Pure and Applied Mathematics Journal
2020; 9(5): 96-100
http://www.sciencepublishinggroup.com/j/pamj
doi: 10.11648/j.pamj.20200905.13
ISSN: 2326-9790 (Print); ISSN: 2326-9812 (Online)

Characterizations of Jordan *-derivations on Banach *-algebras

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To cite this article:

Guangyu An, Ying Yao. Characterizations of Jordan $*$-derivations on Banach $*$-algebras. *Pure and Applied Mathematics Journal*. Vol. 9, No. 5, 2020, pp. 96-100. doi: 10.11648/j.pamj.20200905.13

Received: August 11, 2020; **Accepted:** September 18, 2020; **Published:** October 28, 2020

Abstract: Suppose that $\mathcal{A}$ is a real or complex unital Banach $*$-algebra, $\mathcal{M}$ is a unital Banach $\mathcal{A}$-bimodule, and $G \in \mathcal{A}$ is a left separating point of $\mathcal{M}$. In this paper, we investigate whether the additive mapping $\delta : \mathcal{A} \to \mathcal{M}$ satisfies the condition $A, B \in \mathcal{A},\ AB = G \Rightarrow A\delta(B) + \delta(A)B^* = \delta(G)$ characterize Jordan $*$-derivations. Initially, we prove that if $\mathcal{A}$ is a real unital C^*-algebra and $G = I$ is the unit element in $\mathcal{A}$, then δ (non-necessarily continuous) is a Jordan $*$-derivation. In addition, we prove that if $\mathcal{A}$ is a real unital C^*-algebra and δ is continuous, then δ is a Jordan $*$-derivation. Finally, we show that if $\mathcal{A}$ is a complex factor von Neumann algebra and δ is linear, then δ (non-necessarily continuous) is equal to zero.

Keywords: Jordan *-derivation, Left Separating Point, C^*-algebra, Factor

1. Introduction

Let $\mathcal{R}$ be an associative ring. By an *involution* on $\mathcal{R}$, we mean a mapping $*$ from $\mathcal{R}$ into itself, such that $(AB)^* = B^*A^*$ and $(A^*)^* = A$ for each A, B in $\mathcal{R}$. A ring equipped with an involution is called a $*$-ring. In [1], M. Brešar and J. Vukman give the concept of Jordan $*$-derivations. An additive mapping δ from $\mathcal{R}$ into itself is called a *Jordan $*$-derivation* if

$$\delta(A^2) = \delta(A)A^* + A\delta(A) \tag{1}$$

for every A in $\mathcal{R}$. If $\mathcal{A}$ is a $*$-algebra and $\mathcal{M}$ is an $\mathcal{A}$-bimodule, then it is clear that an additive mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is a Jordan $*$-derivation if and only if

$$\delta(AB) = \delta(A)B^* + A\delta(B) + \delta(B)A^* + B\delta(A) \tag{2}$$

for each A, B in $\mathcal{A}$.

The study of Jordan $*$-derivations has been motivated by the problem of the representability of quasi-quadratic functionals by sesquilinear ones. It turns out that the question of whether each quasi-quadratic functional is generated by some sesquilinear functional is intimately connected with the structure of Jordan $*$-derivations. For the results concerning this problem we refer to [5, 6, 12-15]

In [1], the authors study some algebraic properties of Jordan $*$-derivations. As a special case of [1, Theorem 1], we know that every Jordan $*$-derivation δ from a complex unital $*$-algebra $\mathcal{A}$ into itself is of the form $\delta(A) = TA^* - AT$ for some T in $\mathcal{A}$. For non-unital $*$-algebras, in [2], M. Brešar and B. Zalar prove that every Jordan $*$-derivation δ from an algebra of all compact linear operators on a complex Hilbert space $\mathcal{H}$ into itself is of the form $\delta(A) = TA^* - AT$ for some T in $B(\mathcal{H})$. But it is also an open question whether above result in [2] remains true in the real case.

Roughly speaking, it is much more difficult to study Jordan $*$-derivations on real algebras than on complex algebras.

Nevertheless, in [10], P. Šemrl proves that every Jordan $*$-derivation on $B(\mathcal{H})$ is of the form $\delta(A) = TA^* - AT$ for some T in $B(\mathcal{H})$, where $\mathcal{H}$ is a real Hilbert space $\mathcal{H}$ with $\dim\mathcal{H} > 1$, and in [2], the authors give a new proof of this result. In [11], P. Šemrl shows that every Jordan $*$-derivation from a standard operator algebra $\mathcal{A}$ on $\mathcal{H}$ into $B(\mathcal{H})$ is of the form $\delta(A) = TA^* - AT$ for some T in $B(\mathcal{H})$, where $\mathcal{H}$ is a real or complex Hilbert space $\mathcal{H}$ with $\dim\mathcal{H} > 1$. In [9], X. Zhao and X. Qi prove that, under some mild conditions, an additive mapping δ from a $*$-ring with a symmetric idempotent P into itself satisfies the condition

$$AB = P \Rightarrow \delta(AB) = \delta(A)B^* + A\delta(B) + \delta(B)A^* + B\delta(A) \tag{3}$$

if and only if δ is an additive Jordan $*$-derivation.

2. The Main Results

It is not hard to see that when $\mathcal{A}$ is a complex unital Banach $*$-algebra, every linear Jordan $*$-derivation from $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule is zero. So, the notion of Jordan $*$-derivation studied in this note is more appropriate in the setting of real Banach $*$-algebras, and specially for real operator algebras and real C^*-algebras as illustrated by results in [10, 11].

Theorem 2.1. Suppose that $\mathcal{A}$ is a real unital C^*-algebra and $\mathcal{M}$ is a unital Banach $\mathcal{A}$-bimodule. If δ is an additive mapping (non-necessarily continuous) from $\mathcal{A}$ into $\mathcal{M}$ that satisfies the condition

$$AB = I \Rightarrow A\delta(B) + \delta(A)B^* = \delta(I), \tag{4}$$

then δ is a Jordan $*$-derivation.

proof. It is easy to show that $\delta(I) = 0$. Let A be an invertible element in $\mathcal{A}$. By $AA^{-1} = I$, we can obtain that

$$A\delta(A^{-1}) + \delta(A)(A^*)^{-1} = 0. \tag{5}$$

For any T in $\mathcal{A}$, let n be a positive integer with $n > \|T\| + 1$ and $B = nI + T$. We have that B and $I - B$ are invertible in $\mathcal{A}$. Thus we know that

$$\begin{aligned}
\delta(B) =& - B\delta(B^{-1})B^* = -B\delta(B^{-1}(I-B)^2 - B)B^* \\
=& - B\delta(B^{-1}(I-B)^2)B^* + B\delta(B)B^* \\
=&(I-B)^2\delta((I-B)^{-2}B)(I-B^*)^2 + B\delta(B)B^* \\
=&(I-B)^2\delta((I-B)^{-2} - (I-B)^{-1})(I-B^*)^2 + B\delta(B)B^* \\
=&(I-B)^2\delta((I-B)^{-2})(I-B^*)^2 - (I-B)^2\delta((I-B)^{-1})(I-B^*)^2 + B\delta(B)B^* \\
=& - \delta((I-B)^2) + (I-B)\delta(I-B)(I-B^*) + B\delta(B)B^* \\
=& - \delta(B^2) + B\delta(B) + \delta(B)B^* + \delta(B).
\end{aligned} \tag{6}$$

It implies that $\delta(B^2) = B\delta(B) + \delta(B)B^*$, by $\delta(I) = 0$, we have that

$$\delta(T^2) = T\delta(T) + \delta(T)T^* \tag{7}$$

for every T in $\mathcal{A}$. it follows that δ is a Jordan $*$-derivation.

Recall the definition of the left separating point. For an algebra $\mathcal{A}$ and an $\mathcal{A}$-bimodule $\mathcal{M}$, G in $\mathcal{A}$ is called a left separating point of $\mathcal{M}$ if $GM = 0$ implies $M = 0$ for every M in $\mathcal{M}$.

Theorem 2.2. Suppose that $\mathcal{A}$ is a real unital C^*-algebra, $\mathcal{M}$ is a unital Banach $\mathcal{A}$-bimodule and G is a left separating point of $\mathcal{M}$. If δ is a continuous linear mapping from $\mathcal{A}$ into $\mathcal{M}$ that satisfies the condition

$$AB = G \Rightarrow A\delta(B) + \delta(A)B^* = \delta(G), \tag{8}$$

then δ is a Jordan $*$-derivation.

Proof. Since $GI = G$, it follows that $G\delta(I) + \delta(G) = \delta(G)$. By the definition of the left separating point, we know that $\delta(I) = 0$.

Let A be a non-zero element in $\mathcal{A}$, it is well known that $I - tA$ is invertible in $\mathcal{A}$ for every t in $\mathbb{R}$ with $|t| < (\|A\|)^{-1}$, and we have that

$$(I - tA)^{-1} = \sum_{n=0}^{\infty}(tA)^n = \sum_{n=0}^{\infty} t^n A^n. \tag{9}$$

Since δ is a continuous linear mapping, it is easy to prove that δ is real linear. Thus $\delta(t^n B) = t^n\delta(B)$ for every B in $\mathcal{A}$ and every positive integer n.

By $G(I - tA)(I - tA)^{-1} = G$, it follows that

$$(G - tGA)\delta((I - tA)^{-1}) + \delta(G - tGA)(I - tA^*)^{-1} = \delta(G). \tag{10}$$

By (9) we can obtain that

$$(G - tGA)\delta(\sum_{n=0}^{\infty} t^n A^n) + \delta(G - tGA)\sum_{n=0}^{\infty} t^n (A^*)^n = \delta(G). \tag{11}$$

Since δ is a continuous linear mapping, we have that

$$\begin{aligned}
\delta(G) &= \sum_{n=0}^{\infty} t^n G\delta(A^n) - \sum_{n=0}^{\infty} t^{n+1} GA\delta(A^n) + \sum_{n=0}^{\infty} t^n \delta(G)(A^*)^n - \sum_{n=0}^{\infty} t^{n+1}\delta(GA)(A^*)^n \\
&= \sum_{n=1}^{\infty} t^n [G\delta(A^n) - GA\delta(A^{n-1}) + \delta(G)(A^*)^n - \delta(GA)(A^*)^{n-1}] + G\delta(I) + \delta(G).
\end{aligned} \tag{12}$$

By $\delta(I) = 0$, it implies that

$$\sum_{n=1}^{\infty} t^n [G\delta(A^n) - GA\delta(A^{n-1}) + \delta(G)(A^*)^n - \delta(GA)(A^*)^{n-1}] = 0 \tag{13}$$

for every t in $\mathbb{R}$ with $|t| < (\|A\|)^{-1}$. Consequently,

$$G\delta(A^n) - GA\delta(A^{n-1}) + \delta(G)(A^*)^n - \delta(GA)(A^*)^{n-1} = 0 \tag{14}$$

for all $n = 1,\ 2, \cdots$. In particular, choose $n = 1$ and $n = 2$ in (14), respectively, we have the following two identities:

$$G\delta(A) - GA\delta(I) + \delta(G)A^* - \delta(GA) = 0 \tag{15}$$

and

$$G\delta(A^2) - GA\delta(A) + \delta(G)(A^*)^2 - \delta(GA)A^* = 0. \tag{16}$$

Multiplying A^* from the right side of (15) and by $\delta(I) = 0$, we obtain

$$G\delta(A)A^* + \delta(G)(A^*)^2 - \delta(GA)A^* = 0. \tag{17}$$

Comparing (16) and (17), we have that

$$G\delta(A^2) = GA\delta(A) + G\delta(A)A^*. \tag{18}$$

By the definition of the left separating point, we know that

$$\delta(A^2) = A\delta(A) + \delta(A)A^* \tag{19}$$

for every A in $\mathcal{A}$. Thus δ is a Jordan $*$-derivation.

We note that if $\mathcal{A}$ is a complex unital Banach $*$-algebra and δ is a linear Jordan $*$-derivation from $\mathcal{A}$ into its Banach bimodule $\mathcal{M}$, then δ is equal to zero. In fact, if

$$\delta(AB + BA) = A\delta(B) + \delta(B)A^* + B\delta(A) + \delta(A)B^* \tag{20}$$

for each A, B in $\mathcal{A}$, then by $\delta(I) = 0$, we have that

$$\delta(A(iI) + (iI)A) = (iI)\delta(A) + \delta(A)(-iI). \tag{21}$$

Hence $\delta(A) = 0$ for every A in $\mathcal{A}$. Thus by Theorems 2.1 and 2.2, we have the following two corollaries.

Corollary 2.1. Suppose that $\mathcal{A}$ is a complex unital C^*-algebra and $\mathcal{M}$ is a unital Banach $\mathcal{A}$-bimodule. If δ is a linear mapping (non-necessarily continuous) from $\mathcal{A}$ into $\mathcal{M}$ that satisfies the condition

$$AB = I \Rightarrow A\delta(B) + \delta(A)B^* = \delta(I), \tag{22}$$

then δ is equal to zero.

Corollary 2.2. Suppose that $\mathcal{A}$ is a complex unital C^*-algebra, $\mathcal{M}$ is a unital Banach $\mathcal{A}$-bimodule and G is a left separating point of $\mathcal{M}$. If δ is a continuous linear mapping from $\mathcal{A}$ into $\mathcal{M}$ that satisfies the condition

$$AB = G \Rightarrow A\delta(B) + \delta(A)B^* = \delta(G), \tag{23}$$

then δ is equal to zero.

For complex von Neumann algebras, we have the following result.

Theorem 2.3. Suppose that $\mathcal{A}$ is a complex factor von Neumann algebra, $\mathcal{M}$ is a unital Banach $\mathcal{A}$-bimodule and G is a left separating point of $\mathcal{M}$. If δ is a linear mapping (non-necessarily continuous) from $\mathcal{A}$ into $\mathcal{M}$ that satisfies the condition

$$AB = G \Rightarrow A\delta(B) + \delta(A)B^* = \delta(G), \tag{24}$$

then δ is equal zero.

Proof. Since $GI = G$, it follows that

$$G\delta(I) + \delta(G) = \delta(G). \tag{25}$$

By the definition of the left separating point, we know that $\delta(I) = 0$.

For every projection P in $\mathcal{A}$ and t in $\mathbb{R}$ with $t \neq 1$, it is easy to show that

$$G(I - tP)(I - \frac{t}{t-1}P) = G. \tag{26}$$

It follows taht

$$(G - tGP)\delta(I - \frac{t}{t-1}P) + \delta(G - tGP)(I - \frac{t}{t-1}P) = \delta(G). \tag{27}$$

Multiplying $(t-1)$ from the left and right sides of (27), we have that

$$(G - tGP)\delta((t-1)I - tP) + \delta(G - tGP)((t-1)I - tP) = (t-1)\delta(G). \tag{28}$$

Hence, for any $t \neq 0, 1$, we can obtain that

$$t(GP\delta(P) - \delta(GP) + \delta(GP)P) + (\delta(GP) - G\delta(P) - \delta(G)P) = 0. \tag{29}$$

Hence

$$\delta(GP) = G\delta(P) + \delta(G)P = G\delta(P) + \delta(G)P^*. \quad (30)$$

Since $\mathcal{A}$ is a complex factor von Neumann algebra and by [4, Theorem 3], we know that every element in $\mathcal{A}$ can be written as a complex linear combination of projections in $\mathcal{A}$. Thus we have that

$$\delta(GA) = G\delta(A) + \delta(G)A^* \quad (31)$$

for every A in $\mathcal{A}$.

Let A be an invertible element in $\mathcal{A}$. By $GAA^{-1} = G$, we can obtain that

$$GA\delta(A^{-1}) + \delta(GA)(A^*)^{-1} = \delta(G). \quad (32)$$

By (31) and (32), it implies that

$$A\delta(A^{-1}) + \delta(A)(A^*)^{-1} = 0. \quad (33)$$

For any T in $\mathcal{A}$, let n be a positive integer with $n > \|T\| + 1$ and $B = nI + T$. We have that B and $I - B$ are invertible in $\mathcal{A}$. By (33), we know that

$$\begin{aligned}
\delta(B) =& - B\delta(B^{-1})B^* = -B\delta(B^{-1}(I-B)^2 - B)B^* \\
=& - B\delta(B^{-1}(I-B)^2)B^* + B\delta(B)B^* \\
=&(I-B)^2\delta((I-B)^{-2}B)(I-B^*)^2 + B\delta(B)B^* \\
=&(I-B)^2\delta((I-B)^{-2} - (I-B)^{-1})(I-B^*)^2 + B\delta(B)B^* \\
=&(I-B)^2\delta((I-B)^{-2})(I-B^*)^2 - (I-B)^2\delta((I-B)^{-1})(I-B^*)^2 + B\delta(B)B^* \\
=& - \delta((I-B)^2) + (I-B)\delta(I-B)(I-B^*) + B\delta(B)B^* \\
=& - \delta(B^2) + B\delta(B) + \delta(B)B^* + \delta(B).
\end{aligned} \quad (34)$$

It implies that $\delta(B^2) = B\delta(B) + \delta(B)B^*$ for every B in $\mathcal{A}$, by $\delta(I) = 0$, we have that

$$\delta(T^2) = T\delta(T) + \delta(T)T^* \quad (35)$$

for every T in $\mathcal{A}$. By the discussion preceding Corollary 2.1, we know that δ is equal to zero.

3. Conclusion

Suppose that $\mathcal{A}$ is a real or complex unital Banach $*$-algebra and $\mathcal{M}$ is a unital Banach $\mathcal{A}$-bimodule. An element G in $\mathcal{A}$ is called a *left separating point* of $\mathcal{M}$ if $GM = 0$ implies $M = 0$ for every M in $\mathcal{M}$. It is easy to see that every left invertible element in $\mathcal{A}$ is a left separating point of $\mathcal{M}$.

In Section 2, we let G be in $\mathcal{A}$ that is a left separating point of $\mathcal{M}$, and characterize the additive mapping δ from $\mathcal{A}$ into $\mathcal{M}$ satisfies the following condition

$$A, B \in \mathcal{A},\ AB = G \Rightarrow A\delta(B) + \delta(A)B^* = \delta(G). \quad (36)$$

Initially, we prove that if $\mathcal{A}$ is a real unital C^*-algebra and $G = I$ is the identity in $\mathcal{A}$, then δ (non-necessarily continuous) is a Jordan $*$-derivation. In addition, we prove that if $\mathcal{A}$ is a real unital C^*-algebra and δ is continuous linear, then δ is a Jordan $*$-derivation. Finally, we show that if $\mathcal{A}$ is a complex factor von Neumann algebra and δ is a linear mapping, then δ (non-necessarily continuous) is equal to zero.

Data Availability

The data set supporting the conclusions is included within this article.

Conflicts of Interest

The authors declare that they have no competing interests.

Funding Statement

This research was partly supported by the National Natural Science Foundation of China (Grant No. 11801342); Natural Science Foundation of Shaanxi Province (Grant No. 2020JQ-693); Scientific research plan projects of Shannxi Education Department (Grant No. 19JK0130).

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