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Approaching by D_X-Schemes and Jets to Conformal Blocks in Commutative Moduli Schemes

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Abstract: The D_X-schemes (and their particular tools example *jets*) are related to determine conformal blocks of space-time pieces that are invariant under conformal transformations. All algebras will be commutative and Sym will always denote Sym_{O_X} However, all Hom, and ⊗, will be understood over the base field *k*. This will permit the construction of one formal moduli problem on the base of CAlg_k whose objects are obtained as limits of the corresponding jets in an $\mathrm{Aff}_{\mathrm{Spec}}$. An algebra B, belonging to the D_X-schemes to the required formal moduli problem is the image under a corresponding generalized Penrose transform, in the conformal context, of many pieces of the space-time, having a structure as objects in commutative rings of CAlg_k each one.

Keywords: Cohomologies, Commutative Rings, Conformal Blocks, Jets, Spectrum Functor

1. Introduction

The development of the jets technique as *forgetful* functors can establish equivalences between categories of opposite class as $\mathcal{D}_X$ – algebras and affine $\mathcal{D}_X$ – schemes, being their relation with these categories as co-limints of ring structures modulo an ideal $\mathcal{I}$, that can be a bi-sided ideal $\mathcal{I}^{\pm}$, when we want establish correspondences between two manifolds inside a same context (conformally, holomorphicity, etc) and obtain transformed objects with the same invariance. In the ambit of the commutative algebra [1] these relations can establish equivalences to the construction of one *formal moduli problem* on the base of CAlg_k, whose objects are obtained as limits of the corresponding jets in a $\mathrm{Aff}_{\mathrm{Spec}}$. In this last, results very useful this technique in special with the demonstration of be conformally invariant of some characteristics of the objects in the $\mathcal{D}_X$ – schemes can derive their notorious spectrum.

In this research, are used some properties of the jets as the functores $\mathcal{D}_X-\mathrm{sch} \to \mathcal{O}_X-\mathrm{sch}$, and using some tools as algebras of conformal blocks cohomologies to establish a commutative scheme of a moduli problem to conformal properties of geometrical invariants. These geometrical invariants are obtained as images under a corresponding generalized *Penrose transform* which can derive of the *Verdier duality* in the *cohomological context of the categories* and consigned in the structure of the objects of these categories, that is to say algebraic modules.

Then the homogeneous bundles and their objects can be extrapolated in *homogeneous polynomials* where these polynomials are direct images of the corresponding jets applied to objects belonging to a $\mathcal{O}_X$ – algebra or $\mathcal{D}_X$ – schemes.

2. $\mathcal{D}_X$ – Schemes

Fix a base field k, and a smooth scheme X, over k. A $\mathcal{D}_X$ – scheme is a scheme equipped with a flat connection over X. For an affine scheme, this is equivalent to being the spectrum of a $\mathcal{D}_X$ – algebra. For example, affine $\mathcal{D}_X$ – schemes of finite type have the form:

$$\mathrm{Spec}((\mathrm{Sym}\,\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F})/\mathcal{I}) \qquad (1.1)$$

for some coherent $\mathcal{O}_X$ – sheaf $\mathcal{F}$, and some $\mathcal{D}_X$ – ideal sheaf $\mathcal{I}$. Throughout this talk, we will often pass freely from $\mathcal{D}_X$ – algebras to affine $\mathcal{D}_X$ – schemes and vice-versa (the two categories are opposite in the usual sense).

A very important example of an affine $\mathcal{D}_X$ – scheme is $\mathrm{Spec}(\mathrm{Sym}\,\mathcal{M})$, for any $\mathcal{D}_X$ – module $\mathcal{M}$. This suggests that $\mathcal{D}_X$ – algebras are generalizations of $\mathcal{D}_X$ – modules, which is supported by the following fact: $\mathcal{D}_X$ – modules parametrize solutions of linear differential equations, while $\mathcal{D}_X$ – algebras parametrize solutions of nonlinear differential equations. More precisely, suppose we take the $\mathcal{D}_X$ – schemes $(\mathrm{Sym}\,\mathcal{D}_X^n)$, where the ideal $\mathcal{I}$, is generated (locally) by "polynomials" $P_1,\ldots,P_k \in \mathrm{Sym}\,D_X^n$.

Then giving a map of $\mathcal{D}_X$ – modules:

$$(\mathrm{Sym}\mathcal{D}_X^n)/\mathcal{I} \to \mathcal{O}_x, \tag{1.2}$$

is the same as giving a collection of functions $f_1,\ldots,f_n$, which satisfy the system of nonlinear differential equations:

$$P_1(f_1,\ldots,f_n) = 0, \tag{1.3}$$

A map of $\mathcal{D}_X$ – schemes is one which is a morphism of $\mathcal{D}_X$ – algebras at the level of coordinate rings. A more involved notion is the following:

Definition 1. 1. Given a morphism of $\mathcal{D}_X$ – schemes $\mathcal{Y} \to \mathcal{Z}$, the functor of horizontal sections $\mathrm{HorHom}(\mathcal{Z},\mathcal{Y})$, is given by:

$$S \in \mathrm{Sch} \to \mathrm{HorHom}(\mathcal{Z}\times S, \mathcal{Y}), \tag{1.4}$$

HorHom, consists of horizontal morphisms, i.e. morphisms of $\mathcal{D}_X$ – schemes.

The above definition is completely analogous to that of the functor Sect, replacing $\mathcal{O}_X$ – with $\mathcal{D}_X$. Note that for a morphism of $\mathcal{O}_X$ – –algebras to be a morphism of $\mathcal{D}_X$ – algebras is a closed condition. Since the functor of sections is representable, it follows that the functor of horizontal sections is also representable.

Moreover $\mathrm{HorSect}(\mathcal{Z},\mathcal{Y}) \to \mathrm{Sect}(\mathcal{Z},\mathcal{Y})$, is a closed embedding.

3. Jets

In this section, we will show that the forgetful functor

$$\mathcal{D}_X - \mathrm{sch} \to \mathcal{O}_X - \mathrm{sch}, \tag{2.1}$$

has a right adjoint, which is called the *Jet functor*:

$$\begin{gathered} \mathcal{J} : \mathcal{O}_X - \mathrm{sch} \to \mathcal{D}_x - \mathrm{sch}, \\ Hom_{\mathcal{D}x}(\mathcal{Z},\mathcal{J}\mathcal{Y}) = Hom_{\mathcal{O}_X}(\mathcal{Z},\mathcal{Y}) \end{gathered} \tag{2.2}$$

for any $\mathcal{O}_X$ – scheme $\mathcal{Y}$, and any $\mathcal{D}_X$ – scheme $\mathcal{Z}$. At the level of algebras, this functor will be a left adjoint to the *forgetful functor*:

$$\begin{gathered} \mathcal{J} : \mathcal{O}_X - \mathrm{alg} \to \mathcal{D}_x - \mathrm{sch}, \\ \mathrm{Hom}_{\mathcal{D}x}(\mathcal{J}\mathcal{A},\mathcal{B}) = \mathrm{Hom}_{\mathcal{O}x}(\mathcal{A},\mathcal{B}), \end{gathered} \tag{2.3}$$

for any $\mathcal{O}_X$ – algebra $\mathcal{A}$, and any $\mathcal{D}_X$ – algebra $\mathcal{B}$. Naturally, $\mathrm{Spec}\mathcal{J}\mathcal{A} = \mathcal{J}\mathrm{Spec}(\mathcal{A})$. Basically, there is only one natural construction which will make

$\mathcal{J}$, into a left adjoint:

$$\mathcal{J}\mathcal{A} = \mathrm{Sym}(D_x \otimes_{\mathcal{O}_x} \mathcal{A})/I, \tag{2.4}$$

where $\mathcal{I}$, is the $\mathcal{D}_X$ – ideal generated by $\mathrm{Ker}(\mathrm{Sym}(\mathcal{A} \to \mathcal{A})$. In other words,

$\mathcal{J}\mathcal{A}$, is the $\mathcal{D}_X$ – algebra generated by $\mathcal{A}$. Setting $\mathcal{Z} = X$, in (2. 2) gives us the following:

Proposition 2. 1. For any $\mathcal{O}_X$ – scheme $\mathcal{Y}$, we have:

$$\mathrm{HorSect}(X,\mathcal{J}\mathcal{Y}) = \mathcal{J}\mathrm{SecT}(X,\mathcal{Y}), \tag{2.5}$$

Example 2. 1. For any $\mathcal{O}_X$ – module $\mathcal{N}$, we have:

$$\mathcal{J}(\mathrm{Sym}\mathcal{N}) = \mathrm{Sym}(\mathcal{D}_x \otimes_{\mathcal{O}_x} \mathcal{N}), \tag{2.6}$$

Example 2. 2. Let X, be a smooth projective curve, $C = \mathrm{Spec}((\mathrm{Sym}\,\mathfrak{g})^G)$, as in our previous expositions [2], and consider the fiber bundle $C\omega_X = C \times k^* \omega X$, on X.

Then we have:

$$\mathrm{HorSee}(X, \mathcal{J}C_{\omega_X}) = \mathrm{Hitch}(X), \tag{2.7}$$

and

$$(\mathcal{J}C_{\omega_X})_x = \mathrm{Hitch}_x(X), \tag{2.8}$$

for any closed point $x \in X$. The first equality follows from *Proposition 2. 1*, while the second one follows from *Proposition 3. 1*, in the next section.

Let us now prove that the definition of jets in (2. 4) is the correct one, i.e. that it satisfies property (2. 3). For this, consider the following constructions:

$$(\phi : \mathcal{J}\mathcal{A} \to \mathcal{B}) \to (\phi' : \mathcal{A} \to \mathcal{B}), \qquad \phi'(a) = \phi(1\otimes a),$$

$$.(\phi' : \mathcal{A} \to \mathcal{B}) \leftarrow (\phi' : \mathcal{J}\mathcal{A} \to \mathcal{B}), \quad \phi(d\otimes a) = d\cdot\phi'(a),$$

where ϕ, denotes any map of $\mathcal{D}_X$ – algebras, while ϕ', denotes any map of $\mathcal{O}_X$ –algebras. It's easy to check that the assignments $\phi \to \phi'$, and $\phi' \to \phi$, are well-defined, are inverses to each other and are natural in $\mathcal{A}$, and $\mathcal{B}$.

4. Relation between Morphisms of Affine Schemes and Morphisms of Algebras

This section is not just motivated by etymological questions, but will actually be very useful for us. Our purpose will be to prove the following result:

Proposition 3. 1. Pick a closed point $x \in X$. and let y, be any $\mathcal{O}_X$ – scheme.

Then the fiber of $\mathcal{J}\mathcal{Y}$, over x, is given by:

$$(\mathcal{J}\mathcal{Y})_x = \mathrm{Sect}(\mathrm{Spf}\, \hat{\mathcal{O}}_x, \mathcal{Y}). \tag{3.1}$$

where $\hat{\mathcal{O}}_x$, is the completed local ring of X, at x.

Proof. Let us recall that for any k - scheme S, we define:

$$(\mathrm{Spf}\, \hat{\mathcal{O}}_x)\times S = \varinjlim_n ((\mathrm{Spec}\mathcal{O}_x / \mathfrak{m}_x^n)\times S) \neq (\mathrm{Spec}\, \hat{\mathcal{O}}_x)\times S,$$

Therefore, the structure ring of $\mathrm{Spf}\, \hat{\mathcal{O}}_x \times \mathrm{Spec} C$, is

$$\hat{\mathcal{O}}_x \hat{\otimes} C := \varprojlim_n ((\mathcal{O}_x / \mathfrak{m}_x^n) \hat{\otimes} C) \neq \hat{\mathcal{O}}_x \otimes C, \tag{3.2}$$

The above proposition makes the terminology clear, since a section from the formal disk to $\mathcal{Y}$, is, by definition, an $\mathcal{Y}$−jet at x. By naturality, it will be enough to prove the proposition in the affine case $\mathcal{Y} = \mathrm{Spec}\mathcal{A}$. In the following, C, will denote any algebra and $\mathcal{B}$, will denote any $\mathcal{D}_X$ – algebra. We claim the following functorial bijections hold:

$$\mathrm{Hom}(\mathrm{Spec}\, C, \mathrm{Spec}\, \mathcal{B}_x) \cong \mathrm{Hom}(\mathcal{B}_x, C), \tag{3.3}$$

and

$$\mathrm{Hom}(\mathcal{B}_x, C) \cong \mathrm{Hom}_{\mathcal{D}_x}(\mathcal{B}_x, \hat{\mathcal{O}}_x \hat{\otimes} C), \tag{3.4}$$

Specialize $\mathcal{B} = \mathcal{J}\mathcal{A}$, and we have:

$$\mathrm{Hom}_{\mathcal{D}_x}(\mathcal{J}\mathcal{A}, \hat{\mathcal{O}}_x \hat{\otimes} C) = \mathrm{Hom}_{\mathcal{D}_x}(\mathcal{A}, \hat{\mathcal{O}}_x \hat{\otimes} C), \tag{3.5}$$

and

$$\mathrm{Hom}_{\mathcal{O}_x}(\mathcal{A}, \hat{\mathcal{O}}_x \hat{\otimes} C) = \mathrm{Hom}(\mathrm{Spf}\, \hat{\mathcal{O}}_x \times \mathrm{Spec}\, C, \mathrm{Spec}\, \mathcal{A}), \tag{3.6}$$

This sequence of identifications proves (3. 1) on the level of C− points, and since they hold naturally in C, they are enough to establish Proposition 3. 1.

- Relation (3. 3) is just the bijection between morphisms of affine schemes and morphisms of algebras.
- To prove relation (3. 4), it is enough to verify it in the bigger category of vector spaces and $\mathcal{D}_X$ −modules. Then, we need to verify that for any

$\mathcal{D}_X$ −module $\mathcal{M}$, and any vector space V, we have

$$\mathrm{Hom}(\mathcal{M}_x, V) \cong \mathrm{Hom}_{\mathrm{D}_x}(\mathcal{M}, \hat{\mathcal{O}}_x \hat{\otimes} V), \tag{3.7}$$

The map going from right to left is just evaluation at x. Let's now define the map going from left to right: given any morphism of vector spaces

$\phi: \mathcal{M} \to V$, what does it mean to assign to it a morphism $\mathcal{M} \to \hat{\mathcal{O}}_x \hat{\otimes} V$? It merely means to give morphisms $\phi_n : \mathcal{M} \to \mathcal{O}_x / \mathfrak{m}_x^n \otimes V$, for x, all n, which satisfy the inverse limit compatibilities. We start off with $\phi_1 = \phi$, and then there is a unique way to inductively define each ϕ_n, such that the inverse limit is a morphism of $\mathcal{D}_X$ −algebras.

- Relation (3. 5) is just property (2. 3).
- Relation (3. 6) is just the bijection between morphisms of affine schemes and morphisms of algebras.

5. Conformal Blocks

The functor $k-\mathrm{sch} \to \mathcal{D}_X - \mathrm{sch}$, sending a k - scheme S ,to the "constant" $\mathcal{D}_X$ − scheme $\mathrm{X}\times\mathrm{S}$ (which has coordinate ring $\mathcal{O}_{\mathrm{x}} \otimes_k \mathcal{O}_S$) has a right adjoint functor:

$$\begin{aligned} &\mathrm{H}_\nabla(X,\cdot): \mathcal{D}_x - \mathrm{sch} \to k - \mathrm{sch}, \\ &\mathrm{Hom}(S, \mathrm{H}_\nabla(X, \mathcal{Z})) \cong \mathrm{Hom}_{\mathcal{D}_x}(X\times S, \mathcal{Z}), \end{aligned} \tag{4.1}$$

for any $\mathcal{D}_X$ – scheme $\mathcal{Z}$, and any k - scheme S . Alternatively, we can define this functor for algebras:

$$\begin{aligned} &\mathrm{H}_\nabla(X,\cdot): \mathcal{D}_x - \mathrm{Alg} \to k - \mathrm{Alg}, \\ &\mathrm{Hom}(\mathrm{H}_\nabla(X, \mathcal{B}), C) \cong \mathrm{Hom}_{D_x}(\mathcal{B}, \mathcal{O}_x \otimes_k C), \end{aligned} \tag{4.2}$$

for any $\mathcal{D}_X$ −algebra $\mathcal{B}$, and any k - algebra C. Obviously, $\mathrm{Spec}\, \mathrm{H}_\nabla(X, \mathcal{B}) = \mathrm{H}_\nabla(X, \mathrm{Spec}\mathcal{B})$. The scheme $\mathrm{H}_\nabla(X, \mathcal{Z})$, is called the scheme of conformal blocks of $\mathcal{Z}$, and it is tautologically the largest constant $\mathcal{D}_X$ − subscheme of $\mathcal{Z}$.

Example 4. 1. For any $\mathcal{D}_X$ − scheme $\mathcal{Z}$, we have:

$$\mathrm{H}_\nabla(X, \mathcal{Z}) \cong \mathrm{HorSect}(X, \mathcal{Z}), \tag{4.3}$$

This follows easily by unraveling the definitions.

Example 4. 2. Setting $\mathcal{Z} = \mathcal{J}\mathcal{Y}$, in the above for some $\mathcal{O}_x$ −scheme $\mathcal{Y}$, and combining with the Proposition 2. 1, gives us:

$$\mathrm{H}_\nabla(X, \mathcal{J}\mathcal{Y}) \cong \mathrm{HorSect}(X, \mathcal{Y}), \tag{4.4}$$

6. Cohomologies

In this section we restrict to X, projective of dimension n, and to affine $\mathcal{D}_X$ – algebras. The reason why we denote algebras of conformal blocks by $\mathrm{H}_\nabla(X, \mathcal{B})$, is that they turn out to be some sort of "cohomology algebras" [3] of the $\mathcal{D}_X$ – algebra $\mathcal{B}$. In fact, *Verdier duality* implies the following natural bijection for $\mathcal{D}_X$ − modules:

$$\mathrm{Hom}_{\mathcal{D}_x}(\mathcal{M}, \mathcal{O}_{\mathrm{x}} \otimes_k V), \cong \mathrm{Hom}(\mathrm{H}^{\mathrm{n}}_{\mathrm{dR}}(X, \mathcal{M}), V), \tag{5.1}$$

for any $\mathcal{D}_X$ – module $\mathcal{M}$, and any vector space V . By definition, $\mathrm{H}^\bullet_{\mathrm{dR}}(X, \mathcal{M})$, are the cohomology groups of the complex of sheaves of k - vector spaces:

$$\ldots \to \mathcal{M} \otimes_{\mathcal{O}_X} \Lambda^1 T^\bullet X \to \mathcal{M} \otimes_{\mathcal{O}_X} \Lambda^{i+1} T^\bullet X \to \ldots, \tag{5.2}$$

These cohomology groups coincide with $R^\bullet \pi_*(\mathcal{M})$, where

$\pi : X \to pt$, is the projection to a point. Note that (5. 2) implies that

$$\mathrm{H}_{\nabla}(X, \mathrm{Sym}\mathcal{M}) = \mathrm{Sym}\mathrm{H}^{\mathrm{n}}_{\mathrm{dR}}(X, \mathcal{M}), \qquad (5.3)$$

This can be further re-interpreted as follows. Pick a closed point $x \in X$, let $i : x \to X$, be the closed embedding and $j : X - x \to X$, be the open embedding. Then for any $\mathcal{D}_X$ – module $\mathcal{M}$, we have the exact triangle:

$$i_{\bullet}\mathcal{M}_x[-n] \to j_{\bullet}j^{\bullet}\mathcal{M}, \qquad (5.4)$$

The shift by n happens when we pass from $\mathcal{D}_X$ – modules to quasicoherent $\mathcal{O}_x$ –modules, as we will be doing now. This induces a long exact sequence on cohomology:

$$\begin{aligned} \ldots \to H^{n\text{-}1}_{\mathrm{dR}}(X - x, \mathcal{M}) \xrightarrow{\phi} \mathcal{M}_{\mathrm{x}} \to H^{n\text{-}1}_{\mathrm{dR}}(X, \mathcal{M}) \to \\ \to H^{n}_{\mathrm{dR}}(X - x, \mathcal{M}), \end{aligned} \qquad (5.5)$$

We claim that the last group is 0. To see this, recall that Lichtenbaum's theorem says that the Cěch cohomological dimension of $X - x$, is at most $n - 1$, e. g. $H^n(X - x, \mathcal{F}) = 0$ for any quasi-coherent $\mathcal{F}$. As the $\mathcal{D}_X$ – module $\mathcal{M}$, is a quotient of the form:

$$\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{M}, \qquad (5.6)$$

for some quasicoherent $\mathcal{F}$, and

$$\mathrm{H}^{\mathrm{n}}_{\mathrm{dR}}(X - x, \mathcal{D}_X \otimes_{\mathcal{O}_x} \mathcal{F}) = H^n(X - x, \mathcal{F}) = 0$$

it also follows that $\mathrm{H}^{\mathrm{n}}_{\mathrm{dR}}(X - x, \mathcal{M}) = 0$. Therefore, (5. 3) and (5. 5) imply:

$$\mathrm{H}_{\nabla}(X, \mathrm{Sym}\,\mathcal{M}) = \mathrm{Sym}(\mathcal{M}_{\mathrm{x}} / \mathrm{Im}\,\phi), \qquad (5.7)$$

The above description applies equally well to $\mathcal{D}_X$ – algebras, so we infer:

Corollary 5. 1. For any $\mathcal{D}_X$ – algebra $\mathcal{B}$, we have:

$$\mathrm{H}_{\nabla}(X, \mathcal{B}) \cong \mathcal{B}_x / (\mathrm{Im}\phi), \qquad (5.8)$$

where $\mathrm{Im}\phi$, denotes the ideal generated by the image of the co-boundary mapping

$$\phi : H^{n\text{-}1}(X - x, \mathcal{B}) \to \mathcal{B}_x, \qquad (5.9)$$

We can actually do all of this with any finite number of closed points $\mathrm{x}_1 \ldots, x_k \in X$. The analogue of the co-boundary map is ϕ, given by:

$$\begin{aligned} \ldots \to H^{n\text{-}1}(X - \{x_1, \ldots, x_{\mathrm{k}}\}, \mathcal{B} \xrightarrow{\phi} \mathcal{B}_{x_1} \oplus \ldots \oplus \mathcal{B}_{x_k} \\ \to H^n_{\mathrm{dR}}(X, \mathcal{B}) \to 0, \end{aligned} \qquad (5.10)$$

We will need an algebra, not just a vector space, so define the map:

$$\widetilde{\phi} : H^{n-1}(X - \{x_1, \ldots, x_{\mathrm{k}}\}, \mathcal{B} \xrightarrow{\phi} \mathcal{B}_{x_1} \oplus \ldots \oplus \mathcal{B}_{x_k},$$

$$\widetilde{\phi}(h) = \phi_1^{'}(h)1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes 1 \otimes \phi_k^{1}(h),$$

In the above, $\phi_i{}'$ denotes the projection of the map ϕ', to the ith - factor.

Proposition 5. 1. We have the following natural isomorphism:

$$\mathcal{B}_{x_1} / (\mathrm{Im}\phi) \cong \mathcal{B}_{x_1} \otimes \ldots \otimes \mathcal{B}_{x_K} / (\mathrm{Im}\widetilde{\phi}) \qquad (5.11)$$

where $(\mathrm{Im}\phi)$, denotes the ideal generated by the image of the map $\widetilde{\phi}$.

To prove the *proposition 5. 1*, is necessary to take the natural morphism from left to right sending $b_1 \in \mathcal{B}_{x_1}$, to $b_1 \otimes 1 \otimes \ldots 1$. Its injectivity is immediate, and its surjectivity follows readily from the $k = 2$, case. Since it will also make the explanation clearer, let's just do $k = 2$. We have the following commutative diagram

$$\begin{array}{ccccc} H^{n\text{-}1}(X - x_1, \mathcal{B}) & \xrightarrow{\phi} \mathcal{B}_{x_1} & \xrightarrow{\pi} H^n(X, \mathcal{B}) \to 0 \\ \downarrow & \downarrow & =\downarrow \\ H^{\mathrm{n\text{-}1}}(X - \{x_1, x_2\}, \mathcal{B}) & \xrightarrow{\phi'} \mathcal{B}_{x_1} \oplus \mathcal{B}_{x_2} & \xrightarrow{\pi'} H^n(X, \mathcal{B}) \to 0, \end{array}$$

Take any $b_2 \in \mathcal{B}_{x_2}$, and look at $\pi(b_2) \in H^n(X, \mathcal{B})$. By the above diagram, there exists $a \in \mathcal{B}_1$, such that $\pi(a) = \pi(b_2)$. This means that $(-a, b_2) \in \mathrm{Ker}\pi$, if and only is $(-a, b_2) \in \phi'(h)$, for some h. Take any $b_1 \in \mathcal{B}_{x_1}$, and we have:

$$\begin{aligned} \mathrm{b}_1 \otimes b_2 = b_1 \otimes \phi_2^{'}(h) = (\mathrm{b}_1 \otimes 1)(1 \otimes \phi_2^{'}(h)) = \\ (\mathrm{b}_1 \otimes 1)(\otimes \phi_1^{'}(h) \otimes 1 + 1 \otimes \phi_2^{'}(h)) \\ -(b_1 \cdot \phi_2^{'}(h) \otimes 1 \in (\mathrm{Im}\widetilde{\phi}) + \mathcal{B}_1, \end{aligned}$$

This implies that the map (5. 11) is surjective, and concludes the proof of *proposition 5. 1*. Therefore, the *corollary 5. 1*, implies the following corollary.

Corollary 5. 2. For any $\mathcal{D}_X$ – algebra $\mathcal{B}$, we have:

$$\mathrm{H}_{\nabla}(X, \mathcal{B}) \cong \mathcal{B}_{x1} \otimes \ldots \otimes \mathcal{B}_{x_k} / (\mathrm{Im}\widetilde{\phi}), \qquad (5.12)$$

7. Results

The conformal blocks can be obtained by the apparatus of the *Penrose transform* through their extension, interpreting the invariances under scheme of "CRings" and $\mathcal{D}_X$ – schemes.

$$\mathrm{Hom}_{Dx}(\mathcal{JA}, \mathcal{B}) = \mathrm{Hom}_{\mathrm{Alg}_k}(X, \mathrm{Spec}\mathcal{JA}), \qquad (6.1)$$

where $\mathcal{B}$, is a $\mathcal{D}_X$ – algebra.

Let X, be smooth over $\mathbb{Z}$, locally at x. Let I, be the

$\mathcal{D}_X$ – ideal generated by $\mathrm{Ker}(\mathrm{Sym}(\mathcal{A} \to \mathcal{A}))$, where $\mathcal{A}$, is a $\mathcal{O}_X$ – sub-algebra of $\mathcal{O}_X[1/x]$, generated by the sub-sheaf $x^{-1}\mathcal{I}$. But $\mathcal{Y} = \mathrm{Spec}\mathcal{A}$, to $\mathcal{Y}$, any $\mathcal{O}_X$ – scheme, then $\mathcal{J}\mathcal{Y}$, is a $\mathcal{D}_X$ – scheme. Specializing $\mathcal{B} = \mathcal{J}\mathcal{A}$, we have $\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{J}\mathcal{A}, \hat{\mathcal{O}}_x \hat{\otimes} C)$ by (3. 5), and the left of (6. 1) is proved. To establish the right side of (6. 1) we need use the relations between morphisms of affine schemes and morphisms of algebras to that the category of schemes can be identified with a full subcategory of Sp[1]. Indeed, for any object $\mathcal{Y} = \mathrm{Spec}\mathcal{A}$, and let $\mathrm{Aff}_{\mathcal{Y}}$, denote the full subcategory of $\mathrm{Sp}_{\mathcal{Y}}$, consisting of objects X, over $\mathcal{Y}$, where X, is affine. We define $\mathrm{AffRel}_{\mathcal{Y}}$, to be the full subcategory of $\mathrm{Aff}_{\mathcal{Y}}$, consisting of objects X, whose structure map $X \to \mathcal{Y}$, factors through an affine open sub-scheme of $\mathcal{Y}$. The inclusion $\mathrm{AffRel}_{\mathcal{Y}} \to \mathrm{Sp}_{\mathcal{Y}}$, induces an equivalence between $\mathrm{Sp}_{\mathcal{Y}}$, and the category of sheaves of sets on $\mathrm{AffRel}_{\mathcal{Y}}$. $\mathcal{J}\mathcal{A}$, is the $\mathcal{D}_X$ – algebra generated by $\mathcal{A}$. By (2. 5) we have for any $\mathcal{O}_X$ – scheme $\mathcal{Y}$, we have $\mathrm{HorSect}(X, \mathcal{J}\mathcal{Y})$, which is equivalent to the conformal bock $\mathrm{H}_\nabla(X, \mathcal{J}\mathcal{Y})$. Then by (4. 2) we have $\mathrm{Spec}\ \mathrm{H}_\nabla(X, \mathcal{B}) = \mathrm{H}_\nabla(X, \mathrm{Spec}\mathcal{B})$, and considering the property of the jet a $\mathcal{J}$, given by $\mathrm{Spec}\mathcal{J}\mathcal{A} = \mathcal{J}\mathrm{Spec}(\mathcal{A})$, (mentioned in the *section 2*) we have that the jet carry us to an object in"CRings" defined by an $k-$ .algebra $\mathrm{Spec}R$, where R, is a commutative E_n – ring to $0 \leq n < \infty$. Then is followed (6. 1).

Proposition. 6. 1. For any $\mathcal{D}_X$ – algebra $\mathcal{B}$, we have that every conformal block is the Penrose transform

$$\mathcal{P} : H^0(\mathcal{J}\mathcal{A}, \mathcal{O}) \cong H_\nabla(X, \mathcal{B}) \tag{6.2}$$

An application example of the solution classes given by integrals of (6. 2) are given for the solutions to the field equations of the Bach tensor and Eastwood-Dighton tensor[2]

$$B_{ab} = 0, \quad E_{abc} = 0, \tag{6.3}$$

where the tensors , to the *conformal case* can be designed as elements of a $\mathcal{D}_X$ – algebras or $\mathcal{O}_X$ – algebras.

If we consider the scheme commutative moduli we could think in the *forgetful functor* defined in (2. 3), that is to say, the jet $\mathcal{J} : \mathcal{O}_X - \mathrm{alg} \to \mathcal{D}_x - \mathrm{sch}$, where are verified the relations in the conformal context of the space-time $(\phi : \mathcal{J}\mathcal{A} \to \mathcal{B}) \to (\phi' : \mathcal{A} \to \mathcal{B})$, and $(\phi' : \mathcal{A} \to \mathcal{B}) \leftarrow (\phi' : \mathcal{J}\mathcal{A} \to \mathcal{B})$, to the concrete case of a *conformal factor* between $\mathcal{D}_X$ – algebras and $\mathcal{O}_X$ – algebras.

Indeed, a complex space-time $\mathbb{M}$, satisfying (6. 3) will be called a solution to the conformal gravity equations [4, 5] if $\mathbb{M}$ (as a $4-$ dimensional complex Riemannian manifold) implies that $(\mathbb{M}, g)$, is a solution of (6. 3) with algebraically several Weyl curvature implying that exists a conformal factor α, such that $\tilde{g} = \alpha^2 g$,where their meaning of Ricci curvature of $\tilde{\mathcal{R}}$, satisfies $\tilde{\mathcal{R}}_{ab} = \frac{1}{4}\tilde{g}$.

Due that the Bach and Easwood-Dighton equations haves are both symmetric trace-free tensors and are both *conformally invariants*, with conformal weight -2, meaning that under the transformation

$$g \mapsto \tilde{g} = \alpha^2 g, \tag{6.4}$$

we have

$$B \mapsto \tilde{B} = \alpha^{-2} B, \tag{6.5}$$

and

$$E \mapsto \tilde{E} = \alpha^{-2} E, \tag{6.6}$$

Moreover, they are both invariant under bi-holomorphisms meaning that it

$$\phi : \mathbb{M} \to \tilde{\mathbb{M}}, \tag{6.7}$$

is a bi-holomorphism then these tensors depends upon the metric in such a manner that

$$B(\phi^*(g)) = \phi^*(B(g)), \tag{6.8}$$

where $\phi^*(g) = \mathcal{J}^m\mathcal{A}, \forall m \in \mathbb{N}$ since considering the jet of the metric g[3], in $x \in \mathbb{M}$, we have $B(\mathcal{J}^m\mathcal{A}) \in \mathrm{Spec}(\mathcal{J}^m\mathcal{A})$, spectrum element in a $\mathcal{D}_X$ – sheme. For other side, in the $\mathcal{D}_X$ – algebras, the image $\phi^* B(g) \in \mathcal{J}^m\mathrm{Spec}(\mathcal{A})$, and the similar to[4]

$$E(\phi^*(g)) = \pm\phi^*(E(g)), \tag{6.9}$$

The value at $x \in \mathbb{M}$, of these tensors is a holomorphic function in $\mathrm{Spec}(\mathcal{A})$, of the $m-$ jet of g, at $x, \forall m \in \mathbb{N}$, that is to say

$$\mathcal{J}^m(\mathrm{Spec}(\mathcal{A})) = \mathrm{Spec}(\mathcal{J}^m\mathcal{A}), \tag{6.10}$$

The interesting of this application is the property of the jet of their homogeneous polynomial context. Likewise, if $h(\alpha^2 r, \alpha^3 s, \alpha^4 t, \ldots, \alpha^m u) \in \mathrm{Spec}\mathcal{R}^{\pm}$, then (6. 10) takes the form:

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[1] Sp, denote the category of sheaves of sets on Aff.

Aff, denote the category of affine schemes with certain covering topology whose images cover $\mathrm{Aff}_{\mathcal{Y}}$.

[2] Here $B_{ab} = (\nabla^c\nabla^d + 1/2\mathcal{R}^{cd})C_{abcd}$, where C_{abcd}, denotes the *Weyl curvature*, while that $E_{abc} = \overline{\Psi}_{A'B'C'D'}\nabla^{DD'}\Psi_{ABCD} - \Psi_{ABCD}\nabla^{DD'}\overline{\Psi}_{A'B'C'D'}$.

[3] Jets of metrics of the form:

$$g_{ab} = \delta_{ab} + r_{(ab)(cd)}x^c x^d + s_{(ab)(cde)}x^c x^d + t_{(ab)(cdef)}x^c x^d x^f + \cdots + u_{(ab)(cde\cdots f)}x^c x^d \cdots x^f.$$

[4] $\pm$, sign depends upon the choice of sign for the associated star operator, $* : \wedge^2 \to \wedge^2$.

$$h(\alpha^2 r, \alpha^3 s, \alpha^4 t, \ldots, \alpha^m u) = \alpha^{2+l}(h(r, s, t, \ldots, u)). \quad (6.11)$$

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