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Common Fixed Point for Two Multivalued Mappings on Proximinal Sets in Regular Modular Space

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Abstract: In 2010, Chistyakov, V. V., defined the theory of modular metric spaces. After that, in 2011, Mongkolkeha, C. et. al. studied and proved the new existence theorems of a fixed point for contraction mapping in modular metric spaces for one single-valued map in modular metric space. Also, in 2012, Chaipunya, P. introduced some fixed point theorems for multivalued mapping under the setting of contraction type in modular metric space. In 2014, Abdou, A. A. N., Khamsi, M. A. studied the existence of fixed points for contractive-type multivalued maps in the setting of modular metric space. In 2016, Dilip Jain et. al. presented a multivalued F-contraction and F-contraction of Hardy-Rogers-type in the case of modular metric space with specific assumptions. In this work, we extended these results into the case of a pair of multivalued mappings on proximinal sets in a regular modular metric space. This was done by introducing the notions of best approximation, proximinal set in modular metric space, conjoint F-proximinal contraction, and conjoint F-proximinal contraction of Hardy-Rogers-type for two multivalued mappings. Furthermore, we give an example showing the conditions of the theory which found a common fixed point of a pair of multivalued mappings on proximinal sets in a regular modular metric space. Also, the applications of the obtained results can be used in multiple fields of science like Electrorheological fluids and FORTRAN computer programming as shown in this communication.

Keywords: Common Fixed Point, Multivalued Mappings, Regular Modular Space, Proximinal Set, F-contraction, Δ_2-condition and Δ_M-condition

1. Introduction, Definitions and Preliminaries

The concept of weak contraction was defined by Alber and Guerre-Delabriere in 1997 [1]. Rhoades exhibited that most results in ref. [1] of single valued maps are still true for any Banach space [2].

In 2010, the notion of modular metric space was introduced by Chistyakov [3].

In 2012, Wardowski characterized the idea of F-contraction which generalized the Banach contraction principle in various manners, and he utilized the new concept of contraction to find the fixed point theorem [4].

Authors [5:8] have studied the fixed point property concept in modular metric space.

Furthermore, in 2013, Sgroi et al. obtained a multivalued version of Wardowski's result [9].

In 2014, Abdou et al. investigated the existence of fixed points for multivalued modular contractive mappings in modular metric spaces [10]. Also, their results generalized as well as improved fixed point results of Nadler [11] and Edelstein [12].

In 2016, Dilip Jain et al. presented a multivalued F-contraction in the case of modular metric space with specific assumptions [13]. Their results were an extension of Nadler, Wardowski and Sgroi to the case of modular metric spaces [11, 4 and 9].

In 2018, a common fixed point theorem for a pair of multivalued F-Ψ-proximinal mappings satisfying Ćirić-Wardowski type contraction in partial metric spaces was presented by S. U. Khan et al. [14].

The problem here is to a construct the contraction connecting between two multivalued mappings in modular metric space with codomain over proximinal sets and finds a common fixed point between them. Also, we want to find an example and an application in this case.

In this paper, the study of the modifications in the concept of F-contraction of Hardy-Rogers-type to the case of two multivalued F-proximinal mappings in modular metric spaces will be introduced. Also, we introduce the notions of modular proximinal sets and common fixed point theorems for a pair of multivalued F-proximinal mappings in modular metric spaces.

Definition 1.1 [13]:

Let X be a nonempty set. A function $\omega: (0,\infty)\times X\times X \to [0,\infty]$ is said to be a metric modular on X if it satisfies, for all x, y, z ∈ X, the following conditions: (we will write $\omega_\lambda(x,y)$ instead of $\omega(\lambda,x,y)$)

(i) $\omega_\lambda(x,y)=0$ for all $\lambda>0$ if and only if x = y,

(ii) $\omega_\lambda(x,y)=\omega_\lambda(y,x)$ for all $\lambda>0$,

(iii) $\omega_{\lambda+\mu}(x,y)\le\omega_\lambda(x,z)+\omega_\mu(z,y)$ for all $\lambda,\mu>0$.

If instead of (iii), we have the condition:

(iii)' $\omega_{\lambda+\mu}(x,y)\le\frac{\lambda}{\lambda+\mu}\omega_\lambda(x,z)+\frac{\mu}{\lambda+\mu}\omega_\mu(z,y)$ for all $\lambda,\mu>0$ and x, y, z ∈ X, then ω is called convex metric modular on X.

Also, if instead of (i), we have the condition

(i)' $\omega_\lambda(x,x)=0$ for all $\lambda>0$, then ω is said to be a metric pseudomodular on X.

Definition 1.2 [10]:

Let ω be a pseudomodular on X. Fix $x_0\in X$, the two sets $X_\omega(x_0)=\{x\in X:\lim_{\lambda\to\infty}\omega_\lambda(x,x_0)=0\}$ and $X^*_\omega(x_0)=\{x\in X:\exists\,\lambda=\lambda(x)>0 \text{ such that } \omega_\lambda(x,x_0)<\infty\}$ are said to be modular spaces generated by x_0.

The spaces $X_\omega(x_0)$ and $X^*_\omega(x_0)$ are metric spaces with the metrics $d_\omega(x,y)=\inf\{\lambda>0,\omega_\lambda(x,y)<\lambda\}$ and $d^*_\omega(x,y)=\inf\{\lambda>0,\omega_\lambda(x,y)<1\}$ respectively.

For each x, y ∈ X and $\lambda>0$, Abdou [10] defined $\omega_{\lambda^+}(x,y):=\lim_{\epsilon\to0^+}\omega_{\lambda+\epsilon}(x,y)$ and $\omega_{\lambda^-}(x,y):=\lim_{\epsilon\to0^+}\omega_{\lambda-\epsilon}(x,y)$.

Remarks:

1. A metric modular ω on X is nonincreasing with respect to $\lambda>0$. In fact for any x, y ∈ X and $0<\mu<\lambda$, we have $\omega_\lambda(x,y)\le\omega_{\lambda-\mu}(x,x)+\omega_\mu(x,y)=\omega_\mu(x,y)$.
2. $\omega_{\lambda^+}(x,y)\le\omega_\lambda(x,y)\le\omega_{\lambda^-}(x,y)$.
3. If a metric modular ω on X possesses a finite value for each x, y ∈ X and $\omega_\lambda(x,y)=\omega_\mu(x,y)$ for all $\lambda,\mu>0$, then $d(x,y)=\omega_\lambda(x,y)$ is a metric on X.

Example 1.3 [3]:

For any metric space (X, d), one can define a modular metric $\omega_\lambda(x,y)$ by $\omega_\lambda(x,y)=\frac{d(x,y)}{\varphi(\lambda)}$ for all $x,y\in X$, $\lambda>0$ where $\varphi:(0,\infty)\to(0,\infty)$ is any nondecreasing function.

Definition 1.4 [13]: (Regular Metric modular)

A modular metric ω on X is said to be regular if the following condition satisfies:

x = y if and only if $\omega_\lambda(x,y)=0$ for some $\lambda>0$.

This condition plays a significant role to ensure the existence of fixed point in modular metric space.

Definition 1.5 [10]:

Let ω be a metric modular on X then

(i) The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be ω-convergent if and only if there exists x ∈ X such that $\omega_1(x_n,x)\to0$ as $n\to\infty$.

(ii) The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be ω-Cauchy if $\omega_1(x_m,x_n)\to0$ as $m,n\to\infty$.

(iii) A subset D of X is said to be ω-complete if any ω-Cauchy sequence in D is a convergent sequence and its limit is in D.

(iv) A subset D of X is said to be ω-closed if ω-limits of all ω-convergent sequences of D always belong to D.

(v) A subset D of X is said to be ω-bounded if we have $\delta_\omega(D)=\sup\{\omega_1(x,y);x,y\in D\}<+\infty$.

(vi) A subset D of X is said to be ω-compact if for any $\{x_n\}_{n\in\mathbb{N}}$ in D there exists a subsequence $\{x_{n_k}\}$ and x ∈ D such that $\omega_1(x_{n_k},x)\to0$.

(vii) ω is said to satisfy the Fatou property if and only if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X ω-convergent to x, we have $\omega_1(x,y)\le\lim_{n\to\infty}\inf\omega_1(x_n,y)$ for any y ∈ X.

In general, if $\lim_{n\to\infty}\omega_\lambda(x_n,x)=0$ for some $\lambda>0$, then we may not have $\lim_{n\to\infty}\omega_\lambda(x_n,x)=0$ for all $\lambda>0$.

Definition 1.6 [13]: (Δ_2-condition)

Let (X, ω) be a modular metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. The metric modular ω is said to satisfy the Δ_2-condition if $\lim_{n\to\infty}\omega_\lambda(x_n,x)=0$ for some $\lambda>0$ then $\lim_{n\to\infty}\omega_\lambda(x_n,x)=0$ for all $\lambda>0$.

Definition 1.7 [13]: (Δ_M-condition)

Let (X, ω) be a modular metric space and $\{x_n\}_{n\in\mathbb{N}}$ be sequence in X. The metric modular ω is said to satisfy the Δ_M-condition if $\lim_{n\to\infty}\omega_p(x_{n+p},x_n)=0$ for ($n\in\mathbb{N}$, $p>0$) then $\lim_{n\to\infty}\omega_\lambda(x_{n+p},x_n)=0$ for some $\lambda>0$.

2. Multivalued F-contraction on Modular Metric Space

Throughout this paper, let $\mathcal{CB}(D)$ denote the set of all nonempty closed and bounded subsets of D, $\mathcal{C}(D)$ denotes the set of all nonempty closed subsets of D, and PR(D) denotes the set of all closed proximinal subsets of D.

Let A, B ∈ PR(D), we define the Proximinal Hausdorff metric modular as follows:

$H_{\omega_1}(A,B):=\max\{\sup_{a\in A}\omega_1(a,B),\sup_{b\in B}\omega_1(b,A)\}$ where $\omega_1(a,B):=\inf_{b\in B}\omega_1(a,b)$.

Definition 2.1 [13]:

Let $F:\mathbb{R}^+\to\mathbb{R}$ be a function satisfying the following conditions:

(F1) F is strictly increasing on $\mathbb{R}^+$,

(F2) For every sequence $\{s_n\}$ in $\mathbb{R}^+$, we have $\lim_{n\to\infty}s_n=0$ if and only if $\lim_{n\to\infty}F(s_n)=-\infty$,

(F3) There exists a number k ∈ (0, 1) such that $\lim_{s\to0^+}s^kF(s)=0$.

The family of all functions F satisfying the conditions (F1)-(F3) is denoted by $\mathcal{F}$.

Definition 2.2 [13]: (F-contraction)

Let D be a non-empty ω-bounded subset of a modular metric space (X, ω). For a fixed $F\in\mathcal{F}$ a multivalued mapping T: D → $\mathcal{CB}(D)$ is called F-contraction on X if $\exists\tau \in \mathbb{R}^+$ such that for any x, y ∈ D with y ∈ Tx there exists z ∈ Ty such that $\omega_1(y, z) > 0$ and the following inequality holds:

$$\tau + F(\omega_1(y,z)) \leq F(M(x,y)) \quad (1)$$

where $M(x,y) = \max\{\omega_1(x,y), \omega_1(x,Tx), \omega_1(y,Ty), \omega_1(y,Tx)\}$.

Definition 2.3 [13]: (F-contraction of Hardy-Rogers-type)

Let D be a non-empty ω-bounded subset of a modular metric space (X, ω). A multivalued mapping T: D → $\mathcal{CB}(D)$ is called F-contraction of Hardy-Rogers-type if there exists $F \in \mathcal{F}$, and $\tau \in \mathbb{R}^+$ such that

$$2\tau + F(H_\omega(Tx,Ty)) \leq F(\alpha\omega_1(x,y) + \beta\omega_1(x,Tx) + \gamma\omega_1(y,Ty) + L\omega_1(y,Tx)) \quad (2)$$

Definition 2.4 [15]: (proximinal)

Let X be a Banach space, and E ⊂ X be a closed bounded subset X. The set E is called proximinal in X if for all x ∈ X there is some e ∈ E such that

$$\|x-e\| = \inf\{\|x-y\| : y \in E\}.$$

We will rewrite the following lemmas in the case of PR(X).

Lemma 2.5 [10]:

Let (X, ω) be a modular metric space and D be a nonempty subset of X_ω. Let A, B ∈ PR(D) then for each $\epsilon > 0$ and a ∈ A there exists b ∈ B such that

$$\omega_1(a,b) \leq H_{\omega_1}(A,B) + \epsilon.$$

Moreover, if B is ω-compact and ω satisfies the Fatou property, then for any a ∈ A there exists b ∈ B such that

$$\omega_1(a,b) \leq H_{\omega_1}(A,B).$$

Lemma 2.6 [10]:

Let D be a nonempty subset of a modular metric space (X, ω). Assume that ω satisfies Δ_2-condition and let A_n be a sequence of sets in PR(D) such that $\lim_{n\to\infty} H_{\omega_1}(A_n, A_0) = 0$ where $A_0 \in PR(D)$. If $x_n \in A_n$ and $\lim_{n\to\infty} x_n = x_0$ then $x_0 \in A_0$.

3. Main Results

Definition 3.1: (Best approximation and proximinal set in modular metric space)

Let K be a nonempty set of a modular metric space (X, ω). Let x ∈ X, an element $y_0 \in K$ is called *a best approximation for x in K* if $\omega_1(x,y_0) = \inf\{\omega_1(x,y); y \in K\}$ (we will write $\omega_1(x,K)$ instead of $\inf\{\omega_1(x,y); y \in K\}$). If for each x ∈ X there exists at least one best approximation in K then K is called a *modular proximinal set*.

Definition 3.2: (Conjoint F-proximinal contraction)

Let D be a nonempty ω-bounded subset of a modular metric space (X, ω). For fixed $F \in \mathcal{F}$, we say that two multivalued mappings T, S: D → PR(D) form *conjoint F-proximinal contraction* on X if for 0 < q < 1 all x, y ∈ D such that $H_{\omega_1}(Tx,Sy) > 0$ the following inequality holds:

$$0 < \inf\{F(M_{T,S}(x,y)) - F(H_{\omega_1}(Tx,Sy))\} \quad (3)$$

and $M_{T,S}(x,y) = q\max\{\omega_1(x,y), \omega_1(y,Tx), \omega_1(Tx,x), \omega_1(Sy,y)\}$.

Theorem 3.3: Let D be a non-empty ω-bounded and ω-complete subset of a modular metric space (X, ω). Assume that ω is a regular modular satisfying Δ_M and Δ_2-conditions. If T, S: D → PR(D) form a continuous conjoint F-proximinal contraction then they have a unique common fixed point.

Proof

We construct an iterative sequence {S T(x_n)} generated by x_1 as follows:

Let $x_1 \in D$ be an arbitrary point of D and choose $x_2 \in Tx_1$ one of the best approximation of Tx_1 or $\omega_1(x_1,x_2) = \omega_1(x_1,Tx_1)$. Let $x_3 \in Sx_2$ one of the best approximation of Sx_2 or $\omega_1(x_2,x_3) = \omega_1(x_2,Sx_2)$. Let $x_4 \in Tx_3$ one of the best approximation of Tx_3 or $\omega_1(x_3,x_4) = \omega_1(x_3,Tx_3)$ and so on.

If $M(x_1, x_2) = 0$, then clearly $x_1 = x_2 = x_3$ is a common fixed point of (S, T) and there is nothing to prove and our proof is complete. In ordered to find common fixed point of both T and S for the situation when

$M_{T,S}(x_1,x_2) > 0$ with $x_1 \neq x_2$, and $\tau \in \mathbb{R}^+$such that

$$\tau + F(H_{\omega_1}(Tx_1,Sx_2)) \leq F(M_{T,S}(x_1,x_2)) \quad (4)$$

where $\tau = \inf\{F(M_{T,S}(x_1,x_2)) - F(H_{\omega_1}(Tx_1,Sx_2))\}$.

We can write equation (4), as

$$\tau + F(\omega_1(x_2,x_3)) \leq F(M_{T,S}(x_1,x_2)) \text{ and } x_1 \neq x_2$$

and let $x_4 \in Sx_3$ such that $\omega_1(x_3,Sx_3) = \omega_1(x_4,x_3)$

$$\tau + F(\omega_1(x_3,x_4)) \leq F(M_{T,S}(x_2,x_3)) \text{ and } x_2 \neq x_3$$

Repeating this process, we find that there exists a sequence $\{x_n\}$ with initial point x_1 such that $x_{2n} \in Tx_{2n-1}$, $x_{2n+1} \in Sx_{2n}$ and

$$\tau + F(\omega_1(x_n,x_{n+1})) \leq F(M_{T,S}(x_{n-1},x_n)) \text{ for all } n \in \mathbb{N} \quad (5)$$

This implies

$$F(\omega_1(x_n,x_{n+1})) < F(M_{T,S}(x_{n-1},x_n)) \text{ for all } n \in \mathbb{N}$$

Consequently,

$\omega_1(x_n,x_{n+1}) < M_{T,S}(x_{n-1},x_n)$(Since F is strictly increasing)

$$= q\max\{\omega_1(x_{n-1},x_n), \omega_1(x_n,Tx_{n-1}), \omega_1(x_{n-1},Tx_{n-1}), \omega_1(x_n,Sx_n)\}$$

$$= q\max\{\omega_1(x_{n-1},x_n), \omega_1(x_n,x_{n+1})\}$$

$$< \max\{\omega_1(x_{n-1},x_n), \omega_1(x_n,x_{n+1})\}$$

Obviously, if $\max\{\omega_1(x_{n-1},x_n),\omega_1(x_n,x_{n+1})\}=\omega_1(x_n,x_{n+1})$, then we get $x_n=x_{n+1}$ is a common fixed point of T and S. So,

$$\omega_1(x_n,x_{n+1})<\max\{\omega_1(x_{n-1},x_n),\omega_1(x_n,x_{n+1})\}=\omega_1(x_{n-1},x_n) \tag{6}$$

Consequently, By (F1) and equations (5), (6), we get

$$\tau+F\big(\omega_1(x_n,x_{n+1})\big)\le F\Big(M_{T,S}(x_{n-1},x_n)\Big)<F\big(\omega_1(x_{n-1},x_n)\big)\text{ for all } n\in\mathbb{N} \tag{7}$$

So,

$$F\big(\omega_1(x_n,x_{n+1})\big)\le F\big(\omega_1(x_{n-1},x_n)\big)-\tau\text{ for all } n\in\mathbb{N}$$

Then, we have

$$F\big(\omega_1(x_n,x_{n+1})\big)\le F\big(\omega_1(x_{n-1},x_n)\big)-\tau\le\cdots\le F\big(\omega_1(x_2,x_1)\big)-n\tau\text{ for all } n\in\mathbb{N} \tag{8}$$

hence $\lim_{n\to\infty}F\big(\omega_1(x_n,x_{n+1})\big)=-\infty$. By (F2) we have that $\omega_1(x_n,x_{n+1})\to 0$ as $n\to\infty$.

Now, let $k\in(0,1)$ and condition (F3), we get

$$\lim_{n\to\infty}F\big(\omega_1(x_n,x_{n+1})\big)\,(\omega_1(x_n,x_{n+1}))^k=0$$

Multiply eq. (8) by $(\omega_1(x_n,x_{n+1}))^k$, we get

$$(\omega_1(x_n,x_{n+1}))^k\big[F\big(\omega_1(x_n,x_{n+1})\big)-F\big(\omega_1(x_2,x_1)\big)\big]\le-n\tau\,(\omega_1(x_n,x_{n+1}))^k\le 0$$

then

$$\lim_{n\to\infty}\big(-n\tau\,(\omega_1(x_n,x_{n+1}))^k\big)=0$$

or

$$\lim_{n\to\infty}n\,(\omega_1(x_n,x_{n+1}))^k=0$$

then there exists $n_1\in\mathbb{N}$ such that

$$n\,(\omega_1(x_n,x_{n+1}))^k\le 1\text{ for all } n\ge n_1$$

that is

$$\omega_1(x_n,x_{n+1})\le\frac{1}{n^{1/k}}\text{ for all } n\ge n_1.$$

Now, for all $n\ge n_1$ with $p>0$, we have

$$\omega_p(x_n,x_{n+p})\le\omega_1(x_n,x_{n+1})+\omega_1(x_{n+1},x_{n+2})+\cdots+\omega_1(x_{n+p-1},x_{n+p})\le\frac{1}{n^{1/k}}+\frac{1}{(n+1)^{1/k}}+\cdots+\frac{1}{(n+p-1)^{1/k}}<\sum_{i=n}^{\infty}\frac{1}{i^{1/k}}$$

since the series $\sum_{i=n}^{\infty}\frac{1}{i^{1/k}}$ tends to zero, this implies

$$\lim_{n\to\infty}\omega_p(x_n,x_{n+p})=0\text{ for } p>0.$$

Since ω satisfies Δ_M -condition. Then, we have $\lim_{n\to\infty}\omega_\lambda(x_n,x_{n+p})=0$ for some $\lambda>0$.

Also, since ω satisfies Δ_2 -condition. Then, $\lim_{n\to\infty}\omega_1(x_n,x_{n+p})=0$.

This shows that $\{x_n\}$ is a ω-Cauchy sequence and D is ω-complete, then there exists $v\in D$ such that $x_n\to v$ as $n\to\infty$.

Now, we will prove that v is a fixed point of T and S.

Let Tx_n be a sequence in $PR(D)$. Since T is continuous then we have $Tx_n\to Tv$ or $Tx_{2n}\to Tv$. Also,

$$\lim_{n\to\infty}H_{\omega_1}(Tx_{2n},Tv)=0$$

where $Tv\in PR(D)$. Then since $x_{2n+1}\in Tx_{2n}$ and $\lim_{n\to\infty}x_{2n+1}=v$, then by Lemma 2.6, we get $v\in Tv$. Hence v is a fixed point of T.

Similarly, we can deduce that v is also a fixed point of S.

So, v is a common fixed point of T and S.

Let $v\neq u$ be two common fixed points of T and S. So, $u\in Tu$, $u\in Su$, $v\in Tv$ and $v\in Sv$. Also, since T, S are conjoint F-proximinal contraction then we have the following:

$$\tau+F\Big(H_{\omega_1}(v,u)\Big)\le F\Big(M_{T,S}(u,v)\Big)$$

or

$$\tau + F(\omega_1(v,u)) \le F\left(M_{T,S}(u,v)\right)$$

implies

$$\omega_1(v,u) < M_{T,S}(u,v)$$
$$= q\max\{\omega_1(u,v), \omega_1(u,Tu), \omega_1(v,Sv), \omega_1(v,Tu)\}$$
$$< \max\{\omega_1(u,v), \omega_1(u,Tu), \omega_1(v,Sv), \omega_1(v,Tu)\}$$
$$\le \omega_1(u,v).$$

This gives a contradiction. Then u and v must be equal or T and S have a unique common fixed point.

In the following ensures example, we show how to get the common fixed point for two multivalued functions on a proximinal set.

Example 3.4: Let $X = \mathbb{R}$ be equipped with modular metric $\omega_\lambda(x,y) = \frac{1}{\lambda}|x-y|$. Consider T, S: X → PR(X) as $T(x) = \left[\frac{x}{3}, \frac{x}{2}\right]$ and $S(x) = \left[\frac{x}{3}, \frac{2x}{3}\right]$, then (T, S) is a pair of continuous mappings. Define F: $\mathbb{R}^+ \to \mathbb{R}$ by $Fx = \ln x$. So, for $q = \frac{1}{2}$, $x = 1$, we have $y = \frac{1}{2} \in T(1) = \left[\frac{1}{3}, \frac{1}{2}\right]$ such that $\omega_1(x,y) = \omega_1(x, T(1))$ and

$$H_{\omega_1}\left(T(1), S(\tfrac{1}{2})\right) = H_{\omega_1}\left(\left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{1}{6}, \frac{1}{3}\right]\right) = \max\left\{\left|\frac{1}{3} - \frac{1}{6}\right|, \left|\frac{1}{2} - \frac{1}{3}\right|\right\} = \max\left\{\frac{1}{6}, \frac{1}{6}\right\} = \frac{1}{6}$$

and $M_{T,S}\left(1, \frac{1}{2}\right) = \frac{1}{2}\max\left\{\omega_1\left(1, \frac{1}{2}\right), \omega_1(1, T(1)), \omega_1\left(\frac{1}{2}, S\left(\frac{1}{2}\right)\right), \omega_1\left(\frac{1}{2}, T(1)\right)\right\}$

$= \frac{1}{2}\max\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{6}, 0\right\} = \frac{1}{4}.$

Then, the L.H.S of equation (4) is

$$\text{L.H.S.} = \tau + F\left(\frac{1}{6}\right) = \tau + \ln(\tfrac{1}{6}) = \tau - 1.8, \text{ and R.H.S.} = F\left(\frac{1}{2}\right) = \ln(\tfrac{1}{2}) = -1.4.$$

Then for $\tau = 0.1$ then

$$-1.7 \le -1.4$$

Hence all the hypotheses of Theorem 3.3 are satisfied. Therefore, T and S have a unique common fixed point which is zero.

Definition 3.5:

Let D be a nonempty bounded subset of a modular metric space (X, ω). Two multivalued mappings T, S: D → PR(D) are called *conjoint F- proximinal contraction of Hardy-Rogers-type* on X if there exists $F \in \mathcal{F}$, and

$$0 < \inf\left\{F(\alpha\omega_1(x,y) + \beta\omega_1(x,Tx) + \gamma\omega_1(y,Sy) + L\omega_1(y,Tx)) - F\left(H_{\omega_1}(Tx,Sy)\right)\right\} \tag{9}$$

for all $x, y \in D$ with $H_{\omega_1}(Tx,Sy) > 0$,where $\alpha, \beta, \gamma, L \ge 0, \alpha + \beta + \gamma = 1, \gamma < 1$ and $\beta + L < 1$.

Theorem 3.6: Let D be a nonempty ω-bounded and ω-complete subset of a modular metric space (X, ω). Assume that ω is a regular modular satisfying Δ_M and Δ_2-conditions. Let T, S: D → PR(D) be continuous conjoint F-proximinal contraction of Hardy-Rogers-type on X then T, S have a common fixed point.

Proof

We construct an iterative sequence $\{S\, T(x_n)\}$ generated by x_1 as in Theorem 3.3.

Since $x_2 \in Tx_1$, then

$$\omega_1(x_2, Sx_2) \le H_{\omega_1}(Tx_1, Sx_2)$$

and F is strictly increasing, then we get

$$F(\omega_1(x_2,x_3)) \le F\left(H_{\omega_1}(Tx_1, Sx_2)\right) \text{ and } x_1 \ne x_2$$

with $x_3 \in Sx_2$ and $\omega_1(x_2,x_3) = \omega_1(x_2, Sx_2)$.

And by the same way, we find $x_4 \in Tx_3$ with $\omega_1(x_3,x_4) = \omega_1(x_3, Tx_3)$such that

$$\omega_1(x_4, Sx_4) \le H_{\omega_1}(Tx_3, Sx_4)$$

$$F(\omega_1(x_4,x_5)) \le F\left(H_{\omega_1}(Tx_3, Sx_4)\right) \text{ and } x_3 \ne x_4$$

with $x_5 \in Sx_4$ and $\omega_1(x_4,x_5) = \omega_1(x_4, Sx_4)$.

Repeating this process, we find that there exists a sequence $\{x_n\}$ with initial point x_1 such that $x_{2n} \in Tx_{2n-1}$, $x_{2n+1} \in Sx_{2n}$ and

$$F(\omega_1(x_{n+1}, x_{n+2})) \le F\left(H_{\omega_1}(Tx_n, Sx_{n+1})\right) \text{ for all } n \in \mathbb{N} \tag{10}$$

And since T, S are continuous conjoint F-proximinal contraction of Hardy-Rogers-type on X, then we have

$$F(\omega_1(x_{n+1},x_{n+2})) \leq F\left(H_{\omega_1}(Tx_n,Sx_{n+1})\right)$$

$$< F(\alpha\omega_1(x_n,x_{n+1}) + \beta\omega_1(x_n,Tx_n) + \gamma\omega_1(x_{n+1},Sx_{n+1}) + L\omega_1(x_{n+1},Tx_n))$$

then

$$\omega_1(x_{n+1},x_{n+2}) < \alpha\omega_1(x_n,x_{n+1}) + \beta\omega_1(x_n,Tx_n) + \gamma\omega_1(x_{n+1},Sx_{n+1}) + L\omega_1(x_{n+1},Tx_n)$$

$$< \alpha\omega_1(x_n,x_{n+1}) + \beta\omega_1(x_n,x_{n+1}) + \gamma\omega_1(x_{n+1},x_{n+2})$$

or

$$(1-\gamma)\omega_1(x_{n+1},x_{n+2}) < (\alpha+\beta)\omega_1(x_n,x_{n+1})$$

So,

$$\omega_1(x_{n+1},x_{n+2}) < \omega_1(x_n,x_{n+1}) \tag{11}$$

Also, from eq. (9), then

$$F\left(H_{\omega_1}(Tx_n,Sx_{n+1})\right)$$

$$\leq F(\alpha\omega_1(x_n,x_{n+1}) + \beta\omega_1(x_n,Tx_n) + \gamma\omega_1(x_{n+1},Sx_{n+1}) + L\omega_1(x_{n+1},Tx_n)) - \tau$$

$$\leq F((\alpha+\beta)\omega_1(x_n,x_{n+1}) + \gamma\omega_1(x_{n+1},x_{n+2})) - \tau$$

$$= F((1-\gamma)\omega_1(x_n,x_{n+1}) + \gamma\omega_1(x_{n+1},x_{n+2})) - \tau$$

$$= F(\omega_1(x_n,x_{n+1}) - \gamma(\omega_1(x_n,x_{n+1}) - \omega_1(x_{n+1},x_{n+2}))) - \tau$$

And from equations (10) and (11), we get

$$F(\omega_1(x_{n+1},x_{n+2})) \leq F\left(H_{\omega_1}(Tx_n,Sx_{n+1})\right) < F(\omega_1(x_n,x_{n+1})) - \tau$$

So,

$$F(\omega_1(x_{n+1},x_{n+2})) \leq F(\omega_1(x_n,x_{n+1})) - \tau \leq \cdots \leq F(\omega_1(x_1,x_2)) - n\tau$$

for all $n \in \mathbb{N}$ and hence $\lim_{n\to\infty} F(\omega_1(x_n,x_{n+1})) = -\infty$. Similarly in Theorem 3.3, we get that $\{x_n\}$ is a ω-Cauchy sequence and D is ω-complete, then there exists $v \in D$ such that $x_n \to v$ as $n \to \infty$.

Now, we will prove that v is a common fixed point of T and S. If there exists an increasing sequence $n_k \in \mathbb{N}$ such that $x_{n_k} \in Tv$ and $x_{n_k} \in Sv$ for all $k \in \mathbb{N}$, since Tv and Sv are ω – closed and $x_{n_k} \to v$, we have $v \in Tv$ and $v \in Sv$ and the proof is completed. So, suppose contrarily that for any sequence $n_k \in \mathbb{N}$ it is true that Card $\{k; x_{n_k} \in Tv\}$ is finite. In this case there exists $n_0 \in \mathbb{N}$ such that $x_{n_k} \notin Tv$ or $x_{2n+1} \notin Tv$ for all $n \geq n_0$. This implies that $Tx_{2n} \neq Tv$ for all $n \geq n_0$. Using equation (9), $x_{2n+2} \in Sx_{2n+1}$ with $x = v$ and $y = x_{2n+1}$, we obtain

$$0 < \inf\left\{F(\alpha\omega_1(v,x_{2n+1}) + \beta\omega_1(v,Tv) + \gamma\omega_1(x_{2n+1},Sx_{2n+1}) + L\omega_1(x_{2n+1},Tv)) - F\left(H_{\omega_1}(Tv,Sx_{2n+1})\right)\right\}$$

Or

$$\tau + F(\omega_1(Tv,x_{2n+2})) \leq \tau + F\left(H_{\omega_1}(Tv,Sx_{2n+1})\right)$$

$$\leq F(\alpha\omega_1(v,x_{2n+1}) + \beta\omega_1(v,Tv) + \gamma\omega_1(x_{2n+1},Sx_{2n+1}) + L\omega_1(x_{2n+1},Tv))$$

$$\leq F(\alpha\omega_1(v,x_{2n+1}) + \beta\omega_1(v,Tv) + \gamma\omega_1(x_{2n+1},x_{2n+2}) + L\omega_1(x_{2n+1},Tv))$$

Since F is strictly increasing, we have

$$\omega_1(Tv,x_{2n+2}) < \alpha\omega_1(v,x_{2n+1}) + \beta\omega_1(v,Tv) + \gamma\omega_1(x_{2n+1},x_{2n+2}) + L\omega_1(x_{2n+1},Tv)$$

Letting $n \to \infty$ in the previous inequality as $\beta + L < 1$, we have

$$\omega_1(Tv,v) < (\beta+L)\omega_1(v,Tv) < \omega_1(v,Tv)$$

which gives a contradiction. We obtain that $v \in Tv$, that is v is a fixed point for T.

On the other hand, for any sequence $n_k \in \mathbb{N}$ it is true that Card $\{k; x_{n_k} \in Sv\}$ is finite. In this case there exists $n_0 \in \mathbb{N}$ such that $x_{n_k} \notin Sv$ or $x_{2n+2} \notin Sv$ for all $n \geq n_0$. This implies that $Sx_{2n+1} \neq Sv$.

Now, using equation (9), $x_{2n+1} \in Tx_{2n}$ with $x = x_{2n}$ and $y = v$, we obtain

$$0 < \inf\left\{F\big(\alpha\omega_1(x_{2n}, v) + \beta\omega_1(x_{2n}, Tx_{2n}) + \gamma\omega_1(v, Sv) + L\omega_1(v, Tx_{2n})\big) - F\left(H_{\omega_1}(Tx_{2n}, Sv)\right)\right\}$$

Or

$$\tau + F\left(H_{\omega_1}(Tx_{2n}, Sv)\right)$$
$$\leq F\big(\alpha\omega_1(x_{2n}, v) + \beta\omega_1(x_{2n}, Tx_{2n}) + \gamma\omega_1(v, Sv) + L\omega_1(v, Tx_{2n})\big)$$

where $\tau \in \mathbb{R}^+$

This implies that

$$\tau + F(\omega_1(x_{2n+1}, Sv)) \leq \tau + F\left(H_{\omega_1}(Tx_{2n}, Sv)\right)$$
$$\leq F(\alpha\omega_1(x_{2n}, v) + \beta\omega_1(x_{2n}, Tx_{2n}) + \gamma\omega_1(v, Sv) + L\omega_1(v, Tx_{2n}))$$
$$\leq F(\alpha\omega_1(x_{2n}, v) + \beta\omega_1(x_{2n}, x_{2n+1}) + \gamma\omega_1(v, Sv) + L\omega_1(v, x_{2n+1}))$$

Since F is strictly increasing, we have

$$\omega_1(x_{2n+1}, Sv) < \alpha\omega_1(x_{2n}, v) + \beta\omega_1(x_{2n}, x_{2n+1}) + \gamma\omega_1(v, Sv) + L\omega_1(v, x_{2n+1})$$

Letting $n \to \infty$ in the previous inequality as $\gamma < 1$, we have

$\omega_1(v, Sv) < \gamma\omega_1(v, Sv) < \omega_1(v, Sv)$ which gives a contradiction. We obtain that $v \in Sv$, that is, v is a fixed point for S. Or v is a common fixed point for T and S.

4. Applications

4.1. Application in Integral Equations

In this section, we give an application of *Theorem 3.3* in Volterra type integral equations. Consider the following integrals:

$$u(t) = \int_0^t K_1(t, s, u(s))\, ds + f(t), \tag{12}$$

$$v(t) = \int_0^t K_2(t, s, v(s))\, ds + g(t) \tag{13}$$

for all $t \in [0, a]$. Let $C[0, a]$ be the set of all continuous functions defined on [0, a]. For $u \in C[0, a]$ with the norm

$$\|u\|_\tau = \max_{t\in[0,a]} |u(t)e^{-\tau t}|$$

for arbitrary $\tau > 0$ and the metric

$$\omega_\lambda(u, v) = \frac{1}{\lambda}\|u - v\|_\tau = \frac{1}{\lambda}\max_{t\in[0,a]} |(u(t) - v(t))e^{-\tau t}|$$

for all $u, v \in C[0, a]$. With these setting $(C[0, a], \|.\|_\tau)$ becomes Banach space.

Now we prove the following theorem to ensure the existence of the solution of the system of integral equations.

Theorem 4.1: Consider $K_1, K_2\colon [0, a]_\times[0, a]_\times\mathbb{R} \to \mathbb{R}$, $f, g\colon [0, a] \to \mathbb{R}$ are continuous and $T, S\colon C[0, a] \to PR(C[0, a])$ as

$$Tu(t) = \left\{\left(\int_0^t K_1(t, s, u(s))\, ds + f(t)\right) e^{-nt}\right\}, \tag{14}$$

$$Sv(t) = \left\{\left(\int_0^t K_2(t, s, v(s))\, ds + g(t)\right) e^{-\frac{n}{2}t}\right\}, \tag{15}$$

for every $n \in \mathbb{N} \cup \{0\}$.

If there exists $\tau > 1$, such that

$$\sup_{n,m\in\mathbb{N}\cup\{0\}}\left\{\int_0^t \left|K_1(t,s,u(s))e^{-nt}-K_2(t,s,v(s))e^{-\frac{m}{2}t}\right|ds+\left|f(t)e^{-nt}-g(t)e^{-\frac{m}{2}t}\right|\right\}e^{-\tau t}\le \tau e^{-\tau}\left|M_{T,S}(u(t),v(t))\right|$$

for some $t\in[0,a]$, for every $n,m\in\mathbb{N}\cup\{0\}$ and $u,v\in C[0,a]$ then the system of integral equations (14) and (15) has a solution.

Proof

Choosing x^* and y^* to be among the best approximations of $Sv(t)$ and $Tu(t)$, we have

$$H(Tu(t),Sv(t))=\max\left\{\sup_{x\in Tu(t)}\omega_1(x,Sv(t)),\ \sup_{y\in Sv(t)}\omega_1(y,Tu(t))\right\}$$

$$=\max\left\{\sup_{x\in Tu(t)}\omega_1(x,x^*),\ \sup_{y\in Sv(t)}\omega_1(y,y^*)\right\}$$

$$\le \sup_{x\in Tu(t),y\in Sv(t)}\omega_1(x,y)$$

$$\le \sup_{n,m\in\mathbb{N}\cup\{0\}}\left\{\int_0^t \left|K_1(t,s,u(s))e^{-nt}-K_2(t,s,v(s))e^{-\frac{m}{2}t}\right|ds+\left|f(t)e^{-nt}-g(t)e^{-\frac{m}{2}t}\right|\right\}e^{-\tau t}$$

$$\le \tau e^{-\tau}\int_0^t \left|M_{T,S}(u(t),v(t))\right|e^{-\tau s}e^{\tau s}ds$$

$$\le \left\|M_{T,S}(u,v)\right\|_\tau \tau e^{-\tau}\int_0^t e^{\tau s}ds$$

$$=\left\|M_{T,S}(u,v)\right\|_\tau \tau e^{-\tau}\frac{e^{\tau t}}{\tau}$$

$$=\left\|M_{T,S}(u,v)\right\|_\tau e^{-\tau}e^{\tau t}$$

for any $t\in[0,a]$, for every $n,m\in\mathbb{N}\cup\{0\}$ and $u,v\in C[0,a]$. Dividing by $e^{\tau t}$, we get

$$H(Tu(t),Sv(t))e^{-\tau t}\le e^{-\tau}\left\|M_{T,S}(u,v)\right\|_\tau$$

So,

$$\|H(Tu,Sv)\|_\tau\le e^{-\tau}\left\|M_{T,S}(u,v)\right\|_\tau$$

This implies that

$$\tau+\ln\|H(Tu,Sv)\|_\tau\le \ln\left\|M_{T,S}(u,v)\right\|_\tau$$

So, all the conditions of Theorem 3.3 are satisfied if $F(\alpha)=\ln\alpha$. Hence there exists $r\in C[0,a]$ such that $r(t)\in Tr(t)=\{r(t)e^{-nt}\}$ and $r(t)\in Sr(t)=\{r(t)e^{-\frac{n}{2}t}\}$ or there exists $n_0,n_1\in\mathbb{N}\cup\{0\}$ such that $r(t)=\left(\int_0^t K_3(t,s,r(s))\,ds+h(t)\right)e^{-n_0t}$ and $r(t)=\left(\int_0^t K_3(t,s,r(s))\,ds+h(t)\right)e^{-\frac{n_1}{2}t}$.

So $r(t)$ is a solution of the system of integral equations given in (14) and (15).

4.2. Electrorheological Fluids (ERF)

A lot of researchers take Electrorheological fluids (ERF) into their consideration, especially the applications in various branches of Science by proposed Nemours of modeling solutions [16, 17]. Where ERF are typically the liquids that fast solidify in the existence of an electric field [17]. Furthermore, it is considered as flexible materials that can be concentrated suspensions of polarizable particles in a non-conducting dielectric liquid [16]. Also, These fluids were converted to be anisotropic, and the apparent viscosity (the resistance to flow) in the direction orthogonal to the direction of the electric field sharply increases, in contrast to the apparent viscosity in the direction of the electric field does not modify drastically.

On the other hand, Sobolev et al. articulated that the spaces with a variable exponent. It has been proposed that modular metric spaces might be useful in modeling them [18, 8].

Recently, various works show the results of the modular metric space fixed point which they fit improved to certain types of differential equations [3].

R. H. W. Hoppe and his colleagues [16] studied the general flow problems of an anisotropic fluid with special cases of ERF under nonhomogeneous mixed boundary conditions with certain values of the velocities as well as the surface forces on different parts of that boundary. They examined two cases where the viscosity function considered being continuous and also proposed to be singular for vanishing shear rate. For instance, the problem (the latter case) reduces to a variational inequality. Using methods of nonlinear analysis like fixed point theory, monotonicity, and compactness, they illustrate the achieved results for the problems under consideration. Also, they found specific efficient approaches, typically for the numerical solution of the problems are shown [16], and numerical results for the simulation of the fluid flow in (ERF) shock absorbers are assumed.

Finally, we denote [19] for another comprehensive study of nonlinear superposition operators on modular metric spaces of functions.

4.3. In computer Programming

It is known that the procedure of the theory of generalized Fourier series as employed in the Best Approximation Method can be practical for the approximation of linear operator relations. In order to exhibit the computational results in using this method, numerous example problems where analytic solutions or quasi-analytic solutions exist are deliberated [20]. Applications include two-dimensional problems including the Laplace and Poisson equations, tests for discrepancies in results owing to inner product weighting factors, and applications to nonhomogeneous domain problems.

Hromadka et al., emphases on the weighting factor selection and modeling sensitivity, effects of additional basis functions on computational accuracy, and the effects on modeling results due to adding a lot of collocation points. Around thirteen simple but detailed examples have been used to clarify the approximation results achieved by the method when applied to practical problems [20].

It should be mentioned that the FORTRAN computer program is based on the Best Approximation Method as articulated by the FORTRAN computer program, which calculates approximation boundary values and relative errors (between exact and approximation values) [20].

5. Conclusion

In this research, we show a brief introduction on modular metric space and its properties. Furthermore, the concept of conjoint F-proximinal contraction on regular modular space for a pair of multivalued mappings has been modified and used to find a common fixed point result. In addition, we enhanced our results with an example. Meanwhile, the notion of conjoint F-proximinal contraction of Hardy-Rogers-type on regular modular space for a pair of multivalued mappings was presented. Also, a common fixed point, in this case, was proved. Finally, an application of our main result established the existence of the solution of an integral equation. The extra applications which could use the main concept of our results in other fields of science were illustrated in the final of the article. In the future work, it is recommended to generalize the obtained results for nemours spaces as well as different contractions.

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