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Finite Volume Scheme and Renormalized Solutions for a Noncoercive Elliptic Problem with Measure Data

Arouna Ouédraogo*, **Wendlassida Basile Yaméogo**

Department of Mathematics, Faculty of Sciences and Technology, University Norbert Zongo, Koudougou, Burkina Faso

Email address:
arounaoued2002@yahoo.fr (Arouna Ouédraogo), wendlassbas@gmail.com (Wendlassida Basile Yaméogo)
*Corresponding author

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Abstract: The objective of this paper is to show that the approximate solution, by the finite volumes method, converges to the renormalized solution of elliptic problems with measure data. The methods used are a priori estimates and density arguments. In the first part, we recall formulas and give some notations which are useful for the next of the work. It is also mentioned some definitions and properties on Partial Differentials Equations. In the second part we show the bases principle of the main methods of discretization, more precisely, the finite volume method. In the third part, we study a no coercive elliptic convection-diffusion equation with measure data. In our case, we take a diffuse measure data instead of L^1-data. The main originality in the present work is that we pass to the limit in a "renormalized discrete version". A first difficulty is to establish a discrete version of the estimate on the energy. The second difficulty is to deal with the diffuse measure data. By adapting the strategy developed in the finite volume method, we state and show our main result: the approximate solution converges to the unique renormalized solution. This work ends with a conclusion.

Keywords: Elliptic Problem, Measure Data, Renormalized Solutions, Finite Volume Scheme

1. Introduction

In this work, we consider the discretization by the cell-centered finite volume method of the following convection-diffusion problem:

$$\begin{cases} -\Delta u + div(vu) + bu = \mu \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is an open bounded polygonal subset of $\mathbb{R}^d$, $d \geq 2$, $v \in (L^p(\Omega))^d$, $2 < p < +\infty$ if $d = 2$, $p = d$, if $d \geq 3$, $b \in L^2(\Omega)$, $b \geq 0$ and μ is a diffuse measure.

The space of bounded Radon measures is denote by $\mathcal{M}_b^q(\overline{\Omega})$, with $\frac{1}{q} + \frac{1}{p} = 1$. J. Carrillo and M. Chipot proved that for $\mu \in \mathcal{M}_b^q(\overline{\Omega})$, there exists $f \in L^1(\Omega)$ and $F \in (L^p(\Omega))^d$ such that $\mu = f - divF$, see [2].

For the study of problem (1), the obstacles encountered are the noncoercive character of the operator $u \longmapsto -\Delta u + div(vu) + bu$ and the measure data.

Recall that a renormalized solution of (1) is a measurable function u defined from Ω to $\overline{\mathbb{R}}$, such that u is finite a.e. in Ω and

$$\forall k > 0, T_k(u) \in H_0^1(\Omega), \tag{2}$$

$$\lim_{k \to +\infty} \frac{1}{k} \int_\Omega |\nabla T_k(u)|^2 dx = 0, \tag{3}$$

$$\begin{cases} \forall h \in C_c^1(\mathbb{R}), \forall \psi \in H_0^1(\Omega) \cap L^\infty(\Omega), \\ \displaystyle\int_\Omega \nabla u h(u) \nabla \psi dx + \int_\Omega \nabla u \psi \nabla u h'(u) dx - \int_\Omega u h(u) v.\nabla \psi dx \\ \quad - \displaystyle\int_\Omega u h'(u) \psi v.\nabla u dx + \int_\Omega b u h(u) \psi dx = \int_\Omega \psi h(u)\, d\mu, \end{cases} \tag{4}$$

with T_k the truncate function at height k (see Figure 1 below).

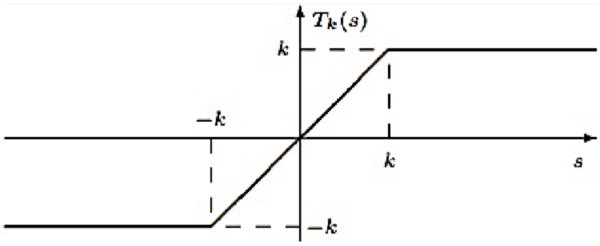

Figure 1. *The function* T_k.

Since h has a compact support, each term of (4) is well defined. Recently, Botti et al used the finite volume method to solve Poroelasticity problems using high order operators, see [3, 4]. The Hybrid High-Order method has also been studied by several authors, see [6, 9]. The existence and the uniqueness of a renormalized solution to (1) for L^1 data and $b \in L^2(\Omega)$ is proved by M. Ben Cheick and O. Guibé, see [1]. Eymard *et al.* [11] studied problem (1) with bounded measure data and $b \in L^2(\Omega)$. In the present case, it is taken a diffuse measure data instead of L^1-data, which was considered by S. Leclavier, see [14]. Passing to the limit in a "renormalized discrete version" is the main originality in the present paper, this is to say that a discrete version of $\varphi h(u)$ is taken as test function in the finite volume scheme. A first difficulty is to establish a discrete version of the estimate on the energy (3). Moreover it is worth noting that in (4) all the terms are "truncated" while a discrete version of $\varphi h(u)$ in the finite volume scheme leads to some residual terms which are not "truncated". The second difficulty is to deal with the diffuse measure data.

The paper is organized as follows. In Section 2, we present the finite volume scheme and the properties of the discrete gradient. Section 3 is devoted to prove several estimates, especially the discrete equivalent to (4) which is crucial to pass to the limit in the finite volume scheme. In Section 4, the proof of the convergence of the cell-centered finite volume scheme via a density argument is concerned. A brief conclusion closes this work in the last section.

2. Finite Volume Scheme

Let us define the admissibility mesh in the present work, see [14].

A family $\mathcal{T}$ of related subsets of $\Omega \subset \mathbb{R}^d$ is called a mesh. Any $K \in \mathcal{T}$ is called control volume. We impose that every $K \in \mathcal{T}$ is opened, the union of the $\overline{\mathcal{T}}$ is $\overline{\Omega}$ and the interface is in some hyperplane. For $K, L \in \mathcal{T}$ two distinct control volumes, their interface is denoted by $K/L := \overline{K} \cap \overline{L}$. Let $K \in \mathcal{T}$, we can write (N for "neighbour"):

$N(K) = \{L \in \mathcal{T}; L \notin K, K/L \neq \emptyset\}$, the set of neighbour of K and $\partial K = \displaystyle\bigcup_{L \in N(K)} K/L$, the edge of K. Finally, the Lebesgue measure in d-dimensional is denoted by $|K|$ and by $|\partial K|$ (respectively K/L) for the $(d-1)$-dimensional measure of ∂K (resp. of K/L).

Thus, set $\mathcal{E}$ a finite family of disjoint subsets of $\overline{\Omega}$ contained in affine hyperplanes, called the "edges", and set $P = (x_K)_{K \in \mathcal{T}}$ a family of points in Ω such that :

1. each $\sigma \in \mathcal{E}$ is a non-empty open subset of ∂K, for some $K \in \mathcal{T}$,
2. by denoting $\mathcal{E}(K) = \{\sigma \in \mathcal{E}; \sigma \in \partial K\}$, one has $\partial K = \bigcup_{\sigma \in \mathcal{E}(K)} \overline{\sigma}$ for all $K \in \mathcal{T}$,
3. for all $K \neq L$ in $\mathcal{T}$, either the measure of $\overline{K} \cap \overline{L}$ is null or $\overline{K} \cap \overline{L} = \overline{\sigma}$ for all $\sigma \in \mathcal{E}$, that we denote then $\sigma = K/L$,
4. for all $K \in \mathcal{T}$, x_k is in the interior of K,
5. for all $\sigma = K/L \in \mathcal{E}$, the line $[x_K, x_L]$ intersects and is orthogonal to σ,
6. for all $\sigma \in \mathcal{E}$, $\sigma \subset \partial\Omega \cap \partial K$, the line which is orthogonal to σ and going through x_k intercepts σ.

We denote by $|K|$ (resp. $|\sigma|$) the Lebesgue measure of $K \in \mathcal{T}$ (resp. of $\sigma \in \mathcal{E}$). The unit normal to $\sigma \in \mathcal{E}(K)$ outward to K is denoted by $\eta_{K,\sigma}$. $\mathcal{E}_{int}$ (resp. $\mathcal{E}_{ext}$) is defined as the set of interior (resp. the boundary) edges. For all $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}(K)$, we denote $d_{K,\sigma}$ the Euclidean distance between x_k and σ.

For any $\sigma \in \mathcal{E}$, d_σ is defined by $d_\sigma = d_{K,\sigma} + d_{L,\sigma}$, if $\sigma = K/L \in \mathcal{E}_{int}$ (in which case d_σ is the Euclidean distance between x_K and x_L) and $d_\sigma = d_{K.\sigma}$, if $\sigma \in \mathcal{E}_{ext} \cap \mathcal{E}(K)$.

The size of the mesh, denoted by $h_\mathcal{T}$, is defined by $h_\mathcal{T} = \sup\limits_{K \in \mathcal{T}} diam(K)$.

There exists $\zeta > 0$ such that for all $K \in \mathcal{T}$ and for all $\sigma \in \mathcal{E}_K$,

$$d_{K.\sigma} \geq \zeta d_\sigma. \tag{5}$$

In the sequel, the discrete $W_0^{1,q}$ norm and the discrete versions of Poincaré and Sobolev inequalities will be useful to solve the problem (see [7]).

Definition 2.1. (discrete $W_0^{1,q}$ norm) Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, $d \geq 2$, and let $\mathcal{T}$ be an admissible mesh. Define $X(\mathcal{T})$ as the set of functions from Ω to $\mathbb{R}$ which are constant over each control volume of the mesh. For $v_\mathcal{T} \in X(\mathcal{T})$ and $q \in [1, +\infty[$, we define the discrete $W_0^{1,q}$ norm by

$$\| v_{\mathcal{T}} \|_{1,q,\mathcal{T}}^{q} = \sum_{\sigma\in\mathcal{E}_{ext}\sigma\in\mathcal{E}(K)} |\sigma| d_\sigma \left|\frac{v_K}{d_\sigma}\right|^q + \sum_{\sigma\in\mathcal{E}_{int}\sigma=K/L} |\sigma| d_\sigma \left|\frac{v_K - v_L}{d_\sigma}\right|^q,$$

where $v_K = \chi_K\, v$.

Definition 2.2. (Discrete finite volume gradient) For all $K \in \mathcal{T}$ and for all $\sigma \in \mathcal{E}(K)$, we define the volume $D_{K,\sigma}$ as the cone of basis σ and of opposite vertex x_K. Then, we define the "diamond-cell" D_σ by:

$$D_\sigma = D_{K,\sigma} \cup D_{L,\sigma} \quad \text{if } \sigma = K/L \in \mathcal{E}_{int},$$
$$D_\sigma = D_{K,\sigma} \quad \text{if } \sigma \in \mathcal{E}_{ext} \cap \mathcal{E}(K).$$

For any $v_{\mathcal{T}} \in X(\mathcal{T}_m)$ and notice that $|D_\sigma| = \dfrac{|\sigma|\, d_\sigma}{d}$, the discrete gradient $\nabla_{\mathcal{T}} v_{\mathcal{T}}$ is defined by:

$$\forall \sigma = K/L \in \mathcal{E}_{int},\ \nabla_{\mathcal{T}} v(x) = |\sigma| \frac{v_L - v_K}{|D_\sigma|}\eta_{K,\sigma} = d\frac{v_L - v_K}{d_\sigma}\eta_{K,\sigma}, \forall x \in D_\sigma,$$
$$\forall \sigma \in \mathcal{E}_{ext} \cap \mathcal{E}(K),\ \nabla_{\mathcal{T}} v(x) = d\frac{0 - v_K}{d_\sigma}\eta_{K,\sigma},\ \forall x \in D_\sigma.$$

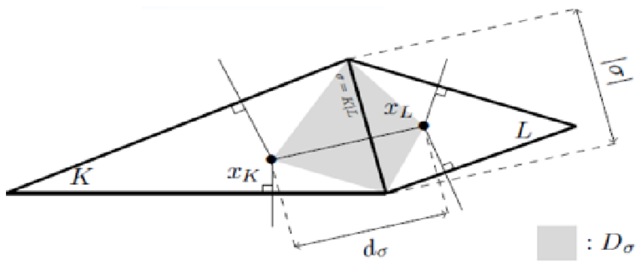

Figure 2. *Example of control volume for the method of finite volume in two dimensions of space.*

Theorem 2.1. (of Rellich) [15] Let Ω be an open set of $\mathbb{R}$ that us supposing boundary and with border enough regular. Then, the injection $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact.

Proposition 2.1. (Discrete Poincaré inequality) Let $\mathcal{T}$ be an admissible mesh and $v_{\mathcal{T}} \in X(\mathcal{T})$. Then, if $1 \le q \le 2$, $\| v_{\mathcal{T}} \|_{L^q(\Omega)} \le diam(\Omega) \| \bigtriangledown v_{\mathcal{T}} \|_{1,q,\mathcal{T}}$.

Proposition 2.2. (Discrete Sobolev inequality) Let $1 \le q \le 2$, $\mathcal{T}$ be an admissible mesh and $\zeta > 0$ satisfying for all $K \in \mathcal{T}$ and all $\sigma \in \mathcal{E}(K)$, $d_{K,\sigma} \ge \zeta d_\sigma$, Then, with $q* = \frac{dq}{d-q}$ if $q < d$ and $q* < \infty$ if $q = d = 2$, there exists $C > 0$ only depending on $(\Omega, q, q*,)$ such that, for all $v_{\mathcal{T}} \in X(\mathcal{T})$,we have $\| v_{\mathcal{T}} \|_{L^{q*}(\Omega)} \le C \| v_{\mathcal{T}} \|_{1,q,\mathcal{T}}$.

Before writing the finite volume scheme, let us define a discrete finite volume gradient (see [13])).

Lemma 2.1. (Weak convergence of the finite volume gradient) Let $(\mathcal{T}_m)_{m\ge1}$ be a sequence of admissible meshes such that there exists $\zeta > 0$ satisfying for all $m > 1$, for all $K \in \mathcal{T}$ and for all $\sigma \in \mathcal{E}(K)$, $d_{K,\sigma} \ge \zeta d_\sigma$, and such that $h_{\mathcal{T}_m} \longrightarrow 0$. Let $v_{\mathcal{T}_m} \in X(\mathcal{T}_m)$ and let us assume that there exists $\alpha \in [1, +\infty[$ and $C > 0$ such that $\| v_{\mathcal{T}_m} \|_{1,\alpha,\mathcal{T}_m} \le C$, and $v_{\mathcal{T}_m}$ and that $L^1(\Omega)$ converges in $v \in W_0^{1,\alpha}(\Omega)$. Then $\nabla \mathcal{T}_m v_{\mathcal{T}_m}$ converges to ∇v weakly to $L^\alpha(\Omega)^d$.

Let $\mathcal{T}$ be an admissible mesh, we can define the finite volume discretization of (1). For $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}(K)$, we define

$$b_K = \frac{1}{|K|}\int_K b dx, \tag{6}$$

$$v_{K,\sigma} = \frac{1}{|D_\sigma|}\int_{D_\sigma} v.\eta_{K,\sigma} dx, \tag{7}$$

$$F_{K,\sigma} = \frac{1}{|D_\sigma|}\int_{D_\sigma} F.\eta_{K,\sigma} dx, \tag{8}$$

$$f_K = \frac{1}{|K|}\int_K f dx. \tag{9}$$

So, we can write the scheme (1) as following:
For all $K \in \mathcal{T}$,

$$\sum_{\sigma\in\mathcal{E}(K)} \frac{|\sigma|}{d_\sigma}(u_K - u_L) + \sum_{\sigma\in\mathcal{E}(K)} |\sigma| v_{K,\sigma} u_{\sigma,+} + |K| b_K u_K = |K| f_K - \sum_{\sigma\in\mathcal{E}(K)} |D_\sigma| F_{K,\sigma} \tag{10}$$

with

$$\forall\sigma = K/L \in \mathcal{E}_{int}, \begin{cases} u_{\sigma,+} = u_K \text{ if } v_{K,\sigma} \geq 0, \\ u_{\sigma,+} = u_L, \text{ otherwise,} \end{cases} \quad (11)$$

$$\forall\sigma \in \mathcal{E}_{ext} \cap \mathcal{E}(K), \begin{cases} u_{\sigma,+} = u_K \text{ if } v_{K,\sigma} \geq 0, \\ u_{\sigma,+} = 0, \text{ otherwise.} \end{cases} \quad (12)$$

We denote by $u_{\sigma,-}$ the downstream choice of u which is such that $\{u_{\sigma,+}, u_{\sigma,-}\} = \{u_K, u_L\}$ (with $u_L = 0$ if $\sigma \in \mathcal{E}_{est} \cap \mathcal{E}(K)$).

3. Estimations

In this section, we first establish in Proposition 3.1 an estimate on $\ln(1+|u_{\mathcal{T}}|)$ which is crucial to control the measure of the set $\{|u_{\mathcal{T}}| > n\}$. Then, we show in Proposition 3.2 an estimate on $T_n(u_{\mathcal{T}})$ and the convergence of $T_n(u_{\mathcal{T}})$ to $T_n(u)$. Finally, we prove in Proposition 3.3 a discrete version of the decay of the energy.

Proposition 3.1. (see [14]) Let $\mathcal{T}$ be an admissible mesh. If $u_{\mathcal{T}} = (u_K)_{K\in\mathcal{T}}$ is a solution to (10), then

$$\| \ln(1 + |u_{\mathcal{T}}|)\|^2_{1,2,\mathcal{T}} \leq C(\mu, \Omega) + d|\Omega|^{\frac{p-2}{p}} \|v\|^2_{L^p(\Omega)^d}, \quad (13)$$

where $C(\mu, \Omega)$ is a constant depending on μ and Ω.

Let us state an easy corollary, which is used in the proof of the estimate of Proposition 3.2.

Corollary 3.1. Let $\mathcal{T}$ be an admissible mesh. if $u_{\mathcal{T}} = (u_K)_{K\in\mathcal{T}}$ is a solution to (10) and, for $n > 0$, $E_n = \{|u_{\mathcal{T}}| > n\}$, then there exists $C > 0$ only depending on (Ω, v, f, d, p) such that

$$|E_n| \leq \frac{C(1 + \|\mu\|_{M^p_b(\overline{\Omega})})}{(\ln(1 + n))^2}. \quad (14)$$

Proposition 3.2. (Estimation on $T_n(u_{\mathcal{T}})$) Let $\mathcal{T}$ be an admissible mesh. if $u_{\mathcal{T}} = (u_K)_{K\in\mathcal{T}}$ is a solution to (10), then there exists $C > 0$ only depending on (Ω, v, μ, n, d) such that

$$||T_n(u_{\mathcal{T}})||_{1,2,\mathcal{T}} \leq C, \qquad \forall n > 0. \quad (15)$$

Moreover, if $(\mathcal{T}_m)_{m\geq 1}$ is a sequence of admissible meshes such that there exists $\zeta > 0$ satisfying for all $m \leq 1$, for all $K \in \mathcal{T}$ and for all $\sigma \in \mathcal{E}(K)$, $d_{K\sigma} \geqslant \zeta d_\sigma$, there exists a measurable function u finite a.e. in Ω such that, up to a sub-sequence $T_n(u_{\mathcal{T}_m})$ converges to $T_n(u)$ weakly in $H^1_0(\Omega)$, strongly in $L^2(\Omega)$ and a.e. in Ω.

Proof The proof is divided into two steps. In Step 1 we derive the estimate (15) on the truncate on $u_{\mathcal{T}}$. The step 2 is devoted to extract a Cauchy sub-sequences in measure.

Step 1: Estimation on $T_n(u_{\mathcal{T}})$

Multiplying each equation of the scheme (2.6) by $T_n(u_K)$, summing over each control volume and reordering the sum, we obtain $S_1 + S_2 + S_3 = S_4 - S_5$ with

$$\begin{aligned} S_1 &= \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma}(u_K - u_L)(T_n(u_K) - T_n(u_L)), \\ S_2 &= \sum_{\sigma\in\mathcal{E}} |\sigma| v_{K,\sigma} u_{\sigma,+}(T_n(u_K) - T_n(u_L)), \\ S_3 &= \sum_{K\in\mathcal{T}} |K| b_k u_K T_n(u_K), \\ S_4 &= \sum_{K\in\mathcal{T}} \int_K f T_n(u_K) dx, \\ S_5 &= \sum_{\sigma\in\mathcal{E}} |D_\sigma| F_{K,\sigma}(T_n(u_K) - T_n(u_L)). \end{aligned}$$

Since b is nonegative and since $rT_n(r) \geqslant 0 \quad \forall r$, we notice that S_3 is nonnegative. Moreover, since T_n is bounded by n, we deduce that

$$|S_4| \leqslant n\|f\|_{L^1(\Omega)}.$$

For the term S_5, Hölder inequality and relation (5) yield

$$\begin{aligned} |S_5| &\leq \sum_{\sigma\in\mathcal{E}} |D_\sigma||F_{K,\sigma}||(T_n(u_K) - T_n(u_L))| \\ &\leq 2n \sum_{\sigma\in\mathcal{E}} |D_\sigma||F_{K,\sigma}| \\ &\leq 2n \sum_{\sigma\in\mathcal{E}} \int_{D_\sigma} |F|\, dx \\ &\leq 2n \sum_{\sigma\in\mathcal{E}} |D_\sigma|^{1/q} \|F\|_{L^p(\Omega)^d} \\ &\leq 2n \sum_{\sigma\in\mathcal{E}(K)} \frac{(|\sigma| d_{K,\sigma})^{1/q}}{d^{1/q}} \|F\|_{L^p(\Omega)^d} \\ &\leq 2n\, d^{1/p} \|F\|_{L^p(\Omega)^d}. \end{aligned} \quad (16)$$

Therefore,

$$S_1 \leqslant n\|f\|_{L^1(\Omega)} + 2n\, d^{1/p} \|F\|_{L^p(\Omega)^d} - S_2.$$

S_2 can be rewritten as,

$$-S_2 = \sum_{\sigma\in\mathcal{E}} |\sigma||v_{K,\sigma}| u_{\sigma,+}(T_n(u_{\sigma,-}) - T_n(u_{\sigma,+})).$$

The subset $\mathcal{A}$ of edges is defined by (see [10]),

$$\begin{aligned} \mathcal{A} = \{\sigma \in \mathcal{E}; u_{\sigma,+} \geqslant u_{\sigma,-}, u_{\sigma,+} < 0\} \\ \cup\{\sigma \in \mathcal{E}; u_{\sigma,+} < u_{\sigma,-}, u_{\sigma,+} \geq 0\} \end{aligned} \quad (17)$$

and since T_n is non decreasing we have

$$-S_2 = \sum_{\sigma\in\mathcal{E}} |\sigma||v_{K,\sigma}| u_{\sigma,+}(T_n(u_{\sigma,-}) - T_n(u_{\sigma,+})).$$

Notice that $\forall\sigma \in \mathcal{A}, |u_{\sigma,+}| \geq n$ implies $|u_{\sigma,-}| \geq n$. So, we deduce that for all σ in $\mathcal{A}$,

$$u_{\sigma,+}(T_n(u_{\sigma,-})-T_n(u_{\sigma,+}))=T_n(u_{\sigma,+})(T_n(u_{\sigma,-})-T_n(u_{\sigma,+})).$$

It follows that

$$-S_2\leq\sum_{\sigma\in\mathcal{A}}|\sigma||v_{K,\sigma}|u_{\sigma,+}T_n(u_{\sigma,+})(T_n(u_{\sigma,-})-T_n(u_{\sigma,+}))$$

$$\leq nd^{\frac{1}{2}}\|v\|_{L^2(\Omega)^d}\Big(\sum_{\sigma\in\mathcal{A}}\frac{|\sigma|}{d_\sigma}(T_n(u_{\sigma,-})-T_n(u_{\sigma,+}))^2\Big)^{\frac{1}{2}}$$

$$\leq\frac{1}{2}n^2d\|v\|^2_{L^2(\Omega)^d}+\frac{1}{2}\sum_{\sigma\in\mathcal{A}}\frac{|\sigma|}{d_\sigma}\big((T_n(u_{\sigma,-})-T_n(u_{\sigma,+}))^2\big)^{\frac{1}{2}}$$

$$\leq\frac{1}{2}n^2d\|v\|^2_{L^2(\Omega)^d}+\frac{1}{2}\sum_{\sigma\in\mathcal{E}}\frac{|\sigma|}{d_\sigma}\big((T_n(u_K)-T_n(u_L))^2\big)^{\frac{1}{2}}.$$

Since $T_n(u_K)-T_n(u_L)\leq u_K-u_L$ (because T_n is 1-Lipschitz function), we have

$$-S_2\leq\frac{1}{2}\sum_{\sigma\in\mathcal{E}}\frac{|\sigma|}{d_\sigma}(u_K-u_L)(T_n(u_K)-T_n(u_L))+\frac{1}{2}n^2d\|v\|^2_{L^2(\Omega)^d}$$

and we can deduce that

$$\frac{1}{2}\sum_{\sigma\in\mathcal{E}}\frac{|\sigma|}{d_\sigma}(u_K-u_L)(T_n(u_K)-T_n(u_L))\leq n\|f\|_{L^1(\Omega)}+2n\,d^{1/p}\,\|F\|_{L^p(\Omega)^d}+\frac{1}{2}n^2d\|v\|^2_{L^2(\Omega)^d}.$$

Therefore, using again the fact that T_n is 1-Lipschitz, we can write :

$$\frac{1}{2}\sum_{\sigma\in\mathcal{E}}\frac{|\sigma|}{d_\sigma}\big(T_n(u_K)-T_n(u_L)\big)^2\leq n\|f\|_{L^1(\Omega)}+2n\,d^{1/p}\,\|F\|_{L^p(\Omega)^d}+\frac{1}{2}n^2d\|v\|^2_{L^2(\Omega)^d}.$$

Applying Lemma 2.1 and the diagonal process, up to a subsequence still denoted by $\mathcal{T}_m$, for any $n\geq 1$, there exist v_n in $H^1_0(\Omega)$ such that $T_n(u_\mathcal{T})\longrightarrow v_n$ and $T_n(u_\mathcal{T})\rightharpoonup v_n$ in the finite volume gradient sense.

Step 2: Up to a subsequence, $u_\mathcal{T}$ is a Cauchy sequence in measure

In this step, we follow the ideas of Dal Maso *et al.* to show that $u_{\mathcal{T}_m}$ converges a.e. to u (see [8]). Let $\omega>0$. For all $n>0$ and all sequences $(\mathcal{T}_m)_{m\geq1}$ and $(\mathcal{T}_p)_{p\geq1}$ of admissible meshes, we have

$$\{|u_{\mathcal{T}_m}-u_{\mathcal{T}_p}|\geqslant\omega\}\subset\{|u_{\mathcal{T}_m}|>n\}\cup\{|u_{\mathcal{T}_p}|>n\}\cup\{|T_n(u_{\mathcal{T}_m})-T_n(u_{\mathcal{T}_p})|>\omega\}.$$

Let $\varepsilon>0$ fixed. By (14), let $n>0$ such that, for all admissible meshes $\mathcal{T}_m$ and $\mathcal{T}_p$,

$$meas(\{|u_{\mathcal{T}_m}|>n\})+meas(\{|u_{\mathcal{T}_p}|>n\})<\frac{\varepsilon}{2}.$$

Once n is chosen, we deduce from Step 1 that $T_n(u_{\mathcal{T}_m})$ is a Cauchy sequence in measure, thus

$$\exists h_0>0;\forall h_{\mathcal{T}_m},h_{\mathcal{T}_p}<h_0,meas(\{|T_n(u_{\mathcal{T}_m})-T_n(u_{\mathcal{T}_p})|>\omega\})<\frac{\varepsilon}{2}.$$

Therefore, we deduce that

$$\forall h_{\mathcal{T}_m},h_{\mathcal{T}_p}<h_0,\quad meas(\{|u_{\mathcal{T}_m}-u_{\mathcal{T}_p}|>\omega\})<\varepsilon.$$

Hence $(u_{\mathcal{T}_m})$ is a Cauchy sequence in measure. Consequently, up to a subsequence still indexed by $\mathcal{T}_m$, there exists a measurable function u such that $u_{\mathcal{T}_m} \longrightarrow u$ a.e. in Ω. Due to Corollary 3.1, u is finite a.e. in Ω. Moreover from convergences obtained in Step 1 we get that

$$T_n \in H_0^1(\Omega) \text{ and } \nabla_{\mathcal{T}} T_n(u_{\mathcal{T}}) \longrightarrow \nabla T_n(u) \text{ in } (L^2(\Omega))^d. \quad (18)$$

In the following proposition we prove a uniform estimate on the truncated energy of $u_{\mathcal{T}}$ (see (19)) which is crucial to pass to the limit in the approximate problem. We explicitly observe that (19) is the discrete version of (3) which is imposed in the definition of the renormalized solution for elliptic equation with measure data. As in the continuous case (19) is related to the regularity of f : $f \in L^1(\Omega)$ and does not charge any zero-Lebesgue set. If we replace $div(vu)$, we also have to uniformly control the discrete version of $\frac{1}{n}\int_\Omega vu\nabla T_n(u)dx$ which is stated in (20).

Proposition 3.3. *(Discrete estimate on the energy)*

Let $(\mathcal{T}_m)m \geq 1$ be a sequence of admissible meshes such that there exists $\zeta > 0$ satisfying $\forall m \geqslant 1, \forall K \in \mathcal{T}$ and $\forall \sigma \in \mathcal{E}(K), d_{K,\sigma} \geq \zeta d_\sigma$.

If $u_{\mathcal{T}_m} = (u_K)_{K\in\mathcal{T}_m}$ is a solution to (10), then

$$\lim_{n\to+\infty} \overline{\lim_{h_{\mathcal{T}_m}\to 0}} \frac{1}{n}\sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma}(u_K - u_L)(T_n(u_K) - T_n(u_L)) = 0 \quad (19)$$

where $u_L = 0$ if $\sigma \in \mathcal{E}_{ext}$, and

$$\lim_{n\to+\infty} \overline{\lim_{h_{\mathcal{T}_m}\to 0}} \frac{1}{n}\sum_{\sigma\in\mathcal{E}} |\sigma||v_{K,\sigma}||u_{\sigma,+}||(T_n(u_{\sigma,+}) - T_n(u_{\sigma,-})| = 0. \quad (20)$$

Proof We first establish (19). Let $\mathcal{T}$ be an admissible mesh and let $u_{\mathcal{T}}$ be a solution of (10). Multiplying each equation of the scheme by $\frac{T_n(u_K)}{n}$, summing on $K \in \mathcal{T}$ and gathering by edges lead to $T_1 + T_2 + T_3 = T_4 - T_5$ with

$$T_1 = \frac{1}{n}\sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma}(u_K - u_L)(T_n(u_K) - T_n(u_L)),$$

$$T_2 = \frac{1}{n}\sum_{\sigma\in\mathcal{E}} |\sigma| v_{K,\sigma} u_{\sigma,+}(T_n(u_K) - T_n(u_L)),$$

$$T_3 = \frac{1}{n}\sum_{K\in\mathcal{T}} |K| b_K u_K T_n(u_K),$$

$$T_4 = \frac{1}{n}\sum_{K\in\mathcal{T}} \int_K f T_n(u_K)dx,$$

$$T_5 = \frac{1}{n}\sum_{\sigma\in\mathcal{E}} |D_\sigma| F_{K,\sigma}(T_n(u_K) - T_n(u_L)).$$

Since b is non-negative and since $rT_n(r) \geq 0 \ \forall r$, we get $T_3 \geq 0$. Due to the definition of $u_{\mathcal{T}}$ we have $T_4 = \int_\Omega f\frac{T_n(u_{\mathcal{T}})}{n}dx$. In view of the point-wise convergence of $u_{\mathcal{T}}$ to u, we obtain that $T_n(u_{\mathcal{T}})$ converges to $T_n(u)$ a.e. and weak * as $h_{\mathcal{T}} \longrightarrow 0$. It follows that $\lim_{h_{\mathcal{T}}\to 0} T_4 = \int_\Omega f\frac{T_n(u)}{n}dx$. Since u is finite a.e. in Ω, $\frac{T_n(u)}{n}$ converges to 0 a.e. and in L^∞ weak, and since f belongs to $L^1(\Omega)$, the Lebesgue dominated convergence theorem implies that

$$\lim_{n\to+\infty}\lim_{h_{\mathcal{T}}\to 0} T_4 = 0. \quad (21)$$

For the term T_5, using Hölder inequality, relation (5) and Definition 2.2 yield

$$\begin{aligned}
|T_5| &\leq \frac{1}{n}\sum_{\sigma\in\mathcal{E}} |D_\sigma||F_{K,\sigma}||(T_n(u_K) - T_n(u_L))| \\
&\leq \frac{1}{n}\sum_{\sigma\in\mathcal{E}} \frac{d_\sigma}{d}\int_{D_\sigma} |F|\,|\nabla_{\mathcal{T}} T_n(u_{\mathcal{T}})|\,dx \\
&\leq \frac{1}{n}\sum_{\sigma\in\mathcal{E}} \frac{d_\sigma}{d}\int_{\Omega} |F|\,|\nabla_{\mathcal{T}} T_n(u_{\mathcal{T}})|\,dx \\
&\leq \frac{1}{n}\sum_{\sigma\in\mathcal{E}} \frac{d_\sigma}{d}\|F\|_{L^2(\Omega)^d}\,\|\nabla_{\mathcal{T}} T_n(u_{\mathcal{T}})\| \\
&\leq \frac{1}{n}\sum_{\sigma\in\mathcal{E}} \frac{d_{K,\sigma}}{d\zeta}\|F\|_{L^2(\Omega)^d}\,\|\nabla_{\mathcal{T}} T_n(u_{\mathcal{T}})\| \\
&\leq \frac{1}{n}\frac{diam(K)}{\zeta}\|F\|_{L^2(\Omega)^d}\,\|\nabla_{\mathcal{T}} T_n(u_{\mathcal{T}})\|. \quad (22)
\end{aligned}$$

Using the strong convergence in $L^2(\Omega)$ and a.e. in Ω of $T_n(u_{\mathcal{T}})$ to $T_n(u)$, and relation (22), give

$$\lim_{n\to+\infty}\lim_{h_{\mathcal{T}}\to 0} T_5 = 0. \tag{23}$$

Using the same techniques as S. Leclavier [14], it follows that

$$-T_2 \leq \frac{1}{n}\frac{r^2 d|v|^2_{L^2(\Omega)^d}}{2} + \frac{1}{2}T_1. \tag{24}$$

From (22), we deduce (19). By the same manage used by S. Leclavier [14], we prove (20).

The following corollary is useful to pass to the limit in the diffusion term.

Corollary 3.2. (see [14]) Let $(\mathcal{T}_m)_{m\geq 1}$ be a sequence of admissible meshes such that there exists $\xi > 0$ satisfying for all $m \leq 1$ for all $K \in \mathcal{T}$ and all $\sigma \in \mathcal{E}(K), d_{k,\sigma} \geq \xi d_\sigma$.

If $u_{\mathcal{T}_m} = (u_K)_{K\in\mathcal{T}_m}$ is a solution to (10), then

$$\lim_{n\to+\infty}\overline{\lim_{h_{\mathcal{T}}\to 0}} \sum_{\substack{\sigma\in\mathcal{E},\\ |u_K|\leq 2n,\\ |u_L>4n}} \frac{|\sigma|}{d_\sigma}|u_L| = 0. \tag{25}$$

4. Convergence Analysis

Let us now state the main result of this paper.

Theorem 4.1. If $\mathcal{T}$ is an admissible mesh, then there exists a unique solution to (10). If $(\mathcal{T}_m)_{m\geq 1}$ is a sequence of admissible meshes such that there exists $\xi > 0$ satisfying for all $m \geq 1$ for all $K \in \mathcal{T}$ and all $\sigma \in \mathcal{E}(K)$, $d_{K,\sigma} \geq \xi d_\sigma$, and such that $h_{\mathcal{T}_m} \to 0$, then if $u_{\mathcal{T}_m} = (u_K)_{K\in\mathcal{T}_m}$ is the solution to (10) with $\mathcal{T} = \mathcal{T}_m, u_{\mathcal{T}_m}$ converges to u in the sense that for all $n > 0$, $T_n(u_{\mathcal{T}_m})$ converges weakly to $T_n(u)$ in $H_0^1(\Omega)$, when u is the unique renormalized solution of (1).

Before proving Theorem 4.1, we recall the following convergence result concerning the function (h_n) defined, for any $n \geq 1$, by (see [14])

$$h_n(s) = \begin{cases} 0, & \text{if } s \leq -2n; \\ \frac{s}{n}+2, & \text{if } -2n \leq s \leq -n, \\ 1, & -n \leq s \leq n, \\ \frac{-s}{n}+2, & \text{if } n \leq s \leq 2n, \\ 0, & \text{if } s \geq 2n. \end{cases} \tag{26}$$

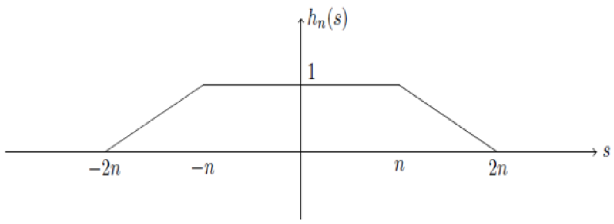

Figure 3. *The function h_n.*

Lemma 4.1. Let $(\mathcal{T}_m)_{m\geq 1}$ be a sequence of admissible meshes such that there exists $\xi > 0$ satisfying for all $m \geq 1$, for all $K \in T$ and for all $\sigma \in \mathcal{E}(K)$, $d_{K,\sigma} \geq \xi d_\sigma$. Let $u_{\mathcal{T}_m} \in X(\mathcal{T}_m)$ be a sequence of solution of (10). We define the function $\widetilde{h}_n$ by

$\sigma \in \mathcal{E}$,$\forall x \in D_\sigma$,$\widetilde{h}_n(x) = \frac{h_n(x_K)+h_n(x_L)}{2}$, then $\widetilde{h}_n \to h_n(u)$ in $L^q(\Omega)$, $\forall q \in [2,+\infty[$ where $h_{\mathcal{T}_m} \to 0$, where u is the limit of $u_{\mathcal{T}_m}$.

Proof Proof Proof of Theorem 4.1 The proof consists into two steps. In Step 1 we prove the existence and the uniqueness of the solution of (10). Concerning the uniqueness of the renormalized solution, the proof is done by Ouédraogo *et al.* [16]. To prove the second point, we adapts one uniqueness techniques developed in the continuous case by several authors (see [5, 12]). It is worth noting that we use here a different method to the one developed by J. Droniou *et al.* [10]. Using the results of Section 3, Step 2 is devoted to pass to the limit in the scheme. It is worth noting that we take in the scheme a discrete version of what is a test function in the renormalized formulation.

4.1. Existence and Uniqueness of the Solution of the Scheme

Since the relation (10) is a linear system of n equations with n unknowns, it is sufficient to show that the solution of the relation (10) with $\mu = 0$ (see [14]).

4.2. Convergence

Let $\varphi \in C_c^\infty(\Omega)$ and h_n the function defined by (26). We denote by $\varphi_{\mathcal{T}}$ the function defined by $\varphi_K = \varphi_K(x_K)$ for all $K \in \mathcal{T}$. Multiplying each equation of the scheme (10) by $\varphi(x_K)h_n(u_K)$ (which is a discrete version of the test function used in the renormalized formulation), summing over the control volumes and gathering by edges, we get $V_1 + V_2 + V_3 = V_4 - V_5$ with

$$V_1 = \sum_{\sigma\in\mathcal{E}} \frac{\sigma}{d_\sigma}(u_K - u_L)\big(\varphi(x_K)h_n(u_K) - \varphi(x_L)h_n(u_L)\big),$$

$$V_2 = \sum_{\sigma\in\mathcal{E}} |\sigma| v_{K,\sigma} u_{\sigma,+}\big(\varphi(x_K)h_n(u_K) - \varphi(x_L)h_n(u_L)\big),$$

$$V_3 = \sum_{K\in\mathcal{E}} |K| b_K u_K \varphi(x_K) h_n(u_K),$$

$$V_4 = \sum_{K\in\mathcal{E}} \int_K f\varphi(x_K)h_n(u_K),$$

$$V_5 = \sum_{K\in\mathcal{E}} |D_\sigma| F_{K,\sigma}\big(\varphi(x_K)h_n(u_K) - \varphi(x_L)h_n(u_L)\big).$$

As far as the term V_4 is concerned, by the regularity of φ, we have $\varphi_{\mathcal{T}} \to \varphi$ uniformly on Ω when $h_{\mathcal{T}} \to 0$. We now pass to the limit as $h_{\mathcal{T}} \to 0$. Since $h_n(u_{\mathcal{T}}) \to h_n(u)$ a.e and L^∞ weak $*$, $\varphi_{\mathcal{T}} \to \varphi$ uniformly, $|f\varphi_{\mathcal{T}}h_n(u_{\mathcal{T}})| \le C_\varphi |f| \in L^1(\Omega)$, the Lebesgue dominated convergence theorem ensures that

$$V_4 = \int_\Omega f\varphi_{\mathcal{T}}h_n(u_{\mathcal{T}})dx \underset{h_{\mathcal{T}}\to 0}{\longrightarrow} \int_\Omega f\varphi h_n(u)dx. \tag{27}$$

Due to the definition of $\nabla_{\mathcal{T}}(\cdot)$ we get for the term V_5:

$$\begin{aligned} V_5 &= \sum_{K\in\mathcal{E}} |D_\sigma| F_{K,\sigma}\big(\varphi(x_K)h_n(u_K) - \varphi(x_L)h_n(u_L)\big) \\ &= \int_\Omega F\cdot\nabla_{\mathcal{T}}\big(\varphi_{\mathcal{T}}(x_{\mathcal{T}})\,h_n(u_{\mathcal{T}})\big)\,dx. \end{aligned} \tag{28}$$

To pass to the limit as $h_{\mathcal{T}} \to 0$ in (28), the following lemma is useful.

Lemma 4.2. Let $\mathcal{T}$ be an admissible mesh, $\varphi \in C_c^\infty(\Omega)$ and h_n the function defined by (26).. if $u_{\mathcal{T}} = (u_K)_{K\in\mathcal{T}}$ is a solution to (10), then there exists $C > 0$ only depending on (Ω, v, μ, n, d) such that

$$\|\varphi_{\mathcal{T}}(x_{\mathcal{T}})\,h_n(u_{\mathcal{T}})\|_{1,2,\mathcal{T}} \le C, \qquad \forall n > 0. \tag{29}$$

According to Lemma 4.2, $\nabla_{\mathcal{T}}\big(\varphi_{\mathcal{T}}(x_{\mathcal{T}})\,h_n(u_{\mathcal{T}})\big)$ converges to $\nabla(\varphi(x)\,h_n(u))$ weakly in $H_0^1(\Omega)$, as $h_{\mathcal{T}} \to 0$. Therefore, passing to the limit in (28) gives

$$\begin{aligned} V_5 &= \int_\Omega F\cdot\nabla_{\mathcal{T}}\big(\varphi_{\mathcal{T}}(x_{\mathcal{T}})\,h_n(u_{\mathcal{T}})\big)\,dx \\ &\underset{h_{\mathcal{T}}\to 0}{\longrightarrow} \int_\Omega F\cdot\nabla(\varphi(x)\,h_n(u))\,dx. \end{aligned} \tag{30}$$

In view of the definition of $b_{\mathcal{T}}$, and since b belongs to $L^1(\Omega)$, $b_{\mathcal{T}} = (b_K)_{K\in\mathcal{T}}$ converges to b in $L^1(\Omega)$ as $h_{\mathcal{T}} \to 0$. With already used arguments we can assert that

$$\begin{aligned} V_3 &= \int_\Omega b_{\mathcal{T}}T_{2n}(u_{\mathcal{T}})\varphi_{\mathcal{T}}h_n(u_{\mathcal{T}})dx \\ &\underset{h_{\mathcal{T}}\to 0}{\longrightarrow} \int_\Omega bT_{2n}(u)\,\varphi\,h_n(u)dx. \end{aligned} \tag{31}$$

We now study the convergence of the diffusion term. We write

$$\begin{aligned} V_1 &= \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma}(u_K - u_L)(\varphi(x_K)h_n(u_K) - \varphi(x_L)h_n(u_L) \\ &= V_{1,1} + V_{1,2} \end{aligned}$$

with

$$\begin{aligned} V_{1,1} &= \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma}h_n(u_K)(u_K - u_L)(\varphi(x_K) - \varphi(x_L)), \\ V_{1,2} &= \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma}\varphi(x_L)(u_K - u_L)(h_n(u_K) - h_n(u_L)). \end{aligned}$$

Using the same techniques as S. Leclavier [14], it follows that

$$\lim_{n\to+\infty}\overline{\lim_{h_{\mathcal{T}}\to 0}} V_{1,2} = 0 \tag{32}$$

and

$$\lim_{h_{\mathcal{T}}\to 0} V_{1,1} = \int_\Omega h_n(u)\nabla T_{4n}(u).\nabla\varphi dx. \tag{33}$$

For the convection term we have

$$\begin{aligned} V_2 = &\sum_{\sigma\in\mathcal{E}} |\sigma| v_{K,\sigma}u_{\sigma,+}\big(\varphi(x_K)h_n(u_K) - \varphi(x_L)h_n(u_L)\big) \\ = &\sum_{\sigma\in\mathcal{E},\ v_{K,\sigma}\ge 0} |\sigma| v_{K,\sigma}u_{\sigma,+}\big(\varphi(x_K)h_n(u_{\sigma,+}) - \varphi(x_L)h_n(u_{\sigma,-})\big) \\ &+ \sum_{\sigma\in\mathcal{E},\ v_{K,\sigma}< 0} |\sigma| v_{K,\sigma}u_{\sigma,+}\big(\varphi(x_K)h_n(u_{\sigma,+}) - \varphi(x_L)h_n(u_{\sigma,-})\big) \\ = &\sum_{\sigma\in\mathcal{E},\ v_{K,\sigma}\ge 0} |\sigma| v_{K,\sigma}u_{\sigma,+}h_n(u_{\sigma,+})(\varphi(x_K) - \varphi(x_L)) \\ &+ \sum_{\sigma\in\mathcal{E},\ v_{K,\sigma}\ge 0} |\sigma| v_{K,\sigma}u_{\sigma,+}\varphi(x_L)(h_n(u_{\sigma,+}) - h_n(u_{\sigma,-})) \\ &- \sum_{\sigma\in\mathcal{E},\ v_{K,\sigma}< 0} |\sigma| v_{K,\sigma}u_{\sigma,+}h_n(u_{\sigma,+})(\varphi(x_L) - \varphi(x_K)) \\ &- \sum_{\sigma\in\mathcal{E},\ v_{K,\sigma}< 0} |\sigma| v_{K,\sigma}u_{\sigma,+}\varphi(x_K)(h_n(u_{\sigma,+}) - h_n(u_{\sigma,-})) \\ = &\ V_{2,1} + V_{2,2} + V_{2,3} \end{aligned}$$

with

$$\begin{aligned}
V_{2,1} &= \sum_{\sigma\in\mathcal{E}} |\sigma| v_{K,\sigma} u_{\sigma,+} h_n(u_{\sigma,+})(\varphi(x_K) - \varphi(x_L)),\\
V_{2,2} &= \sum_{\sigma\in\mathcal{E},\ v_{K,\sigma}\geq 0} |\sigma| v_{K,\sigma} u_{\sigma,+} \varphi(x_L)(h_n(u_{\sigma,+}) - h_n(u_{\sigma,-})),\\
V_{2,3} &= -\sum_{\sigma\in\mathcal{E},\ v_{K,\sigma}< 0} |\sigma| v_{K,\sigma} u_{\sigma,+} \varphi(x_K)(h_n(u_{\sigma,+}) - h_n(u_{\sigma,-})).
\end{aligned}$$

Using the same techniques as S. Leclavier [14], it follows that

$$\lim_{n\to+\infty} \overline{\lim_{h_{\mathcal{T}}\to 0}} (V_{2,2} + V_{2,3}) = 0 \tag{34}$$

and

$$\lim_{h_{\mathcal{T}}\to 0} V_{2,1} = -\int_\Omega T_{2n}(u) h_n(u) v.\nabla\varphi dx. \tag{35}$$

We are now in position to pass to the limit as $h_{\mathcal{T}} \to 0$ in the scheme(10). Gathering equations (27) to (35), we can assert that

$$\begin{aligned}
&\int_\Omega h_n(u)\,\nabla u\cdot\nabla\varphi dx - \int_\Omega u\,h_n(u)\,v\cdot\nabla\varphi dx + \int_\Omega b\,u\,h_n(u)\,\varphi dx - \int_\Omega f\,\varphi\,h_n(u)dx\\
&\qquad + \int_\Omega F\cdot\nabla(\varphi(x)\,h_n(u))\,dx = \lim_{h_{\mathcal{T}}\to 0} T(n,\varphi),
\end{aligned} \tag{36}$$

where $\lim\limits_{h_{\mathcal{T}}\to 0} |T(n,\varphi)| \leq ||\varphi||_{L^\infty(\Omega)}\,\omega(n)$ with $\omega(n)\to 0$ as $n\to+\infty$.

Let $h \in C_c^1(\mathbb{R})$ and $\psi \in C_c^1(\Omega)\cap H_0^1(\Omega)$. In view of the regularity of $T_n(u)$(see (2)) the function $h(u)\psi$ belongs to $L^\infty(\Omega)\cap H_0^1(\Omega)$. By the density of $C_c^\infty(\Omega)$ in $L^\infty(\Omega)\cap H_0^1(\Omega)$ (here any element of $L^\infty(\Omega)\cap H_0^1(\Omega)$ can be approached by a sequence of $C_c^\infty(\Omega)$ which is bounded in $L^\infty(\Omega)$), from (36) we deduce that

$$\begin{aligned}
&\Bigg|\int_\Omega \nabla u\,h_n(u)\,h(u)\,\nabla\psi\,dx + \int_\Omega \nabla u\,h_n(u)\,\psi\,\nabla u\,h'(u)\,dx - \int_\Omega u\,h_n(u)\,h(u)\,v\cdot\nabla\psi\,dx - \int_\Omega u\,h_n(u)\,h'(u)\,\psi\,v\cdot\nabla u\,dx\\
&+\int_\Omega b\,u\,h_n(u)\,h(u)\,\psi\,dx - \int_\Omega \psi\,h(u)\,h_n(u)\,f\,dx + \int_\Omega F\cdot\nabla(\psi\,h(u)\,h_n(u))\,dx\Bigg| \leq \|\varphi\|_{L^\infty(\Omega)}\omega(n).
\end{aligned}$$

Passing to the limit as $n\to+\infty$ in the previous inequality yields that :

$$\begin{aligned}
&\int_\Omega \nabla u\,h(u)\,\nabla\psi\,dx + \int_\Omega \nabla u\,\psi\,\nabla u\,h'(u)\,dx - \int_\Omega u\,h(u)\,v\cdot\nabla\psi\,dx - \int_\Omega u\,h'(u)\,\psi\,v\cdot\nabla u\,dx\\
&\qquad + \int_\Omega b\,u\,h(u)\,\psi\,dx = \int_\Omega \psi\,h(u)\,d\mu,
\end{aligned}$$

which is Equality (4) in the definition of a renormalized solution. It remains to prove that u satisfies the decay (3) of the truncate energy.

Thanks to the discrete estimate on the energy (19) we get,

$$\lim_{n\to+\infty} \overline{\lim_{h_{\mathcal{T}}\to 0}} \frac{1}{n} \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma}(T_{2n}(u_K) - T_{2n}(u_L))^2 = 0$$

and

$$\sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma}(T_{2n}(u_K) - T_{2n}(u_L))^2$$

$$= \sum_{\sigma\in\mathcal{E}} |\sigma| d_\sigma \left(\frac{T_{2n}(u_K) - T_{2n}(u_L)}{d_\sigma}\right)^2$$

$$= \sum_{\sigma \in \mathcal{E}} d|D_\sigma| \left(\frac{T_{2n}(u_K) - T_{2n}(u_L)}{d_\sigma} \right)^2$$

$$= \frac{1}{d} \sum_{\sigma \in \mathcal{E}} |D_\sigma| \left(d \frac{T_{2n}(u_K) - T_{2n}(u_L)}{d_\sigma} \right)^2$$

$$= \frac{1}{d} \int_\Omega |\nabla_\mathcal{T} T_{2n}(u_\mathcal{T})|^2 dx,$$

hence, $\lim\limits_{n \to +\infty} \overline{\lim\limits_{h_\mathcal{T} \to 0}} \frac{1}{n} |\nabla_\mathcal{T} T_{2n}(u_\mathcal{T})|^2 = 0.$

Since $\nabla T_{2n}(u_\mathcal{T})$ converges weakly in $L^2(\Omega)^d$, we have also

$$\frac{1}{n} \int_\Omega |\nabla T_{2n}(u)|^2 \, dx \leq \liminf_{h_\mathcal{T} \to 0} \frac{1}{n} \int_\Omega |\nabla_\mathcal{T} T_{2n}(u_\mathcal{T})|^2 dx,$$

which leads to

$$\lim_{n \to 0} \frac{1}{n} \int_\Omega |\nabla T_{2n}(u)|^2 \, dx = 0.$$

Since the renormalized solution u is unique, we conclude that the whole sequence $u_{\mathcal{T}_m}$ converges to u in the sense that for all $n > 0$, $T_n(u_{\mathcal{T}_m})$ converges weakly to $T_n(u)$ in $H_0^1(\Omega)$.

5. Conclusion

In this paper, the finite volumes method has been used to prove that the approximate solution converges to the renormalized solution of elliptic problems with measure data. A first difficulty is to establish a discrete version of the estimate on the energy (3). Moreover it is worth noting that in (4) all the terms are "truncated" while a discrete version of $\varphi h(u)$ in the finite volume scheme leads to some residual terms which are not "truncated". The second difficulty is to deal with the diffuse measure data. Firstly, we presented the finite volume scheme and the properties of the discrete gradient. Secondly, we are proven several estimates, especially the discrete equivalent to (4) which is crucial to pass to the limit in the finite volume scheme. At last, we established the convergence of the cell-centered finite volume scheme via a density argument.

References

[1] M. Ben Cheick and O. Guibé, *Nonlinear and non-coercive elliptic problems with integrable data*. Adv. Math. Sci. Appl. 16 $n°1$, 275-297, 2006.

[2] L. Boccardo and T. Gallouët, *On some nonlinear elliptic and parabolic equations involving measure data*. J. Funct. Anal., 87, 149-169, 1989.

[3] M. Botti, D. A. Di Pietro, O. Le Maître, P. Sochala. *Numerical Approximation of Poroelasticity with Random Coefficients Using Polynomial Chaos and High-Order Polyhedral Methods*. Comput. Meth. Appl. Mech. Eng. 361 (2020). https://dx.doi.org/10.1016/j.cma.2019.112736.

[4] M. Botti, D. A. Di Pietro, P. Sochala, *A Hybrid High-Order discretization method for nonlinear poroelasticity*. Comput. Meth. Appl. Math. (2019). https://doi.org/10.1515/cmam-2018-0142.

[5] J. Carrillo and M. Chipot, *On some nonlinear elliptic equations involving derivatives of the nonlinearity*. Proc. Roy. Soc. Edinburgh Sect $A100$, $n°3-4$, 281-294, 1985.

[6] D. Castanon Quiroz, D. A. Di Pietro. *A Hybrid High-Order Method for the Incompressible Navier-Stokes Problem Robust for Large Irrotational Body Forces* (2020). https://dx.doi.org/10.1016/j.camwa.2019.12.005.

[7] C. Chainais-Hillairet and J. Droniou, *Finite volume schemes for noncoercive elliptic problems with Neumann boundary conditions*. IMA J. Numer. Anal. 31, $n°1$, 61-85, 2011.

[8] G. Dal Maso, F. Murat, L. Orsina and A. Prignet, *Renormalized solutions of elliptic equations with general measure data*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 28(4), 741-808, 1999.

[9] D. A. Di Pietro, J. Droniou, *The Hybrid High-Order method for polytopal meshes*, Modeling, Simulation and Application, vol. 19, Springer International Publishing, 2020.

[10] J. Droniou, T. Gallouët and R. Herbin, *A finite volume scheme for a noncoercive elliptic equation with measure data*. SIAM J. Numer. Anal., 6, 1997-2031, 2003.

[11] R. Eymard, T. Gallouët and R. Herbin, *Convergence of finite volume approximations to the solutions of semilinear convection diffusion reaction equations*. Numer. Math., 82, 91-116 (1999).

[12] O. Guibé and A. Mercaldo, *Uniqueness results for noncoercive nonlinear elliptic equations with two lower order terms*. Commun. Pure Appl. Anal. 7, $n^{\circ}1$, 163-192, 2008.

[13] A. Larcher and J. C. Latché, Convergence analysis of a finite element-finite volume scheme for a RANS turbulence model. Institut de Radioprotection et de Sureté Nucléaire (IRSN), France, 2012.

[14] S. Leclavier, *Finite volume scheme and renormalized solutions for a noncoercive elliptic problem with L^1-data*, Comput. Methods Appl. Math. 17, No. 1, 85-104 (2017).

[15] F. Murat, Soluciones renormalizadas de EDP elipticas no lineales. Technical Report R93023, Laboratoire d'Analyse Numérique, Paris VI, 1993.

[16] A. Ouédraogo, T. Valea and W. B. Yaméogo, *Renormalized solutions for a non-coercive elliptic problem with measure data*. Submitted.