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Derivations of First Type of Algebra of Second Class Filiform Leibniz Algebras of Dimension Derivation (*n*+2)

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Abstract: This paper describes the derivations of first type of algebra from the second class filiform Leibniz algebras of dimension derivation (*n*+2). The set of all derivations of an algebra L is denoted by $Der(L)$ From the description of the derivations, we found the basis of the space $Der(L_n(a))$ of the algebra.

Keywords: Filiform Leibniz Algebra, Leibniz Algebra, Gradation, Natural Gradation, Derivation

1. Introduction and Preliminaries

In mathematics and in particular in the theory of Lie algebra, a Leibniz algebra (which was first introduced by L. Loday in 1993, [5]) is an algebra L over a field K satisfying the following Leibniz identity

$$[x,[y,z]]=[[x,y],z]-[[x,z],y].$$

for any $x,y,z\in L$, where [.,.] denotes the multiplication in L. Leibniz algebra is a generalization of Lie algebra. Onwards, all algebras are assumed to be over the field of complex numbers C. Now, let L be a Leibniz algebra, we put: $L^1=L, L^{k+1}=[L^k,L]$ for $k\geq 1$.

Following [7], a Leibniz algebra L is said to be nilpotent if there exists $s\in N$ such that $L^1\supset L^2\supset\ldots\supset L^s=0$. A Leibniz algebra L is said to be filiform if $dimL^i=n-i$, where $n=dimL$ and $2\leq i\leq n$, see [1]. Let d be a K-linear transformation of an algebra L, d is called a derivation of L ([6]) if

$$d([x,y])=[d(x),y]+[x,d(y)] \quad \text{for all } x,y\in L.$$

The set of all derivations of an algebra L is denoted by $Der(L)$. We also, denote by Leibniz the set of all (n+1)-dimensional filiform Leibniz algebras. We now look at the following theorem from [3] which splits the set of fixed dimension filiform Liebniz algebras into three disjoint subsets. However we just take the result of this theorem regarding only $FLeib_{n+1}$.

Theorem 1.1 Any $(n+1)$-dimensional complex filiform Leibniz algebra L admits a basis $e_0,e_1,\ldots,e_n$ called adapted, such that the table of multiplication of L has the following forms, where non defined products are zero:

$$SLeib_{n+1}=\begin{cases}[e_0,e_0]=e_2,\\ [e_i,e_0]=e_{i+1},\ 2\leq i\leq n-1,\\ [e_0,e_1]=\beta_3e_3+\beta_4e_4+\ldots+\beta_{n-2}\,e_{n-1}+\beta_{n-1}e_n,\\ [e_1,e_1]=\gamma e_{n-1}\\ [e_j,e_1]=\beta_3e_{j+2}+\beta_4e_{j+3}+\ldots+\beta_{n+1-j}e_n, 2\leq j\leq n-2,\end{cases}$$

for $\beta_3,\beta_4,\ldots,\beta_n,\beta\in C$.

Lemma 1.1 [6] Let $d\in Der(L_n)$. In this case $d=d_0+d_1+d_2+\ldots+d_{n-1}$ where $d_k\in End(L_n)$ and $d_k(L_i)\subseteq L_{i+k}$ for $1\leq i\leq n$.

The purpose of this paper is to study the low dimension of algebras in order to get the basis of the space $Der(L_n(a))$. We attempt to find the basis of the derivation for this algebra and the relationship between the algebra and its derivations, by studying the table of this algebra from dimension 5 to 15.

2. Algebra $L_n(a)$ of the Second Class Filiform Leibniz Algebras

$$L_n(a)=\begin{cases}[e_0,e_0]=e_2,\\ [e_i,e_0]=e_{i+1},\ 2\le i\le n-1.\\ [e_0,e_1]=e_n\ .\end{cases}$$

We observe the derivations of this type of algebra in low dimension in the following table:

Table 1. *Derivations of* $Der(L_n(a))$ *with low dimension.*

dimension	equation(dim Der)	dim Der	No. of equations
5	$d_1(e_0)=e_0,\ d_1(e_1)=3e_1,$ $d_1(e_i)=ie_i, 2\le i\le 4$ $d_2(e_0)=e_1, d_2(e_2)=e_4,$ $d_3(e_0)=e_2, d_3(e_i)=e_{i+1}, 2\le i\le 3$ $d_4(e_0)=e_3,\ d_4(e_2)=e_4,$ $d_5(e_0)=e_4, d_6(e_1)=e_4.$	6	14
6	$d_1(e_0)=e_0,\ d_1(e_1)=4e_1,$ $d_1(e_i)=ie_i, 2\le i\le 5$ $d_2(e_0)=e_1,\ d_2(e_2)=e_5\ d_3(e_0)=e_2,$ $d_3(e_i)=e_{i+1}, 2\le i\le 4$ $d_4(e_0)=e_3,$ $d_4(e_i)=e_{i+1}, 2\le i\le 3$ $d_5(e_0)=e_4,\ d_5(e_2)=e_5,$ $d_6(e_0)=e_5,\ d_7(e_1)=e_5.$	7	19
7	$d_1(e_0)=e_0, d_1(e_1)=5e_1,$ $d_1(e_i)=ie_i, 2\le i\le 6\ \ d_2(e_0)=e_1,$ $d_2(e_2)=e_6\ d_3(e_0)=e_2,\ d_3(e_i)=e_{i+1}, 2\le i\le 5$ $d_4(e_0)=e_3,\ d_4(e_i)=e_{i+1}, 2\le i\le 4$ $d_5(e_0)=e_4,\ d_5(e_i)=e_{i+1}, 2\le i\le 3$ $d_6(e_0)=e_4,\ d_6(e_2)=e_6,$ $d_7(e_0)=e_6,\ d_8(e_1)=e_6.$	8	25
8	$d_1(e_0)=e_0, d_1(e_1)=6e_1,$ $d_1(e_i)=ie_i, 2\le i\le 7\ \ d_2(e_0)=e_1,$ $d_2(e_2)=e_7\ d_3(e_0)=e_2,\ d_3(e_i)=e_{i+1}, 2\le i\le 6$ $d_4(e_0)=e_3,\ d_4(e_i)=e_{i+1}, 2\le i\le 5$ $d_5(e_0)=e_4,\ d_5(e_i)=e_{i+1}, 2\le i\le 4$ $d_6(e_0)=e_5,\ d_6(e_i)=e_{i+1}, 2\le i\le 3$ $d_7(e_0)=e_6,\ d_7(e_2)=e_7,$ $d_8(e_0)=e_7,\ d_9(e_1)=e_7.$	9	32
9	$d_1(e_0)=e_0, d_1(e_1)=7e_1,$ $d_1(e_i)=ie_i, 2\le i\le 8\ \ d_2(e_0)=e_1,\ d_2(e_2)=e_8$ $d_3(e_0)=e_2,\ d_3(e_i)=e_{i+1}, 2\le i\le 7$ $d_4(e_0)=e_3,\ d_4(e_i)=e_{i+1}, 2\le i\le 6$ $d_5(e_0)=e_4,\ d_5(e_i)=e_{i+1}, 2\le i\le 5$ $d_6(e_0)=e_5,\ d_6(e_i)=e_{i+1}, 2\le i\le 4$ $d_7(e_0)=e_6,\ d_6(e_i)=e_{i+1}, 2\le i\le 3$ $d_8(e_0)=e_7,\ d_8(e_2)=e_8,$ $d_9(e_0)=e_8,\ d_{10}(e_1)=e_8.$	10	40

dimension	equation(dim Der)	dim Der	No. of equations
11	$d_1(e_0)=e_0, d_1(e_1)=9e_1,$ $d_1(e_i)=ie_i, 2\le i\le 10\ \ d_2(e_0)=e_1,$ $d_2(e_2)=e_{10}\ d_3(e_0)=e_2,\ d_3(e_i)=e_{i+1}, 2\le i\le 9$ $d_4(e_0)=e_3,\ d_4(e_i)=e_{i+1}, 2\le i\le 8$ $d_5(e_0)=e_4,\ d_5(e_i)=e_{i+1}, 2\le i\le 7$ $d_6(e_0)=e_5,\ d_6(e_i)=e_{i+1}, 2\le i\le 6$ $d_7(e_0)=e_6,\ d_6(e_i)=e_{i+1}, 2\le i\le 5$ $d_8(e_0)=e_7,\ d_8(e_i)=e_{i+1}, 2\le i\le 4$ $d_9(e_0)=e_8,\ d_9(e_i)=e_{i+1}, 2\le i\le 3$ $d_{10}(e_0)=e_9,\ d_{10}(e_2)=e_{10},$ $d_{11}(e_0)=e_{10},\ d_{12}(e_1)=e_{10}.$	12	59
12	$d_1(e_0)=e_0, d_1(e_1)=10e_1,$ $d_1(e_i)=ie_i, 2\le i\le 11\ \ d_2(e_0)=e_1,$ $d_2(e_2)=e_{11}\ d_3(e_0)=e_2,\ d_3(e_i)=e_{i+1}, 2\le i\le 10$ $d_4(e_0)=e_3,\ d_4(e_i)=e_{i+1}, 2\le i\le 9$ $d_5(e_0)=e_4,\ d_5(e_i)=e_{i+1}, 2\le i\le 8$ $d_6(e_0)=e_5,\ d_6(e_i)=e_{i+1}, 2\le i\le 7$ $d_7(e_0)=e_6,\ d_6(e_i)=e_{i+1}, 2\le i\le 6$ $d_8(e_0)=e_7,\ d_8(e_i)=e_{i+1}, 2\le i\le 5$ $d_9(e_0)=e_8,\ d_9(e_i)=e_{i+1}, 2\le i\le 4$ $d_{10}(e_0)=e_9,\ d_{10}(e_i)=e_{i+1}, 2\le i\le 3$ $d_{11}(e_0)=e_{10},\ d_{11}(e_2)=e_{11},$ $d_{12}(e_0)=e_{11},\ d_{13}(e_1)=e_{11}.$	13	70
13	$d_1(e_0)=e_0, d_1(e_1)=11e_1,$ $d_1(e_i)=ie_i, 2\le i\le 12\ \ d_2(e_0)=e_1,$ $d_2(e_2)=e_{12}$ $d_3(e_0)=e_2,\ d_3(e_i)=e_{i+1}, 2\le i\le 11$ $d_4(e_0)=e_3,\ d_4(e_i)=e_{i+1}, 2\le i\le 10$ $d_5(e_0)=e_4,\ d_5(e_i)=e_{i+1}, 2\le i\le 9$ $d_6(e_0)=e_5,\ d_6(e_i)=e_{i+1}, 2\le i\le 8$ $d_7(e_0)=e_6,\ d_6(e_i)=e_{i+1}, 2\le i\le 7$ $d_8(e_0)=e_7,\ d_8(e_i)=e_{i+1}, 2\le i\le 6$ $d_9(e_0)=e_8,\ d_9(e_i)=e_{i+1}, 2\le i\le 5$ $d_{10}(e_0)=e_9,\ d_{10}(e_i)=e_{i+1}, 2\le i\le 4$ $d_{11}(e_0)=e_{10},\ d_{11}(e_i)=e_{i+1}, 2\le i\le 3$ $d_{12}(e_0)=e_{11},\ d_{12}(e_2)=e_{12},$ $d_{13}(e_0)=e_{12},\ d_{14}(e_1)=e_{12}.$	14	82
14	$d_1(e_0)=e_0, d_1(e_1)=12e_1,$ $d_1(e_i)=ie_i, 2\le i\le 13\ \ d_2(e_0)=e_1,$ $d_2(e_2)=e_{13}\ d_3(e_0)=e_2,\ d_3(e_i)=e_{i+1}, 2\le i\le 12$ $d_4(e_0)=e_3,\ d_4(e_i)=e_{i+1}, 2\le i\le 11$ $d_5(e_0)=e_4,\ d_5(e_i)=e_{i+1}, 2\le i\le 10$ $d_6(e_0)=e_5,\ d_6(e_i)=e_{i+1}, 2\le i\le 9$ $d_7(e_0)=e_6,\ d_6(e_i)=e_{i+1}, 2\le i\le 8$ $d_8(e_0)=e_7,\ d_8(e_i)=e_{i+1}, 2\le i\le 7$ $d_9(e_0)=e_8,\ d_9(e_i)=e_{i+1}, 2\le i\le 6$ $d_{10}(e_0)=e_9,\ d_{10}(e_i)=e_{i+1}, 2\le i\le 5$ $d_{11}(e_0)=e_{10},\ d_{11}(e_i)=e_{i+1}, 2\le i\le 4$ $d_{12}(e_0)=e_{11},\ d_{12}(e_i)=e_{i+1}, 2\le i\le 3$ $d_{13}(e_0)=e_{12},\ d_{13}(e_2)=e_{13},$ $d_{14}(e_0)=e_{13},\ d_{15}(e_1)=e_{13}.$	15	95

dimension	equation(dim Der)	dim Der	No. of equations
15	$d_1(e_0)=e_0, d_1(e_1)=13e_1,$ $d_1(e_i)=ie_i, 2\le i\le 14$ $d_2(e_0)=e_1,$ $d_2(e_2)=e_{14}$ $d_3(e_0)=e_2,$ $d_3(e_i)=e_{i+1}, 2\le i\le 13$ $d_4(e_0)=e_3,$ $d_4(e_i)=e_{i+1}, 2\le i\le 12$ $d_5(e_0)=e_4,$ $d_5(e_i)=e_{i+1}, 2\le i\le 11$ $d_6(e_0)=e_5,$ $d_6(e_i)=e_{i+1}, 2\le i\le 10$ $d_7(e_0)=e_6,$ $d_6(e_i)=e_{i+1}, 2\le i\le 9$ $d_8(e_0)=e_7,$ $d_8(e_i)=e_{i+1}, 2\le i\le 8$ $d_9(e_0)=e_8,$ $d_9(e_i)=e_{i+1}, 2\le i\le 7$ $d_{10}(e_0)=e_9,$ $d_{10}(e_i)=e_{i+1}, 2\le i\le 6$ $d_{11}(e_0)=e_{10},$ $d_{11}(e_i)=e_{i+1}, 2\le i\le 5$ $d_{12}(e_0)=e_{11},$ $d_{12}(e_i)=e_{i+1}, 2\le i\le 4$ $d_{13}(e_0)=e_{12},$ $d_{13}(e_i)=e_{i+1}, 2\le i\le 3$ $d_{14}(e_0)=e_{13},$ $d_{14}(e_2)=e_3,$ $d_{15}(e_0)=e_{14},$ $d_{16}(e_1)=e_{14}.$	16	109

Notes:

From Table 1, we obtain the main result of this paper: We are able to find a basis of the space $Der(L_n(a))$ and this is given in the proposition 2.1. Therefore we can find the dim Der of this algebra for any dimension by using this rule: $dimDer(L_n)=dim(L_n)=n+2,$ for $n\ge 3$. Finally, we observe that the number of equations that can be calculated for any dimension obtained from derivations the following rule: = $\frac{n(n+1)}{2}+4$.

Let $d\in Der(L_n(a))$. By using Lemma 2.1 , we have $d=d_0+d_1+d_2+d_3+d_4+...+d_{n-2}+d_{n-1}$.

In Lemma 3.1, we give definition for $d_0\in Der(L_n(a))$.

In Lemma 3.2 , we give definition for $d_k\in Der(L_n(a)), 1\le k\le (n-2)$.

In Lemma 3.3 , we give definition for $d_{n-1}\in Der(L_n(a))$.

We are going to find the basis of algebra $L_n(a)$.

For this purpose, we present the following Lemma (2.1, 2.2 and 2.3).

Lemma 2.1 Let $t_1=\lambda_0 e_0+(n-1)e_1+\sum_{i=2}^{n}\lambda_i e_i$. Then, we have

$$t_1(e_0)=e_0,\ t_1(e_1)=(n-1)e_1\ \ and\ \ t_1(e_i)=ie_i,\ 2\le i\le n-1.$$

Proof. Consider $d_0\in Der(L_n(a))$ which is defined by

$$d_0(e_i)=\begin{cases}\alpha_0 e_0,\ i=0,\\ \alpha_1(n-1)e_1\ \ i=1,\\ \alpha_i ie_i,\ 2\le i\le n,\end{cases}\tag{1}$$

where α_i , $0\le i\le n$, are scalars. Consider the family of derivations

$$d_0([e_i,e_j])=[d_0(e_i),e_j]+[e_i,d_0(e_j)].$$

We now look at the problem case by case. In each case, we repeatedly use algebra $L_n(a)$ and (1).

Case 1: if $i=0$ and $j=0$, then

$$d_0([e_0,e_0])=[d_0(e_0),e_0]+[e_0,d_0(e_0)]$$

and so,

$$d_0(e_2)=[\alpha_0 e_0,e_0]+[e_0,\alpha_0 e_0]$$

which implies

$$2\alpha_2 e_2=\alpha_0 e_2+\alpha_0 e_2$$

and thus

$$\alpha_2=\alpha_0.\tag{2}$$

Case 2: if $2\le i\le n, j=0$ then

$$d_0([\sum_{i=2}^{n}e_i,e_0])=[d_0(\sum_{i=2}^{n}e_i),e_0]+[\sum_{i=2}^{n}e_i,d_0(e_0)]$$

so that,

$$d_0(\sum_{i=2}^{n-1}e_{i+1})=[\sum_{i=2}^{n}\alpha_i e_i,e_0]+[\sum_{i=2}^{n}e_i,\alpha_0 e_0]$$

which implies

$$\sum_{i=2}^{n-1}\alpha_{i+1}(i+1)e_{i+1}=\sum_{i=2}^{n-1}i\alpha_i e_{i+1}+\sum_{i=2}^{n-1}\alpha_0 e_{i+1}$$

$$\sum_{i=2}^{n-1}\alpha_{i+1}(i+1)e_{i+1}=\sum_{i=2}^{n-1}(i\alpha_i+\alpha_0)e_{i+1}$$

and so,

$$(i+1)\alpha_{i+1}=i\alpha_i+\alpha_0,\quad 2\le i\le n-1.\tag{3}$$

If i=2 in (3), then $3\alpha_3=2\alpha_2+\alpha_0$. By (2) we obtain

$$\alpha_3 = \alpha_2 \tag{4}$$

If i=3 in (3) then $4\alpha_4 = 3\alpha_3 + \alpha_0$. By (2) and (4), we obtain

$$\alpha_4 = \alpha_3 \tag{5}$$

Similarly, if i=n-1 in (3) then

$$(n)\alpha_n = (n-1)\alpha_{n-1} + \alpha_0.$$

But $\alpha_0 = \alpha_{n-1}$. Thus,

$$\alpha_n = \alpha_{n-1} \tag{6}$$

From (2), (4), (5) and (6) we obtain

$$\alpha_0 = \alpha_2 = \alpha_3 = \alpha_4 = \cdots = \alpha_n. \tag{7}$$

Thus, using (1) and (7), we get

$$d_0(\sum_{i=0}^{n}\lambda_i e_i) = d_0(\lambda_0 e_0) + d_0(\lambda_1 e_1) + d_0(\lambda_2 e_2) + \sum_{i=3}^{n} d_0(\lambda_i e_i)$$

$$= \lambda_0(\alpha_0 e_0) + \lambda_1(\alpha_0(n-1)e_1) + 2\lambda_2(\alpha_2 e_2) + \sum_{i=3}^{n} i\lambda_i(\alpha_i e_i)$$

$$= \alpha_0[\lambda_0 e_0 + \lambda_1(n-1)e_1 + 2\lambda_2 e_2 + \sum_{i=3}^{n} i\lambda_i e_i]$$

$$= \alpha_0 t_1 .$$

From this we obtain

$$t_1(e_0) = e_1, \ \ t_1(e_1) = (n-1)e_1 \ \text{ and } \ t_1(e_i) = ie_i, \ 2 \le i \le n-1. \tag{8}$$

Lemma 2.1 gives some vectors of the basis. In the next we will complete this basis.

Lemma 2.2 Consider $d_k \in Der(L_n(a)), \ 1 \le k \le n-2$ *such that* d_k *is defined as*

$$d_k(e_i) = \begin{cases} \lambda_0 e_k, \ i = 0, \ 1 \le k \le n \\ \lambda_i e_{k+i-1}, \ 2 \le i \le n-1 \\ \qquad 2 \le k \le n-1 \end{cases} \tag{9}$$

where λ_0 and $\lambda_i, 1 \le i \le n-k$ are constants.

Then $d_k(e_0) = e_k$ and $d_k(e_i) = e_{i+k-1}$ for $2 \le i \le n$ and $2 \le k \le n-1$.

Proof. Consider the family of derivations

$$d_k([e_i, e_j]) = [d_k(e_i), e_j] + [e_i, d_k(e_j)].$$

Similarly, we calculate case by case. In which we repeatedly $L_n(a)$ and (9).

Case 1: if $i = 0, j = 0$, then

$$\sum_{k=1}^{n-1} d_k([e_0, e_0]) = [\sum_{k=1}^{n-1} d_k(e_0), e_0] + [e_0, \sum_{k=1}^{n-1} d_k(e_0)].$$

Then

$$\sum_{k=1}^{n-1} d_k(e_2) = [\sum_{k=1}^{n} \lambda_0 e_k, e_0] + [e_0, \sum_{k=1}^{n} \lambda_0 e_k]$$

which implies

$$\sum_{k=1}^{n-1} \lambda_2 e_{k+1} = \sum_{k=1}^{n-1} \lambda_0 e_{k+1} + [e_0, \lambda_0 e_1]$$

$$\sum_{k=1}^{n-1} \lambda_2 e_{k+1} = \sum_{k=1}^{n-2} \lambda_0 e_{k+1} + \lambda_0 e_n$$

$$\sum_{k=1}^{n-1} \lambda_2 e_{k+1} = \sum_{k=1}^{n-2} \lambda_0 e_{k+1} + \lambda_0 e_n$$

$$\sum_{k=1}^{n-1} \lambda_2 e_{k+1} = \sum_{k=1}^{n-1} \lambda_0 e_{k+1}$$

to obtain

$$\lambda_2 = \lambda_0. \tag{10}$$

Case 2: if $1 \le i \le n-1, j = 0$ and $2 \le k \le n-1$, then

$$\sum_{k=2}^{n-1}\sum_{i=1}^{n-1} d_k([e_i, e_0]) = [\sum_{k=2}^{n-1}\sum_{i=1}^{n-1} d_k(e_i), e_0] + [\sum_{i=1}^{n-1} e_i, \sum_{k=2}^{n-1} d_k(e_0)]$$

$$\sum_{k=1}^{n-1}\sum_{i=2}^{n-1} d_k(e_{i+1}) = [\sum_{k=1}^{n-1}\sum_{i=1}^{n-1} \lambda_i e_{k+i-1}, e_0] + [\sum_{i=1}^{n-1} e_i, \sum_{k=1}^{n-1} d_k(e_0)]$$

$$\sum_{k=1}^{n-1}\sum_{i=2}^{n-1} \lambda_{i+1}(e_{i+k}) = [\sum_{k=1}^{n-1}\sum_{i=1}^{n-1} \lambda_i e_{k+i-1}, e_0] + [\sum_{i=1}^{n-1} e_i, \sum_{k=1}^{n-1} \lambda_0(e_k)]$$

$$\sum_{k=1}^{n-1}\sum_{i=2}^{n-1} \lambda_{i+1} e_{i+k} = \sum_{k=1}^{n-1}\sum_{i=1}^{n-1} \lambda_i e_{k+i}$$

Thus,

$$\lambda_{i+1} = \lambda_i. \tag{11}$$

From (11), if $i = 2$, then $\lambda_3 = \lambda_2$. Also, if $i = 3$ then $\lambda_4 = \lambda_3$. Similarly, if $i = n-1$, then $\lambda_n = \lambda_{n-1}$. From (10) and (11) this implies

$$\lambda_0 = \lambda_2 = \lambda_3 = \ldots = \lambda_{n-1} = \lambda_n \tag{12}$$

and hence, by (9) and (7), we get

$$d_k(\sum_{i=0}^{n}\alpha_i e_i) = d_k(\alpha_0 e_0) + d_k(\alpha_1 e_1) + \sum_{i=2}^{n} d_k(\alpha_i e_i)$$
$$= \alpha_0(\lambda_0 e_k) + \sum_{i=2}^{n}\lambda_i(\alpha_i e_{k+i-1})$$
$$= \alpha_0(\lambda_0 e_k + \sum_{i=2}^{n-k}\lambda_i e_{k+i}).$$

Thus,

$$d_k(e_0) = e_k \text{ and } d_k(e_i) = e_{k+i},\ 2 \le i \le n-k. \quad (13)$$

This conclude the proof.

Lemma 2.2 gives a second part of the basis of Algebra. Up to now, we obtained two parts of the basis. This imcomplete basis will be completed by other vectors will be given Lemma 2.3

Lemma 2.3 Let $t_2 = \lambda_0 e_n$ *. Then* $t_2(e_0) = e_n$.

Proof. Consider $d_{n-1} \in Der(L_n(a))$ where d_{n-1} is defined by

$$d_{n-1}(e_0) = \pi_0 e_n, \quad (14)$$

where π_0 is a constant. Consider the family of derivations

$$d_{n-1}([e_i, e_j]) = [d_{n-1}(e_i), e_j] + [e_i, d_{n-1}(e_j)].$$

Case 1: if $i = 0, j = 0$ then

$$d_{n-1}([e_0, e_0]) = [d_{n-1}(e_0), e_0] + [e_0, d_{n-1}(e_0)]$$

by $L_n(a)$ and (13), thus

$$d_{n-1}(e_2) = [\pi_0 e_n, e_0] + [e_0, \pi_0 e_n].$$

If $\eta_0 \ne 0$ then

$$0 = 0 + 0.$$

Hence, using (14), we get

$$d_{n-1}\left(\sum_{i=0}^{n}\lambda_i e_i\right) = d_{n-1}(\lambda_0 e_0) + d_{n-1}(\lambda_1 e_1) + \sum_{i=2}^{n} d_{n-1}(\lambda_i e_i)$$
$$= \lambda_0(\pi_0 e_n)$$
$$= \pi_0(\lambda_0 e_n)$$
$$= (\pi_0 t_2)$$

and we obtain

$$t_2(e_0) = e_n. \quad (15)$$

This conclude the proof of Lemma 2.3.

In Lemma 2.4 Fullfilment to conclude our study we will now turn to show that the set of vectors given in Lemma 2.1, Lemma 2.2 and Lemma2.3 form a basis of Algebra.

For that, we will show that the set vectors are *linearly independent*.

Lemma 2.4. The mappings t_1, t_2 *and* d_k *for* $1 \le k \le (n-2)$ *are linearly independent.*

Proof. Consider that

$$\alpha_1 t_1(e_i) + \alpha_2 t_2(e_i) + \sum_{k=1}^{n-2}\beta_k d_k(e_i) = 0 \quad (16)$$

where $e_i \in L_n(a), i = 0,1,2,3,...,n-1$. We will show that $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = ... = \beta_k = 0$ for $1 \le k \le n-2$.

$$\sum_{i=0}^{n-1}[\alpha_1 t_1(e_i) + \alpha_2 t_2(e_i) + \Sigma_{k=1}^{n-2}\beta_k d_k(e_i)]$$

$$= \sum_{i=0}^{n-1}[\alpha_1 t_1(e_i) + \alpha_2 t_2(e_i) + \beta_1 d_1(e_i) + \beta_2 d_2(e_i) + ...$$
$$+ \beta_{n-3} d_{n-3}(e_i) + \beta_{n-2} d_{n-2}(e_i)]$$
$$= \alpha_1 t_1(e_0) + \alpha_2 t_2(e_0) + \beta_1 d_1(e_0) + \beta_2 d_2(e_0) + ... + \beta_{n-3} d_{n-3}(e_0) + \beta_{n-2} d_{n-2}(e_0)$$
$$+[\alpha_1 t_1(e_1) + \alpha_2 t_2(e_1) + \alpha_3 t_3(e_1) + \beta_1 d_1(e_1) + \beta_2 d_2(e_1) + ... + \beta_{n-3} d_{n-3}(e_1) + \beta_{n-2} d_{n-2}(e_1)]$$
$$+[\alpha_1 t_1(e_2) + \alpha_2 t_2(e_2) + \alpha_3 t_3(e_2) + \beta_1 d_1(e_2) + \beta_2 d_2(e_2) + ... + \beta_{n-3} d_{n-3}(e_2) + \beta_{n-2} d_{n-2}(e_2)]$$
$$+[\alpha_1 t_1(e_3) + \alpha_2 t_2(e_3) + \alpha_3 t_3(e_3) + \beta_1 d_1(e_3) + \beta_2 d_2(e_3) + ... + \beta_{n-3} d_{n-3}(e_3) + \beta_{n-2} d_{n-2}(e_3)]$$
$$+[\alpha_1 t_1(e_4) + \alpha_2 t_2(e_4) + \alpha_3 t_3(e_4) + \beta_1 d_1(e_4) + \beta_2 d_2(e_4) + ... + \beta_{n-3} d_{n-3}(e_4) + \beta_{n-2} d_{n-2}(e_4)]$$
$$+...$$
$$+...$$

$$+[\alpha_1 t_1(e_{n-2})+\alpha_2 t_2(e_{n-2})+\alpha_3 t_3(e_{n-2})+\beta_1 d_1(e_{n-2})+\beta_2 d_2(e_{n-2})+...+\beta_{n-3}d_{n-3}(e_{n-2})$$

$$+\beta_{n-2}d_{n-2}(e_{n-2})]+[\alpha_1 t_1(e_{n-1})+\alpha_2 t_2(e_{n-1})+\alpha_3 t_3(e_{n-1})+\beta_1 d_1(e_{n-1})+\beta_2 d_2(e_{n-1})+...$$

$$+\beta_{n-3}d_{n-3}(e_{n-1})+\beta_{n-2}d_{n-2}(e_{n-1})]$$

$$=0.$$

This implies

$$(\alpha_1 e_0+\beta_1 e_2+\beta_1 e_1+\beta_2 e_2+\beta_3 e_3+\beta_4 e_4+...+\beta_{n-3}e_{n-3}+\beta_{n-2}e_{n-2}+\beta_{n-1}e_{n-1}+\alpha_2 e_n)$$

$$+(\alpha_1(n-1)e_1)$$

$$+(2\alpha_1 e_2+\beta_2 e_3+\beta_3 e_4+\beta_4 e_5+\beta_5 e_6+...+\beta_{n-3}e_{n-2}+\beta_{n-2}e_{n-1}+\beta_{n-1}e_n)$$

$$+(3\alpha_1 e3+\beta_2 e_4+\beta_3 e_5+\beta_4 e_6+\beta_5 e_7+...+\beta_{n-3}e_{n-1}+\beta_{n-2}e_n)$$

$$+(4\alpha_1 e4+\beta_2 e_5+\beta_3 e_6+\beta_4 e_7+\beta_5 e_8+...+\beta_{n-4}e_{n-1}+\beta_{n-3}e_n)$$

$$+...$$

$$+...$$

$$+((n-2)\alpha_1 e_{n-2}+\beta_2 e_{n-1}+\beta_3 e_n)$$

$$+((n-1)\alpha_1 e_{n-1}+\beta_2 e_n)$$

$$=0$$

We thus have

$$\alpha_1 e_0+(\alpha_1(n-1)+\beta_1)e_2+(2\alpha_1+\beta_2)e_2+(3\alpha_1+\beta_2+\beta_3)e_3+(4\alpha_1+\beta_2+\beta_3+\beta_4)e_4$$

$$+(5\alpha_1+\beta_2+\beta_3+\beta_4+\beta_5)e_5+(6\alpha_1+\beta_2+\beta_3+\beta_4+\beta_5+\beta_6)e_6$$

$$+(7\alpha_1+\beta_2+\beta_3+\beta_4+\beta_5+\beta_6+\beta_7)e_7$$

$$+...$$

$$+...$$

$$+((n-2)\alpha_1+\beta_2+\beta_3+\beta_4+...+\beta_{n-2})e_{n-2}$$

$$+((n-1)\alpha_1+\beta_2+\beta_3+\beta_4+...+\beta_{n-1})e_{n-1}$$

$$+(\alpha_2+\beta_2+\beta_3+\beta_4+...+\beta_{n-1})e_n$$

$$=0.$$

Here we have these following results:

1. $\alpha_1 e_1=0$ which implies $\alpha_1=0$.
2. $(\alpha_1(n-1)+\beta_1)e_2=0$ which implies $\alpha_1(n-1)+\beta_1=0$, but since $\alpha_1=0$ then $\beta_1=0$.
3. $(3\alpha_1+\beta_2+\beta_3)e_3=0$ which implies $3\alpha_1+\beta_2+\beta_3=0$, but since $\alpha_1=\beta_2=0$ then $\beta_3=0$.
4. $(4\alpha_1+\beta_2+\beta_3+\beta_4)e_4=0$ which implies $4\alpha_1+\beta_2+\beta_3+\beta_4=0$ but since $\alpha_1=\beta_2=\beta_3=0$, then $\beta_4=0$.
5. $(5\alpha_1+\beta_2+\beta_3+\beta_4+\beta_5)e_5=0$ which implies $5\alpha_1+\beta_2+\beta_3+\beta_4+\beta_5=0$, but since $\alpha_1=\beta_2=\beta_3=\beta_4=0$ then $\beta_4=0$.
6. Similarly, $(\alpha_2+\beta_2+\beta_3+\beta_4+...+\beta_{n-1})e_n=0$ which implies $\alpha_2+\beta_2+\beta_3+\beta_4+...+\beta_{n-1}=0$ but since $\alpha_2=\beta_2=\beta_3=\beta_4=\beta_5=\beta_6=...=\beta_{n-1}=0$. Then, we get $\alpha_2=0$.

From of the above we will obtain

$$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = ... = \beta_{n-1} = 0. \quad (17)$$

This conclude our proof and the mappings are linearly independent.

Lemma 2.5 The linear mappings $d_0, d_k, d_{n-1} \in Der(L_n(a)), 1 \le k \le n-2$ *defined by (1), (9) and (14) are linearly composition.*

Proof. Let

$$x = \eta_0 e_0 + \eta_1 e_1 + \eta_2 e_2 + \eta_3 e_3 + \eta_4 e_4 + ... + \eta_{n-1} e_{n-1}.$$

First we observe that by (1), (9) and (14),

$$d_k(x) = \begin{cases} 2\eta_0(\alpha_1 e_1) + \eta_1(\alpha_0(n-1))e_1 + \sum_2^n \alpha_i(\eta_i i e_i) \\ + \sum_{k=1}^{n} \eta_0 \lambda_0 e_k + \sum_{k=2}^{n-1}\sum_{i=2}^{n-1} \lambda_i \eta_i e_{k+i-1} + \eta_0 \pi_0 e_n. \end{cases} \quad (18)$$

Hence by using (7) and (12),

$$d(x) = \alpha_0[\eta_0(e_1) + \eta_1(n-1)e_1 + \sum_2^n \eta_i i e_i]$$

$$+\lambda_0[\sum_{k=1}^{n} \eta_0 e_k + \sum_{k=2}^{n-1}\sum_{i=2}^{n-1} \eta_i e_{k+i-1}] + \pi_0 \eta_0 e_n.$$

Thus

$$d = \alpha_0 t_1 + \pi_0 t_2 + \lambda_0 d_k \quad for \ 1 \le k \le n-2.$$

The linear composition of the mappings is proved.

Lemma 2.6 The mappings t_1, t_2, d_k *for* $1 \le k \le n-2$ *are derivations.*

Proof. Consider that

$$x = \alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + ... + \alpha_{n-1} e_{n-1}$$

and

$$y = \beta_0 e_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3 + ... + \beta_{n-1} e_{n-1}$$

Then

$$x.y = \beta_0(\alpha_0 . e_2 + \alpha_2 e_3 + \alpha_3 e_4 + ... + (\alpha_{n-1} + \alpha_1)e_n)$$

and so,

$$t_1(x.y) = \beta_0(2\alpha_0 e_2 + 3\alpha_2 e_3 + 4\alpha_3 e_4 + ... + n(\alpha_{n-1} + \alpha_1)e_n \quad (19)$$

Thus,

$$t_1(x) = \alpha_0 e_0 + \alpha_1(n-1)e_1 + 2\alpha_2 e_2 + 3\alpha_3 e_3 + 4\alpha_4 e_4 + ... + \alpha_{n-1}(n-1)e_{n-1}$$

Hence

$$t_1(x).y = \beta_0[\alpha_0 e_2 + 2\alpha_2 e_3 + 3\alpha_3 e_4 + 4\alpha_4 e_5 + ... + (n-1)(\alpha_{n-1} + \alpha_1)e_n] \quad (20)$$

and thus,

$$t_1(y) = \beta_0 e_0 + \beta_1(n-1)e_1 + 2\beta_2 e_2 + 3\beta_3 e_3 + 4\beta_4 e_4 + ... + \beta_{n-1}(n-1)e_{n-1}$$

Therefore,

$$x.t_1(y) = \beta_0[\alpha_0 e_2 + \alpha_2 e_3 + \alpha_3 e_4 + ... + (\alpha_{n-1} + \alpha_1)e_n] \quad (21)$$

By adding (20) to (21) we will obtain (19). This implies t_1 is a derivation.

We now show that t_2 is also a derivation.

$$t_2(x) = \alpha_0 e_n$$

and

$$t_2(y) = \beta_0 e_n$$

From easy calculation we have $t_2(x).y = 0$, $x.t_2(y) = 0$ and $t_2(x.y) = 0$ and thus t_2 is a derivation.

Now, since

$$d_k(x) = \sum_{k=1}^{n-1}[\alpha_0 d_k(e_0) + \alpha_1 d_k(e_1) + \alpha_2 d_k(e_2) + ... + \alpha_{n-1} d_k(e_{n-1})]$$

thus,

$$\sum_{k=1}^{n-1} d_k(x) = \sum_{k=1}^{n-1}[\alpha_0 e_k + \sum_{i=2}^{n-k} \alpha_i e_{k+i-1}]$$

then,

$$[d_k(x).y] = \beta_0[\sum_{k=1}^{n-1}(e_{k+1} + \sum_{k=1}^{n-1}(\alpha_2 e_{k+2} + \alpha_3 e_{k+3}$$

$$+ ... + \alpha_{n-k} e_{n-k}))]. \quad (22)$$

In addition,

$$\sum_{k=1}^{n-1} d_k(y) = \sum_{k=1}^{n-1}[\alpha_0 e_k + \sum_{i=2}^{n-k} \alpha_i e_{k+i-1}]$$

and we have

$$[x, \sum_{k=1}^{n-1} d_k(y)] = 0 \tag{23}$$

Also,

$$\sum_{k=1}^{n-1} d_k(x.y) = \sum_{k=1}^{n-1} \beta_0(\alpha_0 e_{k+1} + \alpha_2 e_{k+2} + \alpha_3 e_{k+3} + \ldots + \alpha_{n-k} e_{n-1}.) \tag{24}$$

By adding (22) to (23) we will get (24), thus d_k is a derivation.

This complete the proof of the derivation.

We recall that the t_1, t_2, and d_k for $1 \le k \le n-2$ are defined in Lemma $2.4, \ldots,$ Lemma 2.6.

We will conclude our discussion with the following Proposition.

Proposition 2.1 Let $L_n(a)$ be $e_0 e_0 = e_2, e_i e_0 = e_{i+1}$, $2 \le i \le n-1$ and $e_1 e_0 = e_n$. Then t_1, t_2 and d_k for $1 \le k \le n-2$ form a basis of the space Der($L_n(a)$).

Proof.: The proof follows from Lemma 3.4 to Lemma 3.6

3. Conclusion

Finally, the purpose of this manuscript is fourfold:

1. This algebra $L_n(a)$, is nilpotent, but it is not characteristically nilpotent.
2. This algebra $L_n(a)$ work with basis derivations from five dimension and above.
3. We can find number derivations of this algebra on any dimension by this rule:

$$dimDer(L_n(a)) = n + 2.$$

4. We can determine the number of equations from the result of derivations by this rule, i.e, the number of equations is equal to $\frac{n(n+1)}{2} + 4$.

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