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A Study of Some Generalizations of Local Homology

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Abstract: Tate local cohomology and Gorenstein local cohomology theory, which are important generalizations of the classical local cohomology, has been investigated. It has been found that they have such vanishing properties and long exact sequences. However, for local homology, what about the duality? In this paper we are concerned with Tate local homology and Gorenstein local homology. In the first part of the paper we generalize local homology as Tate local homology, and study such vanishing properties, artinianness and some exact sequence of Tate local homology modules. We find that for an artianian R-module M and a finitely generated R-module N with finite Gorenstein projective dimension, the Tate local homology module of M and N with respect to an ideal I is also an artinian module. In the second part of the paper we consider Gorenstein local homology modules as Gorenstein version. We discuss vanishing properties and some exact sequences of Gorenstein local homology modules and obtain an exact sequence connecting Gorenstein, Tate and generalized local homology. Finally, as an applicaton of the exact sequence connecting these local homology modules, we find that for finitely generated R-modules with finite projective dimension and admitting Gorenstein projective proper resolution respectively, Gorenstein local homology coincides with generalized local homology in certain cases.

Keywords: Tate Homology, Local Homology, Generalized Local Homology, Artinian Module

1. Introduction

Local cohomology functors were introduced by Grothendieck [11] in local algebra originally. Let I be an ideal of R. It is defined as the right derived functors of the torsion functor $\Gamma_I(-)$ which is naturally equivalent to the functor $\varinjlim_t \mathrm{Hom}_R(R/I^n, -)$. It is denoted by $\mathrm{H}_I^i(-)$ [3]. The local homology functor were first studied by Matlis [17, 18] for ideal I generated by a regular sequence. Then the work of Greenlees and May [10], and Tarrio and coauthers [22] show the strong connection between local homology and local cohomlogy. In detail, consider the I-adic completion functor $\Lambda^I(-) = \varprojlim_t (R/I^n \otimes_R -)$. Left deriving $\Lambda^I(-)$ we can obtain the so-called local homology functors denoted $\mathrm{H}_i^I(-)$. Later, Herzog and Zamani [13] introduced the definition of the generalized local cohomology which is an extension of local cohomology of Grothendieck. Let M, N be R-modules, then

$$\mathrm{H}_I^i(M, N) = \varinjlim_t \mathrm{Ext}_i^R(M/I^t M, N)$$

is called the i-th generalized local cohomology module of M, N with respect to I. Also the generalized homology module of M, N with respect to I

$$\mathrm{H}_i^I(M, N) = \varprojlim_t \mathrm{Tor}_R^i(M/I^t M, N)$$

was investigated as a dual of the generalized local cohomology module of M, N with respect to I [19]. Some properties of them such as vanishing properties, artianness, noetherianness are investigated, see [9, 19]. Recently, the local cohomology and local duality to Notherian connected cochain DG algebras are investigated and they found that the functor can be used to detect the Gorensteinness of a homologically smooth DG algebra [16].

Avramov and Martsinkovsky [2] studied the Tate cohmology theory in the subcategory of modules of finite G-dimension and study the interaction of the absolute, the relative and the Tate cohomology theories. Parallel to the theory of Tate cohomology, they also noted Tate homology. That is for a right R-module M admitting a complete projective resolution $\mathbf{T} \to \mathbf{P} \to M$, define $\widehat{\operatorname{Tor}}_i^R(M, N) = \mathrm{H}_i(\mathbf{T} \otimes_R N)$ for each R-module N and each $n \in \mathbb{Z}$. Later, Asadollahi and Salarian [1] made an intensive study of the relative and Tate cohomology of modules of finite Gorenstein injective dimension. Let N be an R-module with finite Gorenstein projective dimension. The Tate torsion functors are defined by means of a complete projective dimension of N: if $\mathbf{V} \to \mathbf{P} \to N$ is a complete projective resolution for each $i \in \mathbb{Z}$, let $\overline{\operatorname{Tor}}_i^R(M, N) = \mathrm{H}_i(M \otimes_R \mathbf{V})$[15]. The balance of Tate homology was discussed. Christensen and Jorgensen [5, 6] extended Tate homology to complexes. They established a depth formula that holds for every pair of Tate Tor-independent modules over a Gorenstein local ring which subsumes previous generalizations of Auslander's formula.

Furthermore, Asadollahi and Salarian [1] presented a theory of Gorenstein local cohomology theory, using the Gorenstein injective version of the relative cohomology. This is a natural generalization of a notion introduced in the article [13] which generalizes the classical local cohomology. They also introduced the Tate local cohomology and studied properties of these cohomology theories. We pay attention to the dual version of these cohomology theories-Tate local homology and Gorenstein local homology. We will study some properties of them and show that these two variations of local homology are tightly connected to the generalized local homology modules introduced by Nam [19].

2. Preliminaries and Basic Facts

Throughout this paper, R is a noetherian commutative ring with non-zero identity. We first review some basic facts. For terminology we follow [2, 7].

Gorenstein projective module. [7] An R-module M is said to be Gorenstein projective if there is a $\operatorname{Hom}_R(-, Proj)$ exact exact sequence

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of projective modules such that $M = \operatorname{Ker}(P^0 \to P^1)$. The class of all Gorenstein projective modules is denoted by $\mathcal{GP}$.

Complete projective resolution.[2] A complex $\mathbf{T}$ is acyclic if it is exact, with modules of T is projective, and $\operatorname{Hom}_R(\mathbf{T}, Q)$ is exact for any projective R-module Q. A complete projective resolution of R-module M is a diagram $\mathbf{T} \xrightarrow{\tau} \mathbf{P} \xrightarrow{\pi} M$, where $\mathbf{T}$ is a totally acyclic complex of projective R-modules, π is a projective resolution of M, τ is a morphism of complexes and τ_n is bijective for $n \gg 0$.

Gorenstein projective dimension.[7] A resolution $\mathbf{P} \to M$ is called a $\mathcal{GP}$-resolution if P_i belongs to $\mathcal{GP}$ for all $i \in \mathbb{Z}$. A module M has finite Gorenstein projective dimension if there is a Gorenstein projective resolution of M of the form

$$0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0.$$

If n is the least with this property then we set $\operatorname{Gpd}_R M = n$.

A complex $\mathbf{C}$ is called $\mathcal{GP}$-exact if the induced complexe $\operatorname{Hom}_R(G, \mathbf{C})$ is exact for all $G \in \mathcal{GP}$. A $\mathcal{GP}$-proper resolution $\mathbf{P} \to M$ is a $\mathcal{GP}$-resolution such that $\mathbf{P}$ is $\mathcal{GP}$-proper exact. Let $\overline{\mathcal{GP}}$ denote the full subcategory of the category of R-modules whose objects are the modules admitting some $\mathcal{GP}$-proper resolution. Every module of finite Gorenstein projective dimension has a $\mathcal{GP}$-proper resolution.

3. Tate Local Homology

Definition 3.1. Let I be an ideal of the ring R and N be an R-module with $\operatorname{Gpd}_R N < \infty$. Consider the complete projective resolution $\mathbf{T} \to \mathbf{P} \to N$ of N. For any $i \in \mathbb{Z}$, the i-th Tate local homology module of M and N with respect to I is defined by the formula

$$\varprojlim_t \overline{\operatorname{Tor}}_i^R(M/I^tM, N) = \varprojlim_t \mathrm{H}_i(M/I^tM \otimes \mathbf{T})$$

where M is an arbitrary R-module. These modules will be denoted $\widehat{\mathrm{H}}_i^I(M, N)$.

Proposition 3.1. Let M be an R-module. Then for any R-module N of finite projective dimension, $\widehat{\mathrm{H}}_i^I(M, N) = 0$ for all $i > 0$.

Proof If $\operatorname{pd}_R N < \infty$, then a complete projective resolution of N is $\mathbf{0} \to \mathbf{P} \to N$, where $\mathbf{P}$ is bounded projective resolution of N. So $\widehat{\mathrm{H}}_i^I(M, N) = 0$ for all $i > 0$.

In general, the generalized I-adic completion functor $\Lambda_I(M, -) = \varprojlim_t(M/I^tM \otimes -)$ is neither left nor right exact. Nam [21] introduced the category of R-modules, denoted by $\mathcal{C}_{\Lambda_I}(R)$, such that $L_0\Lambda_I(M, N) \cong \Lambda_I(M, N)$ for all R-modules N, equivalently, the functor $\Lambda_I(M, -)$ is right exact. [21, Theorem2.3] shows that artinian modules are in $\mathcal{C}_{\Lambda_I}(R)$. Consider module M and N as definition 3.1. Apply the right exact functor $\Lambda_I(M, -)$ to the complex $\mathbf{T}$ and take its ith homology module. In fact, for all artinian module M, this homology coincides with $\widehat{\mathrm{H}}_i^I(M, N)$ in certain cases.

Proposition 3.2. Let M be an artinian module and N an finitely generated R-module with $\operatorname{Gpd}_R N < \infty$. Then $\widehat{\mathrm{H}}_i^I(M, N) = \mathrm{H}_i(\varprojlim_t(M/I^tM \otimes \mathbf{T}))$.

Proof Since N is a finitely generated R-module with $\operatorname{Gpd}_R N < \infty$, then we can take the complete projective resolution $\mathbf{T} \to \mathbf{P} \to N$ of N such that each P_i and T_i are finitely generated free. Hence $M/I^tM \otimes \mathbf{T}$ is degreewise artinian for all $t > 0$. It should be noted that the inverse limit $\varprojlim_t$ is exact on artinian R-modules by [12, 9.1]. Therefore it commutes with homology functor H_i and the proof is complete.

Theorem 3.1. Let N be a finitely generated R-module of finite Gorenstein projective dimension. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules. If M is artianian, then there exists a long exact sequence of Tate local homology groups

$$\cdots \to \widehat{\mathrm{H}}_{i+1}^I(M', N) \to \widehat{\mathrm{H}}_{i+1}^I(M, N) \to$$

$$\widehat{\mathrm{H}}^I_{i+1}(M'',N) \to \widehat{\mathrm{H}}^I_i(M',N) \to \cdots .$$

Proof By assumption, there exists a complete projective resolution $\mathbf{T} \to \mathbf{P} \to N$ of N such that each T_i is finitely generated free. So the functor $-\otimes T_i$ is exact. Hence we obtain an exact sequence of complexes

$$0 \to M'/I^tM' \otimes \mathbf{T} \to M/I^tM \otimes \mathbf{T} \to M''/I^tM'' \otimes \mathbf{T} \to 0.$$

By [7, Proposition 2.3.7], M' and M'' are artinian. Hence $M'/I^tM' \otimes \mathbf{T}$,$M'/I^tM \otimes \mathbf{T}$ and $M'/I^tM'' \otimes \mathbf{T}$ are complexes of artianian modules for all $t > 0$. It should be noted that the inverse limit $\varprojlim\limits_t$ is exact on artinian R-modules by [12, 9.1]. We get the following exact sequence of complexes

$$0 \to \varprojlim_t (M'/I^tM' \otimes \mathbf{T}) \to \varprojlim_t (M/I^tM \otimes \mathbf{T})$$
$$\to \varprojlim_t (M''/I^tM'' \otimes \mathbf{T}) \to 0.$$

It induces a long exact sequence of homology groups as desired by Proposition 3.2.

Theorem 3.2. Let $0 \to N' \to N \to N'' \to 0$ be finitely generated R-modules of finite Gorenstein projective dimension. Then for every artinian module M, there is an exact sequence of R-modules

$$\cdots \to \widehat{\mathrm{H}}^I_{i+1}(M,N') \to \widehat{\mathrm{H}}^I_{i+1}(M,N) \to \widehat{\mathrm{H}}^I_{i+1}(M,N'')$$
$$\to \widehat{\mathrm{H}}^I_i(M,N') \to \cdots .$$

Proof By [2, Lemma 5.5], there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{T}' \arrow[r] \arrow[d] & \mathbf{T} \arrow[r] \arrow[d] & \mathbf{T}'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbf{P}' \arrow[r] \arrow[d] & \mathbf{P} \arrow[r] \arrow[d] & \mathbf{P}'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N'' \arrow[r] & 0
\end{tikzcd}$$

with exact rows such that the columns are complete projective resolutions of N', N and N'' respectively. The sequence $0 \to \mathbf{T}' \to \mathbf{T} \to \mathbf{T}'' \to 0$ is degreewise split. So the sequence

$$0 \to M/I^tM \otimes \mathbf{T}' \to M/I^tM \otimes \mathbf{T} \to M/I^tM \otimes \mathbf{T}'' \to 0$$

is exact for all $t > 0$. Hence we get

$$0 \to \varprojlim_t (M/I^tM \otimes \mathbf{T}') \to \varprojlim_t (M/I^tM \otimes \mathbf{T})$$
$$\to \varprojlim_t (M/I^tM \otimes \mathbf{T}'') \to 0.$$

Therefore we obtain the long exact sequence of Tate local homology modules. The proof is complete.

Now we list some properties of Tate local homology.

Proposition 3.3. Let M be an artianian R-module.

(1) If N is a finitely generated R-module of finite Gorenstein projective dimension, then $\widehat{\mathrm{H}}^I_i(M,N)$ is an artinian module for all $i \geq 0$.

(2) If N is a finitely generated R-module of finite Gorenstein projective dimension over a local ring $(R,\mathfrak{m})$, then $\widehat{\mathrm{H}}^{\mathfrak{m}}_i(M,N)$ has finite length for all $i \geq 0$.

Proof (1) As M is artianian, there is a positive integer n such that $I^tM = I^nM$ for all $t \geq n$. Then $\varprojlim\limits_t \overline{\mathrm{Tor}}^R_i(M/I^tM,N) \cong \overline{\mathrm{Tor}}^R_i(M/I^nM,N)$. Since N is a finitely generated R-module of finite Gorenstein projective dimension, there is a complete projective resolution $\mathbf{T} \to \mathbf{P} \to N$ of N in which each P_i and T_i are finitely generated free. Hence $\overline{\mathrm{Tor}}^R_i(M/I^nM,N) = \mathrm{H}_i(M/I^nM \otimes \mathbf{T})$ is an artianian module of all $i \geq 0$.

(2) As M is artianian, there is a positive integer r such that $\mathfrak{m}^tM = \mathfrak{m}^rM$ for all $t \geq r$. Then $\widehat{\mathrm{H}}^{\mathfrak{m}}_i(M,N) \cong \overline{\mathrm{Tor}}^R_i(M/\mathfrak{m}^rM,N) = \mathrm{H}_i(M/\mathfrak{m}^rM \otimes \mathbf{T})$. As $M/\mathfrak{m}^rM$ has finite length and N is finitely generated, $\overline{\mathrm{Tor}}^R_i(M/\mathfrak{m}^rM,N) = \mathrm{H}_i(M/\mathfrak{m}^rM \otimes \mathbf{T})$ is finitely generated. Moreover, we have $\mathfrak{m}^r\mathrm{H}_i(M/\mathfrak{m}^rM \otimes \mathbf{T}) = 0$. Hence $\widehat{\mathrm{H}}^{\mathfrak{m}}_i(M,N)$ has finite length for all $i \geq 0$.

Proposition 3.4. Let M be an R-module and N be an R-module of finite Gorenstein projective dimension. Then Tate local homology module $\widehat{\mathrm{H}}^I_i(M,N)$ is I-separated for $i \geq 0$, i.e., $\bigcap\limits_{t>0} I^t\widehat{\mathrm{H}}^I_i(M,N) = 0$.

Proof Since $I^s\overline{\mathrm{Tor}}^R_i(M/I^tM,N) = I^s\mathrm{H}_i(M/I^tM \otimes \mathbf{T}) = 0$ for all $s \geq t$, then

$$\begin{aligned}
\bigcap_{s>0} I^s\widehat{\mathrm{H}}^I_i(M,N) &\cong \varprojlim_{s>0} I^s \varprojlim_{t>0} \overline{\mathrm{Tor}}^R_i(M/I^tM,N) \\
&\subseteq \varprojlim_{s>0}\varprojlim_{t>0} I^s\overline{\mathrm{Tor}}^R_i(M/I^tM,N) \\
&\cong \varprojlim_{t>0}\varprojlim_{s>0} I^s\overline{\mathrm{Tor}}^R_i(M/I^tM,N) \\
&= 0.
\end{aligned}$$

4. Gorenstein Local Homology Modules

The main purpose of this part is to introduce and study Gorenstein local homology.

Definition 4.1. Let M be an R-module in $\mathcal{C}_{\Lambda_I}(R)$. Choose $N \in \mathcal{GP}$ and take a $\mathcal{GP}$-proper resolution $\mathbf{G} \to N$. Applying the right exact functor $\Lambda_I(M,-)$ on it, take the i-th homology module of the later complex; the result

$$\frac{\mathrm{Ker}(\varprojlim\limits_t (M/I^tM \otimes \partial_i))}{\mathrm{Im}(\varprojlim\limits_t (M/I^tM \otimes \partial_{i+1}))},$$

which is independent (up to isomorphism) of the choice of $\mathcal{GP}$-proper resolution of N by standard facts of homological algebra, will be denoted by $GH^I_i(M,N)$ and will be called the ith Gorenstein local homology module of M and N with respect to I.

Remark 4.1. Since $\Lambda_I(M,-)$ is right exact, $GH_0^I(M,N)$ is naturally equivalent to $\Lambda_I(M,N)$. Especially, if M is an artinian module, then as the proof of Proposition 3.2, for any $i\in\mathbb{Z}$, there is an isomorphism

$$GH_i^I(M,N)\cong\varprojlim_t \operatorname{Tor}_i^{\mathcal{GP}}(M/I^tM,N),$$

where $\operatorname{Tor}_i^{\mathcal{GP}}(M/I^tM,N)$ is the Gorenstein torsion functor computed by the $\mathcal{GP}$-proper resolution of N, that is, $\operatorname{Tor}_i^{\mathcal{GP}}(M/I^tM,N)=\mathrm{H}_i(M/I^tM\otimes_R\mathbf{G})$.

Lemma 4.1. ([8]) Let M be a Gorenstein flat R-module. Then for any ideal I of R, M is Λ^I-acyclic, that is, $\mathrm{H}_i^I(M)=0$.

Proof There is an exact sequence

$$\cdots\to F_1\to F_0\to F_{-1}\to F_{-2}\to\cdots$$

of flat modules such that $M\cong\operatorname{Im}(F_0\to F_{-1})$. Set $L_0:=M$ and $L_i:=\operatorname{Im}(F_i\to F_{i-1})$ for $i<0$. Then for each $i<0$, the exact sequence

$$0\to L_{i+1}\to F_i\to L_i\to 0,$$

yields the following long exact sequence of local homology modules

$$\cdots\to\mathrm{H}_{j+1}^I(F_i)\to\mathrm{H}_{j+1}^I(L_i)\to\mathrm{H}_j^I(L_{i+1})\to\mathrm{H}_j^I(F_i)\to\cdots.$$

Note that $\mathrm{H}_i^I(F)=0$ for any flat R-module F by [17, Corollary 4.5]. Hence $\mathrm{H}_{j+1}^I(L_i)\cong\mathrm{H}_j^I(L_{i+1})$ for all $i<0$ and all $j\geq 1$. Let $n:=\operatorname{cd}_I(R)$. Then by [8, Lemma 2.1], we have

$$\mathrm{H}_j^I(M)\cong\mathrm{H}_{j+1}^I(L_{i-1})\cong\cdots\cong\mathrm{H}_{j+n}^I(L_{-n})=0,$$

for all $j>0$.

Remark 4.2. (1) It is easy to see that a resolution of Λ^I-acyclic R-modules can be used to computed local homology. So if every flat R-module has finite projective dimension, then by [14, Proposition 3.4], every Gorenstein projective R-module also is Gorenstein flat. Hence by Lemma 4.1 one can compute local homology modules using a $\mathcal{GP}$-proper resolution $\mathbf{G}\to N$ of N, where each G_i is Gorenstein flat. That is $GH_i^I(R,N)=H_i^I(N)$. Hence our definition is in fact a generalization of usual local homology functor.

(2) There is a generalization of local homology given by Nam [19]. For each $i\geq 0$, define $\mathrm{H}_i^I(M,N)=\varprojlim_t\operatorname{Tor}_i^R(M/I^tM,N)$, where M and N are R-modules.It is clear that $\mathrm{H}_i^I(R,-)$ is naturally equivalent to the functor $\mathrm{H}_i^I(-)$. If every flat R-module has finite projective dimension and N is a finitely generated module with finite projective dimension, then every projective resolution of N is a $\mathcal{GP}$-proper resolution. So in this case, $GH_i^I(M,N)=H_i^I(M,N)$ for any R-module M.

Proposition 4.1. Let $0\to N'\to N\to N''\to 0$ be a $\mathcal{GP}$-proper exact sequence of R-modules admitting some $\mathcal{GP}$-proper resolutions. Then for any R-module M in $\mathcal{C}_{\Lambda_I}(R)$, there is a long exact sequence of Gorenstein local homology modules

$$\cdots\to\mathrm{GH}_1^I(M,N)\to\mathrm{GH}_1^I(M,N'')\to$$

$$\mathrm{GH}_0^I(M,N')\to\mathrm{GH}_0^I(M,N)\to\mathrm{GH}_0^I(M,N'')\to 0.$$

Proof Since the functor $\Lambda_I(M,-)$ is a covariant and right exact functor, the result follows from [7, Theorem 8.2.3].

The following theorem provides a tight connection between Gorenstein, Tate and generalized local homology.

Theorem 4.1. Let N be a finitely generated R-module with $\operatorname{Gpd}_R N=d<\infty$. For each artinian R-module M there is an exact sequence

$$0\to\widehat{\mathrm{H}}_d^I(M,N)\to\mathrm{H}_d^I(M,N)\to\mathrm{GH}_d^I(M,N)\to\cdots$$

$$\to\mathrm{GH}_2^I(M,N)\to$$

$$\widehat{\mathrm{H}}_1^I(M,N)\to\mathrm{H}_1^I(M,N)\to\mathrm{GH}_1^I(M,N)\to 0.$$

Proof By [2, Construction 3.8], for R-module N there is a degreewise split-exact sequence of complexes of R-modules

$$0\to\Sigma^{-1}\mathbf{G}\to\mathbf{T}^\flat\to\mathbf{P}\to 0.$$

So it induces an exact sequence

$$0\to M/I^tM\otimes\Sigma^{-1}\mathbf{G}\to M/I^tM\otimes\mathbf{T}^\flat\to M/I^tM\otimes\mathbf{P}\to 0$$

of complexes. Its homology exact sequence has the form

$$\cdots\to\mathrm{H}_{i+1}(M/I^tM\otimes\Sigma^{-1}\mathbf{G})\to\mathrm{H}_{i+1}(M/I^tM\otimes\mathbf{T}^\flat)\to$$

$$\mathrm{H}_{i+1}(M/I^tM\otimes\mathbf{P})\to\mathrm{H}_i(M/I^tM\otimes\Sigma^{-1}\mathbf{G})\to\cdots.$$

Since $\mathbf{P}\to N$ is a projective resolution, we have

$$\mathrm{H}_{i+1}(M/I^tM\otimes\mathbf{P})=\operatorname{Tor}_{i+1}^R(M/I^tM,N)$$

for $i\in\mathbb{Z}$.

The right exactness of $M/I^tM\otimes_R-$ yields

$$\mathrm{H}_i(M/I^tM\otimes\mathbf{T}^\flat)=0$$

for $i\leq 0$ and

$$\mathrm{H}_{i+1}(M/I^tM\otimes\mathbf{T}^\flat)=\overline{\operatorname{Tor}}_{i+1}^R(M/I^tM,N).$$

As $\mathbf{G}\to N$ is a proper resolution, we have

$$\mathrm{H}_{i+1}(M/I^tM\otimes\Sigma^{-1}\mathbf{G})=\mathrm{H}_{i+2}(M/I^tM\otimes\mathbf{G})$$

$$=\operatorname{Tor}_{i+2}^{\mathcal{GP}}(M/I^tM,N).$$

By assumption, $M/I^tM\otimes\Sigma^{-1}\mathbf{G}$, $M/I^tM\otimes\mathbf{T}^\flat$ and $M/I^tM\otimes\mathbf{P}$ are degreewise artinian. Hence the above exact sequence of homology groups is degreewise artinian. Then pass it to the inverse limit.

Corollary 4.1. Let N be a finitely generated R-module with $\mathrm{pd}_R N = d < \infty$. Then for each artinian R-module M, $\mathrm{H}_i^I(M,N) \cong \mathrm{GH}_i^I(M,N)$ for $i \leq d$. In particular, for $i > d$,

$$\mathrm{H}_i^I(M,N) = \mathrm{GH}_i^I(M,N) = \widehat{\mathrm{H}}_i^I(M,N) = 0.$$

Proof It is obvious by Theorem 4.1 and Proposition 3.1.

From now on R is an artianian ring.

Lemma 4.2. Let M and N be finitely generated R-modules such that $\mathrm{pd}_R M < \infty$ and N be Gorenstein flat. Then for any $i \in \mathbb{Z}$, $\mathrm{H}_i^I(M,N) = 0$.

Proof Since N is Gorenstein flat, for any $i \geq 0$, there exists exact sequence

$$0 \to L_{i+1} \to F_i \to L_i \to 0,$$

where the middle terms are all flat and $L_{-1} = N$. Note that $\mathrm{H}_j^I(M,F) = \varprojlim_t \mathrm{Tor}_j^R(M/I^tM,F) = 0$ for any flat R-module F and $j > 0$. Hence by [21, Corollary 3.7], for any $j > 0$,

$$\mathrm{H}_j^I(M,N) \cong \mathrm{H}_{j+1}^I(M,L_{i-1}) \cong \cdots \cong \mathrm{H}_{j+n}^I(M,L_{-n}) = 0.$$

By [20, Corollary 2.2] the generalized local homology modules vanishes after $ara(\widehat{I}) + \mathrm{pd}_R M$. So the result follows.

Theorem 4.2. Let M and N be finitely generated R-modules such that $\mathrm{pd}_R M < \infty$ and $N \in \overline{\mathcal{GP}}$. If every flat R-module has finite projective dimension, then for any $i \in \mathbb{Z}$,

$$\mathrm{GH}_i^I(M,N) \cong \mathrm{H}_i^I(M,N).$$

Proof There exists a $\mathcal{GP}$-proper exact sequence of R-modules

$$0 \to L \to P \to N \to 0,$$

where P is Gorenstein projective. Hence Gorenstein flat and $L \in \overline{\mathcal{GP}}$. So by Lemma 4.2, we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{GH}_1^I(M,N) \arrow[r] \arrow[d] & \mathrm{GH}_0^I(M,L) \arrow[r] \arrow[d] & \mathrm{GH}_0^I(M,P) \arrow[r] \arrow[d] & \mathrm{GH}_0^I(M,N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{H}_1^I(M,N) \arrow[r] & \mathrm{H}_0^I(M,L) \arrow[r] & \mathrm{H}_0^I(M,P) \arrow[r] & \mathrm{H}_0^I(M,N) \arrow[r] & 0,
\end{tikzcd}$$

in which the upper exact row follows from Proposition 4.1, the lower exact row follows from [21, Corollary3.7]. For each $i > 0$, $\mathrm{GH}_i^I(M,L) \cong \mathrm{GH}_{i+1}^I(M,N)$ and $\mathrm{H}_i^I(M,L) \cong \mathrm{H}_{i+1}^I(M,N)$. Since the last three vertical maps are isomorphisms by Remark 4.1, so is the first. Since $L \in \overline{\mathcal{GP}}$, the same arguments imply by the isomorphism $\mathrm{GH}_1^I(M,L) \cong \mathrm{H}_1^I(M,L)$. Now use the induction.

5. Conclusion

In this paper we mainly study some generalizations of local homology as the duality of local cohomology. Firstly, Tate local homology is introduced. Such vanishing properties, artinianness and some exact sequence of Tate local homology modules are obtained. Then we consider Gorenstein local homology modules as Gorenstein version. We present vanishing properties and some exact sequences of Gorenstein local homology models and obtain an exact sequence connecting Gorenstein, Tate and generalized local homology. As an application of the exact sequence, we obtain when Gorenstein local homology coincides with generalized local homology. However the vanishing of Tate homology is a sufficient condition implying the depth formula to hold for some modules[4], we may further give a new sufficient condition implying the depth formula to hold for certain modules by vanishing of Tate local homology introduced here.

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