$$\left(\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r |q_k(x_k + y_k)| \right)^{p_r} \right)^{1/M}$$

$$\le \left(\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r (|q_k x_k| + |q_k y_k|) \right)^{p_r} \right)^{1/M}$$

$$\le \left(\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k| \right)^{p_r} \right)^{1/M} + \left(\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r |q_k y_k| \right)^{p_r} \right)^{1/M}$$

which shows that g is subadditive.

Finally we have to check the continuity of scalar multiplication. From the definition of $ces(p,q)$, we have $\inf p_r > 0$. So, we may assume that $\inf p_r \equiv \rho > 0$. Now for any complex λ with $||\lambda|| < 1$, we have

$$g(\lambda x) = \left(\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r |\lambda q_k x_k| \right)^{p_r} \right)^{1/M}$$

$$= |\lambda|^{p_r/M} \left(\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k| \right)^{p_r} \right)^{1/M}$$

$$\le \sup_r ||\lambda||^{\frac{p_r}{M}} g(x)$$

$$\le ||\lambda||^{\frac{\rho}{M}} g(x) \to 0 \text{ as } \lambda \to 0$$

It is quite routine to show that $ces(p,q)$ is a metric space with the metric $d(x,y) = g(x-y)$ provided that $x, y \in ces(p,q)$, where g is defined by (2). And using a similar method to that in [3] one can show that $ces(p,q)$ is complete under the metric mentioned above.

2. Kothe-Toeplitz Duals

If X is a sequence space we define ([2], [5])

$$X^{|+|} = X^\alpha = \left\{ \begin{array}{c} a = (a_k) \in \omega : \sum_k |a_k x_k| < \infty, \\ for\ every\ x \in X \end{array} \right\}$$

$$X^+ = X^\beta$$
$$= \left\{ a = (a_k) \right.$$
$$\left. \in \omega : \sum_k a_k x_k \text{ is convergent, for every } x \in X \right\}$$

Now we are going to give the following theorem by which the generalized Kothe-Toeplitz dual $ces^+(p,q)$ will be determined.

Theorem 1: If $1 < p_r \le \sup_r p_r < \infty$ and $\frac{1}{p_r} + \frac{1}{t_r} = 1, for\ r = 0,1,2,\ldots$ then

$$ces^+(p,q) = [ces(p,q)]^\beta$$

$$= \left\{ a = (a_k) : \sum_{r=0}^{\infty} \left(Q_{2^r} \max_r \left| \frac{a_k}{q_k} \right| \right)^{t_r} E^{-t_r} < \infty, \text{ for some integer } E > 1 \right\}.$$

Proof: Let $1 < p_r \leq \sup_r p_r < \infty$ and $\frac{1}{p_r} + \frac{1}{t_r} = 1$, for $r = 0,1,2, \ldots$ Define

$$\mu(t) = \left\{ a = (a_k) : \sum_{r=0}^{\infty} \left(Q_{2^r} \max_r \left| \frac{a_k}{q_k} \right| \right)^{t_r} E^{-t_r} < \infty, \text{ for some integer } E > 1 \right\} \quad (3)$$

We want to show that $ces^+(p,q) = \mu(t)$.

Let $x \in ces(p,q)$ and $a \in \mu(t)$. Then using inequality (1) we get

$$\sum_{k=1}^{\infty} |a_k x_k| = \sum_{r=0}^{\infty} \sum_r |a_k x_k|$$

$$= \sum_{r=0}^{\infty} \sum_r \left| \frac{a_k}{q_k} q_k x_k \right|$$

$$= \sum_{r=0}^{\infty} \sum_r \left| \frac{a_k}{q_k} \right| |q_k x_k|$$

$$\leq \sum_{r=0}^{\infty} \max_r \left| \frac{a_k}{q_k} \right| \sum_r |q_k x_k|$$

$$= \sum_{r=0}^{\infty} Q_{2^r} \max_r \left| \frac{a_k}{q_k} \right| \frac{1}{Q_{2^r}} \sum_r |q_k x_k|$$

$$\leq E \sum_{r=0}^{\infty} \left\{ \left(Q_{2^r} \max_r \left| \frac{a_k}{q_k} \right| \right)^{t_r} E^{-t_r} + \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k| \right)^{p_r} \right\}$$

$$= E \left\{ \sum_{r=0}^{\infty} \left(Q_{2^r} \max_r \left| \frac{a_k}{q_k} \right| \right)^{t_r} E^{-t_r} + \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k| \right)^{p_r} \right\}$$

$$< \infty$$

which implies that the series $\sum_{k=1}^{\infty} a_k x_k$ convergent.

Therefore,

$a \in dual\ of\ ces(p,q) = ces^+(p,q)$

This shows, $\mu(t) \subset ces^+(p,q)$

Conversely, suppose that $\sum a_k x_k$ is convergent for all $x \in ces(p,q)$ but $a \notin \mu(t)$. Then

$\sum_{r=0}^{\infty} \left(Q_{2^r} \max_r \left| \frac{a_k}{q_k} \right| \right)^{t_r} E^{-t_r} = \infty$, for every integer $E > 1$

So we can define a sequence $0 = n(0) < n(1) < n(2) < \cdots$,

such that $\gamma = 0, 1, 2, \ldots$, we have

$$M_\gamma = \sum_{r=n(\gamma)}^{n(\gamma+1)-1} \left(Q_{2^r} \max_r \left| \frac{a_k}{q_k} \right| \right)^{t_r} (\gamma+2)^{-t_r/p_r} > 1$$

Now we define a sequence $x = (x_k)$ in the following way:

$$x_{N(r)} = Q_{2^r}^{t_r} \left| a_{N(r)} \right|^{t_r - 1} sgn\ a_{N(r)} (\gamma+2)^{-t_r} M_\gamma^{-1}$$

for $n(\gamma) \leq r \leq n(\gamma+1) - 1, \gamma = 0, 1, 2, \ldots$ and $x_k = 0$ for $k \neq N(r)$, where $N(r)$ is such that $\left| a_{N(r)} \right| = \max_r \left| \frac{a_k}{q_k} \right|$, the maximum is taken with respect to k in $[2^r, 2^{r+1})$.

Therefore,

$$\sum_{k=2^{n(\gamma)}}^{2^{n(\gamma+1)-1}} a_k x_k = \sum_{r=n(\gamma)}^{n(\gamma+1)-1} \left(Q_{2^r} \left| a_{N(r)} \right| \right)^{t_r} (\gamma+2)^{-t_r} M_\gamma^{-1}$$

$$= M_\gamma^{-1} (\gamma+2)^{-1} \sum_{r=n(\gamma)}^{n(\gamma+1)-1} \left(Q_{2^r} \left| a_{N(r)} \right| \right)^{t_r} (\gamma+2)^{-t_r/p_r}$$

$$= M_\gamma^{-1} M_\gamma (\gamma+2)^{-1}$$

$$= (\gamma+2)^{-1}$$

It follows that

$\sum_{k=1}^{\infty} a_k x_k = \sum_{\gamma=0}^{\infty} (\gamma+2)^{-1}$ diverges.

Moreover

$$\sum_{r=n(\gamma)}^{n(\gamma+1)-1} \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k| \right)^{p_r}$$

$$= \sum_{r=n(\gamma)}^{n(\gamma+1)-1} Q_{2^r}^{(t_r-1)p_r} \left| a_{N(r)} \right|^{(t_r-1)p_r} (\gamma+2)^{-t_r/p_r} M_\gamma^{-p_r}$$

$$\leq (\gamma+2)^{-2} M_\gamma^{-1} \sum_{r=n(\gamma)}^{n(\gamma+1)-1} Q_{2^r}^{t_r} \left| a_{N(r)} \right|^{t_r} (\gamma+2)^{-t_r/p_r}$$

$$= (\gamma+2)^{-2} M_\gamma^{-1} M_\gamma$$

$$= (\gamma+2)^{-2}$$

Therefore

$$\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k| \right)^{p_r} \leq (\gamma+2)^{-2} < \infty$$

That is, $x \in ces(p,q)$ which is a contradiction to our assumption. Hence

$a \in \mu(t)$. That is, $\mu(t) \supset ces^+(p,q)$.

Then combining the two results, we get

$ces^+(p,q) = \mu(t)$.

The continuous dual of $ces(p,q)$ is determined by the following theorem.

Theorem 2: Let $1 < p_r \leq \sup_r p_r < \infty$. Then continuous dual $ces^*(p,q)$ is isomorphic to $\mu(t)$, which is defined by (3)

Proof: It is easy to check that each $x \in ces(p,q)$ can be written in the form

$$x = \sum_{k=1}^{\infty} x_k e_k, \text{ where } e_k = (0,0,0, \ldots 0,1,0, \ldots)$$

and the 1 appears at the k-th place. Then for any $f \in ces^*(p,q)$ we have

$$f(x) = \sum_{k=1}^{\infty} x_k f(e_k) = \sum_{k=1}^{\infty} x_k a_k. \quad (4)$$

Where, $f(e_k) = a_k$. By theorem 1, the convergence of $\sum a_k x_k$ for every x in $ces(p,q)$ implies that $a \in \mu(t)$.

If $x \in ces(p,q)$ and if we take $a \in \mu(t)$, then by theorem 1, $\sum a_k x_k$ converges and clearly defines a linear functional on $ces(p,q)$. Using the same kind of argument as in theorem 1, it is easy to check that

$$\sum_{k=1}^{\infty}|a_k x_k| \leq E\left(\sum_{r=0}^{\infty}\left(Q_{2^r}\frac{max}{r}\left|\frac{a_k}{q_k}\right|\right)^{t_r} E^{-t_r}+1\right)g(x)$$

whenever $g(x) \leq 1$, where $g(x)$ is defined by (2).

Hence $\sum a_k x_k$ defines an element of $ces^*(p,q)$.

Furthermore, it is easy to see that representation (4) is unique. Hence we can define a mapping

$$T\colon ces^*(p,q) \to \mu(t).$$

by $T(f) = (a_1, a_2, \ldots)$ where the a_k appears in representation (4). It is evident that T is linear and bijective. Hence $ces^*(p,q)$ is isomorphic to $\mu(t)$.

3. Matrix Transformations

In the following theorems we shall characterize the matrix classes $(ces(p,q), l_\infty)$ and $(ces(p,q), c)$. Let $A = (a_{n,k})n, k = 1,2, \ldots$ be an infinite matrix of complex numbers and *X, Y* two subsets of the space of complex sequences. We say that the matrix A defines a matrix transformation from X into Y and denote it by $A \in (X, Y)$ if for every sequence $x = (x_k) \in X$ the sequence $A(x) = A_n(x)$ is in Y, where

$$A_n(x) = \sum_{k=1}^{\infty} a_{n,k}\, x_k$$

provided the series on the right is convergent.

Theorem 3: Let $1 < p_r \leq \frac{sup}{r} p_r < \infty$. Then $A \in (ces(p,q), l_\infty)$ if and only if there exists an integer $E > 1$, such that $U(E) < \infty$..

Where $U(E) = \frac{sup}{n} \sum_{r=0}^{\infty}\left(Q_{2^r}A_r(n)\right)^{t_r} E^{-t_r}$ and $\frac{1}{p_r}+\frac{1}{t_r} = 1, r = 0, 1, 2, \ldots$

Proof: Sufficiency: Suppose there exists an integer $E > 1$, such that $U(E) < \infty$. Then by inequality (1), we have

$$\sum_{k=1}^{\infty}|a_{n,k}x_k| = \sum_{r=0}^{\infty}\sum_{r}\left|\frac{a_{n,k}}{q_k}\, q_k x_k\right| = \sum_{r=0}^{\infty}\sum_{r}\left|\frac{a_{n,k}}{q_k}\right||q_k x_k|$$

$$\leq \sum_{r=0}^{\infty} Q_{2^r}\frac{max}{r}\left|\frac{a_{n,k}}{q_k}\right|\frac{1}{Q_{2^r}}\sum_{r}|q_k x_k|$$

$$\leq E\left(\sum_{r=0}^{\infty}(Q_{2^r}A_r(n))^{t_r}\ E^{-t_r}+\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r}|q_k x_k|\right)^{p_r}\right) < \infty$$

Therefore, $A \in (ces(p,q), l_\infty)$.

Necessity: Suppose that $A \in (ces(p,q), l_\infty)$, but

$$\frac{sup}{n}\sum_{r=0}^{\infty}\left(Q_{2^r}A_r(n)\right)^{t_r} E^{-t_r} = \infty\, for\ every\ integer\ E > 1.$$

Then $\sum_{k=1}^{\infty} a_{n,k}x_k$ converges for every $n\ and\ x \in ces(p,q)$, whence $\left(a_{n,k}\right)_{k=1,2,\ldots} \in ces^+(p,q)$ for every n.

By theorem 1, it follows that each A_n defined by

$$A_n(x) = \sum_{k=1}^{\infty} a_{n,k}x_k$$

is an element of $ces^*(p,q)$. Since $ces(p,q)$is complete and since $\frac{sup}{n}|A_n(x)| < \infty$ on $ces(p,q)$, by the uniform boundedness principle there exists a number L independent of n and a number $\delta < 1$, such that

$$|A_n(x)| \leq L \tag{5}$$

for every n and $x \in S[\theta, \delta]$ where $S[\theta, \delta]$ is the closed sphere in $ces(p,q)$ with centre at the origin θ and radius δ.

Now choose an integer $G > 1$, such that

$G\delta^M > L$.

Since

$$\frac{sup}{n}\sum_{r=0}^{\infty}(Q_{2^r}A_r(n))^{t_r}\ G^{-t_r} = \infty$$

there exists an integer $m_0 > 1$, such that

$$R = \textstyle\sum_{r=0}^{m_0}(Q_{2^r}A_r(n))^{t_r}\, G^{-t_r} > 1 \tag{6}$$

Define a sequence $x = (x_k)$ as follows:

$$x_k = 0\ if\ k \geq 2^{m_0+1}$$

$$x_{N(r)} = Q_{2^r}^{t_r}\delta^{M/p_r}(sgn\ a_n, N(r))\left|a_{n,N(r)}\right|^{t_r-1}R^{-1}G^{-t_r/p_r}$$

and $x_k = 0\ if\ k \neq N(r)$ for $0 \leq r \leq m_0$, where $N(r)$ is the smallest integer such that

$$\left|a_{n,N(r)}\right| = \frac{max}{r}\left|\frac{a_{n,k}}{q_k}\right|$$

Then one can easily show that $g(x) \leq \delta$ but $|A_n(x)| > L$, which contradicts (5). This complete the proof of the theorem.

Corollary 3.1 (see [10]). Let $1 < p_r \leq \frac{sup}{r} p_r < \infty$. Then $A \in (ces(p), l_\infty)$ if and only if there exists an integer E >1 such that $U(E) < \infty$, where

$$U(E) = \frac{sup}{n}\sum_{r=0}^{\infty}(2^r A_r(n))^{t_r}\ and\ \frac{1}{p_r}+\frac{1}{t_r} = 1, r = 0, 1, 2, \ldots$$

Proof: If $q_n = 1$ for every n in the above theorem, then we obtain the result.

Corollary 3.2 (see [9]). Let $1 < p < \infty$. Then $A \in (ces_p, l_\infty)$ if and only if $\frac{sup}{n}(\sum_{r=0}^{\infty}(2^r A_r(n))^t)^{1/t} < \infty\ where\ \frac{1}{p}+\frac{1}{t} = 1$

Proof: If $q_n = 1$ and $p_n = p$ for all n in the above theorem, then we get the result.

Theorem 4. Let $1 < p_r \leq \frac{sup}{r} p_r < \infty$. Then $A \in (ces(p,q), c)$ if and only if

(i) $a_{n,k} \to \alpha_k\ (n \to \infty, k\ is\ fixed)$ and (ii)there exists an integer $E > 1$, such that $U(E) < \infty$, where

$$U(E) = \sup_n \sum_{r=0}^{\infty} \left(Q_{2^r} A_r(n)\right)^{t_r} E^{-t_r}$$

and $\frac{1}{p_r} + \frac{1}{t_r} = 1, r = 0, 1, 2, \ldots$

Proof: Necessity. Suppose $A \in (ces(p,q), c)$. Then $A_n(x)$ exists for each $n \geq 1$ *and* $\lim_{n \to \infty} A_n(x)$ exists for every $x \in ces(p,q)$. Therefore by an argument similar to that in theorem 3 we have condition (ii). Condition (i) is obtained by taking $x = e_k \in ces(p,q)$, where e_k is a sequence with 1 at the k-th place and zeros elsewhere.

Sufficiency. The conditions of the theorem imply that

$$\sum_{r=0}^{\infty} \left(Q_{2^r} \max_r \left|\frac{\alpha_k}{q_k}\right|\right)^{t_r} E^{-t_r} \leq U(E) < \infty \quad (7)$$

By (7) it is easy to check that $\sum_k \alpha_k x_k$ is absolutely convergent for each $x \in ces(p,q)$. For each $x \in ces(p,q)$ and $\varepsilon > 0$, we can choose an integer $m_0 > 1$, such that

$$g_{m_0}(x) = \sum_{r=m_0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k|\right)^{p_r} < \varepsilon^M$$

Then by inequality (1), we have

$$\sum_{k=2^{m_0}}^{\infty} |a_{n,k} - \alpha_k||x_k|$$

$$\leq E\left(\sum_{r=m_0}^{\infty} (Q_{2^r} B_r(n))^{t_r} E^{-t_r} + 1\right)\left(g_{m_0}(x)\right)^{1/M}$$

$$< E(2U(E) + 1)\varepsilon,$$

where $B_r(n) = \max_r \left|\frac{a_{n,k} - \alpha_k}{q_k}\right|$ and

$$\sum_{r=m_0}^{\infty} \left(Q_{2^r} B_r(n)\right)^{t_r} E^{-t_r} \leq 2U(E) < \infty$$

It follows immediately that

$$\lim_{n \to \infty} \sum_{k=1}^{\infty} a_{n,k} x_k = \sum_{k=1}^{\infty} \alpha_k x_k$$

This shows that $A \in (ces(p,q), c)$ which proved the theorem.

Corollary 4.1 (see [10]). Let $1 < p_r \leq \sup_r p_r < \infty$. Then $A \in (ces(p), c)$ if and only if

(i) $a_{n,k} \to \alpha_k (n \to \infty, k\ is\ fixed), and$

(ii)there exists an integer E >1 such that $U(E) < \infty$, where

$$U(E) = \sup_n \sum_{r=0}^{\infty} (2^r A_r(n))^{t_r} E^{-t_r}\ and\ \frac{1}{p_r} + \frac{1}{t_r} = 1, r = 0, 1, 2, \ldots$$

Proof: If $q_n = 1$ for all n in the above theorem, then statements (i) and (ii) follow.

Corollary 4.2 (see [9]). Let $1 < p < \infty$. Then $A \in (ces_p, c)$ if and only if

(i) $a_{n,k} \to \alpha_k (n \to \infty, k\ is\ fixed), and$

(ii) $\sup_n (\sum_{r=0}^{\infty} 2^{rt} A_r^t(n))^{1/t} < \infty, where\ \frac{1}{p} + \frac{1}{t} = 1.$

Proof: If $q_n = 1$ and $p_n = p$ for all n in the above theorem, then we get the results.

Corollary 4.3. Let $1 < p_r \leq sup_r < \infty$. Then $A \in (ces(p,q), c_0)$ if and only if

- the condition of Theorem 3 holds, and
- $a_{n,k} \to 0\ (n \to \infty, k\ is\ fixed)$, where c_0 is the space of all null sequences.

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