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On semi-invariant submanifolds of a generalized Kenmotsu manifold admitting a semi-symmetric non-metric connection

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Abstract: In this paper, semi-invariant submanifolds of a generalized Kenmotsu manifold endowed with a semi-symmetric non-metric connection are studied. Necessary and sufficient conditions are given on a submanifold of a generalized Kenmotsu manifold to be semi-invarinat submanifold with semi-symmetric non-metric connection. Morever, we studied the integrability condition of the distribution on semi-invariant submanifolds of generalized Kenmotsu manifold with semi-symmetric non-metric connection.

Keywords: Generalized Kenmotsu Manifolds, Semi-Invariant Submanifolds, Semi-Symmetric Non-Metric Connection

1. Introduction

In 1963, Yano [11] introduced an f-structure on a C^∞ m-dimensional manifold M, defined by a non-vanishing tensor field φ of type (1,1) which satisfies $\varphi^3+\varphi=0$ and has constant rank r . It is know that in this case r is even, r=2n. Moreover, TM splits into two complementary subbundles $Im\varphi$ and $\ker\varphi$ and the restriction of φ to $Im\varphi$ determines a complex structure on such subbundle. It is also known that the existence of an f-structure on M is equivalent to a reduction of the structure group to $U(n)\times O(s)$ [1] where $s=m-2n$.

In [2] , K. Kenmotsu has introduced a Kenmotsu manifold. In [9], present autours have introduced a generalized Kenmotsu manifold.

Semi-invariant submanifolds are studied by some authours (for examples, M. Kobayashi [3], B. Prasad [6] and B.B. Sinha, A.K. Srivastava [7]). In [5] S. A. Nirmala and R.C. Mangala have introduced a semi-symmetric non-metric connection, they studied some properties of the curvature tensor with respect to the semi-symmetric non-metric connection.

Let ∇ be a linear connection in a n -dimensional differentiable manifold M. The torsion tensor T of ∇ is given by

$$T(X,Y)=\nabla_X Y-\nabla_Y X-[X,Y].$$

The connection ∇ is symmetric if torsion tensor T vanishes, otherwise it is non-symmetric. A lineer connection ∇ is said to be semi-symmetric connection if it torsion tensor T is of the form

$$T(X,Y)=\eta(Y)X-\eta(X)Y,$$

where η is a 1-form. The connection ∇ is metric connection if there is a Riemannian metric g in M such that $\nabla g=0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection.

The paper is organized as follows : In section 2, we give a brief introduction of generalized Kenmotsu manifold. We defined a generalized Kenmotsu manifold with a semi-symmetric non-metric connection. In section 3, we give some basic results for semi-invariant submanifolds of generalized Kenmotsu manifold with a semi-symmetric non-metric connection. In last section, we obtained some necessary and sufficient conditions for integrability of certain distributions on semi-invariant submanifolds of generalized

Kenmotsu manifold with a semi-symmetric non-metric connection.

2. Preliminaries

In *[4],* a (2n+s)-dimensional differentiable manifold *M* is called metric *f*-manifold if there exist an (1,1)-type tensor field φ, s-vector fields $\xi_1,\ldots,\xi_s$, s 1-forms $\eta^1,\ldots,\eta^s$ and a Riemannian metric g on *M* such that

$$\varphi^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i, \qquad \eta^i(\xi_j) = \delta_{ij} \tag{1}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \sum_{i=1}^{s} \eta^i(X)\eta^i(Y), \tag{2}$$

for any $X, Y \in \Gamma(TM)$, $i, j \in \{1,\ldots,s\}$. In addition, we have

$$\eta^i(X) = g(X,\xi_i), g(X,\varphi Y) = -g(\varphi X, Y). \tag{3}$$

Then, a 2 -form Φ is defined by $\Phi(X,Y) = g(X,\varphi Y)$, for any $X, Y \in \Gamma(TM)$, called the *fundamental* 2 *-form.* Moreover, a framed metric manifold is *normal* if

$$[\varphi,\varphi] + 2\sum_{i=1}^{s} d\eta^i \otimes \xi_i = 0,$$

where $[\varphi,\varphi]$ is denoting the Nijenhuis tensor field associated to φ.

In [10], let *M (2n+s)*-dimensional metric *f*-manifold. If there exists 2 -form Φ such that $\eta^1 \wedge \ldots \wedge \eta^s \wedge \Phi^n \neq 0$ on *M*, then *M* is called an almost *s*-contact metric structure.

The almost *s*-contact metric manifold $\overline{M}$ is called a generalized Kenmotsu manifold if it satisfies the condition

$$(\overline{\overline{\nabla}}_X \varphi)Y = \sum_{i=1}^{s} \{g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X\} \tag{4}$$

where $\overline{\overline{\nabla}}$ denotes the Riemannian connection with respect to g [9].

From the formula (4) we have

$$\overline{\overline{\nabla}}_X \xi_i = -\varphi^2 X. \tag{5}$$

Definition 2.1 An *(2n+s)*-dimensional Riemannian submanifold *M* of a generalized Kenmotsu manifold $\overline{M}$ is called a semi-invariant submanifold if ξ_i are tangent to $\overline{M}$ and there exists on *M* a pair of orthogonal distribution $\{D, D^\perp\}$ such that

(i) $TM = D \oplus D^\perp \oplus Sp\{\xi_1,\ldots,\xi_s\}$.

(ii) The distribution D is invariant under φ, that is $\varphi D_x = D_x$, for all $x \in M$

(iii) The distribution $D^\perp$ is anti-invariant under φ, that is $\varphi D_x^\perp \subset T_x M^\perp$, for all $x \in M$, where $T_x M$ and $T_x M^\perp$ are the tangent space of M at x.

The distribution D (resp. $D^\perp$) is called horizontal (resp. vertical) distribution. A semi-invariant submanifold *M* is said to be an invariant (resp. anti-invariant) submanifold if we have $D_x^\perp = \{0\}$ (resp. $D_x = \{0\}$) for each $x \in M$. We say that *M* is a proper semi-invariant submanifold, which is neither an invariant nor an anti-invariant submanifold.

Let $\overline{\overline{\nabla}}$ be the Levi-Civita connection of $\overline{M}$ with respect to the induced metric g. Then Gauss and Weingarten formulas are given by

$$\overline{\overline{\nabla}}_X Y = \nabla^*_X Y + h(X,Y) \tag{6}$$

$$\overline{\overline{\nabla}}_X N = \nabla^{*\perp}_X N - A_N X \tag{7}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$. $\nabla^\perp$ is the connection in the normal bundle, h is the second fundamental from of $\overline{M}$ and A_N is the Weingarten endomorphism associated with N. The second fundamental form h and the shape operator A related by

$$g(h(X,Y),N) = g(A_N X, Y). \tag{8}$$

Now, a semi-symmetric non-metric connection $\overline{\nabla}$ is defined as

$$\overline{\nabla}_X Y = \overline{\overline{\nabla}}_X Y + \sum_{i=1}^{s} \eta^i(Y)X \tag{9}$$

such that

$$(\overline{\nabla}_X g)(Y,Z) = -\sum_{i=1}^{s} \{g((X,Y)\eta^i(Z) + g(X,Z)\eta^i(Y)\} \tag{10}$$

for any $X, Y \in TM$, where $\overline{\nabla}$ is induced connection on M.

From (4) and (9), we have

$$(\overline{\nabla}_X \varphi)Y = \sum_{i=1}^{s} \{g(\varphi X, Y)\xi_i - 2\eta^i(Y)\varphi X\} \tag{11}$$

which is condition for almost *s*-contact metric manifold to be generalized Kenmotsu manifold with semi-symmetric non-metric connection.

Corollary 2.2 Let M be a semi-invariant submanifold of a generalized Kenmotsu manifold $\overline{M}$ with semi-symmetric non-metric connection, then

$$\overline{\nabla}_X \xi_j = 2X - \sum_{j=1}^{s} \eta^j(X)\xi_j \tag{12}$$

for all $X, Y \in TM$.

Proof. Putting $Y = \xi_j$ and $Z = \xi_j$ in (10)

$$X[g(\xi_j,\xi_j)] - g(\overline{\nabla}_X \xi_j, \xi_j) - g(\xi_j, \overline{\nabla}_X \xi_j) = 2\eta^j(X). \text{So}$$

$$g(\overline{\nabla}_X \xi_j, \xi_j) = \eta^j(X). \quad (13)$$

Now, using (11)

$$(\overline{\nabla}_X \varphi)\xi_j = \sum_{i=1}^{s}\{g(\varphi X, \xi_j)\xi_i - 2\eta^i(\xi_j)\varphi X\}$$

or

$$-\varphi\overline{\nabla}_X \xi_j = -2\varphi X$$

from (1) and (13)

$$\overline{\nabla}_X \xi_j - \sum_{j=1}^{s}\eta^j(\overline{\nabla}_X \xi_j)\xi_j = -2(-X + \sum_{j=1}^{s}\eta^j(X)\xi_j).$$

We denote by same symbol g both metrices on $\overline{M}$ and M. Let $\overline{\nabla}$ be the semi-symmetric non-metric connection on $\overline{M}$ and ∇ be the induced connection on M with respect to unit normal N. Then,

$$\overline{\nabla}_X Y = \nabla_X Y + m(X,Y) \quad (14)$$

where m is a tensor field of type (0,2) on semi-invariant submanifold M. Using (6) and (9) we have,

$$\nabla_X Y + m(X,Y) = \nabla^*_X Y + h(X,Y) + \eta^i(Y)X.$$

So equation tangential and normal components from both the sides, we get

$$m(X,Y) = h(X,Y)$$

and

$$\nabla_X Y = \nabla^*_X Y + \sum_{i=1}^{s}\eta^i(Y)X. \quad (15)$$

From (15) and (7)

$$\nabla_X N = \nabla^*_X N + \sum_{i=1}^{s}\eta^i(N)X$$

$$= -A_N X + \sum_{i=1}^{s}\eta^i(N)X$$

$$= (-A_N + a)X$$

where $a = \sum_{i=1}^{s}\eta_i(N)$ is a function on M.

Now, Gauss and Weingarten formulas for a semi-invariant submanifolds of a generalized Kenmotsu manifold with a semi-symmetric non-metric connection is

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad (16)$$

and

$$\overline{\nabla}_X N = (-A_N + a)X + \nabla^{\perp}_X N \quad (17)$$

for all $X, Y \in \Gamma(TM)$, $N \in \Gamma(TM^{\perp})$, h second fundamental form of M and A_N is the Weingarten endomorphism associated with N. The second fundamental form h and the shape operator A related by

$$g(h(X,Y),N) = g(A_N X, Y). \quad (18)$$

The projection morphisms of TM to D and $D^{\perp}$ are denoted by P and Q respectively. For any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$, we have

$$X = PX + QX + \sum_{i=1}^{s}\eta^i(X)\xi_i \quad (19)$$

and

$$\varphi N = BN + CN \quad (20)$$

where BN (resp. CN) denotes the tangential (resp. normal) component of φN.

Theorem 2.3 The connection induced on semi-invariant submanifolds of a generalized Kenmotsu manifold with semi-symmetric non-metric connection is also a semi-symmetric non-metric connection.

3. Basic Results

Lemma 3.1 Let M be a semi-invariant submanifold of a generalized Kenmotsu manifold $\overline{M}$ with semi-symmetric non-metric connection, then we have

$$(\overline{\nabla}_X \varphi)Y = (\nabla_X P)Y + (-A_{QY} + a)X - Bh(X,Y)$$
$$+ (\nabla_X Q)Y + h(X,PY) - Ch(X,Y) \quad (21)$$

$$(\overline{\nabla}_X \varphi)N = (\nabla_X B)N + (-A_{CN} + a)X + P(-A_N + a)X$$
$$+ (\nabla_X C)N + h(X,BN) + Q(-A_N + a)X \quad (22)$$

for all $X, Y \in TM$; $N \in \Gamma(TM)^{\perp}$ where $a = \sum_{i=1}^{s}\eta^i(CN) = 0.$

Proof. Using (19) and (20), necessary arrangements are made to obtain the desired.

Lemma 3.2 Let M be a semi-invariant submanifold of a generalized Kenmotsu manifold $\overline{M}$ with semi-symmetric non-metric connection, we have

$$(\nabla_X P)Y + (-A_{QY} + a)X - Bh(X,Y) = -2\sum_{i=1}^{s}\eta^i(Y)PX$$

$$(\nabla_X Q)Y + h(X,PY) - Ch(X,Y) = -2\sum_{i=1}^{s}\eta^i(Y)QX$$

$$(\nabla_X B)N + (-A_{CN} + a)X + P(-A_N + a)X = 0$$

$$(\nabla_X C)N + h(X, BN) + Q(-A_N + a)X = 0$$

$$g(PX, Y) = 0$$

$$g(QX, Y) = 0$$

for all $X, Y \in \Gamma(TM)$, $N \in \Gamma(TM)^{\perp}$.

Proof. Using (11) in (21) and (22), completes the proof.

Corollary 3.3 Let M be a semi-invariant submanifold of a generalized Kenmotsu manifold $\overline{M}$ with semi-symmetric non-metric connection such that $\xi_i \in TM$, we have

$$(\nabla_X P)\xi_i = -2PX$$

$$(\nabla_X Q)\xi_i = -2QX$$

$$(\nabla_{\xi_i} B)N = 0, \nabla_{\xi_i} B = 0$$

$$(\nabla_{\xi_i} C)N = 0, \nabla_{\xi_i} C = 0.$$

For $X, Y \in \Gamma(TM)$, we put

$$u(X, Y) = \nabla_X \varphi PY - A_{\varphi QY} X.$$

We begin with the following lemma.

Lemma 3.4 Let M be a semi-invariant submanifold of a generalized Kenmotsu manifold $\overline{M}$ with semi-symmetric non-metric connection, then we have

$$P(u(X, Y)) = \sum_{i=1}^{s} \{g(\varphi X, Y)P\xi_i - 2\eta^i(Y)\varphi PX\} + \varphi P\nabla_X Y$$

$$Q(u(X, Y)) = \sum_{i=1}^{s} \{g(\varphi X, Y)Q\xi_i - 2\eta^i(Y)\varphi QX\} + Bh(X, Y)$$

$$\varphi Q\nabla_X Y + Ch(X, Y) = h(X, \varphi PY) + \nabla_X^{\perp} \varphi QY$$

$$\eta^i(u(X, Y)) = g(\varphi X, Y)$$

for all $X, Y \in \Gamma(TM)$.

Proof. Easily shown using (11), (16), (17), (19) and (20).

Lemma 3.5 Let M be a semi-invariant submanifold of a generalized Kenmotsu manifold $\overline{M}$ with semi-symmetric non-metric connection such that $\xi_i \in TM$, we have

$$\nabla_X \xi_i = 2X - \sum_{i=1}^{s} \eta^i(X)\xi_i, h(X, \xi_i) = 0 \quad (23)$$

$$\nabla_{\xi_i} \xi_i = 0, \qquad h(\xi_i, \xi_i) = 0, \qquad A_N \xi_i = 0. \quad (24)$$

Proof. Using (12) and (15) for (22). And

$$0 = g(h(X, \xi_i), N) = g(A_N X, \xi_i) = g(A_N \xi_i, X).$$

4. Integrability of Distribution on a Semi-Invariant Submanifolds a Generalized Kenmotsu Manifold with Semi-SymmetricNon-Metric Connection

Theorem 4.1 Let M be a semi-invariant submanifold of a generalized Kenmotsu manifold $\overline{M}$ with semi-symmetric non-metric connection. Then the distribution D is integrable.

Proof. We have for all $X, Y \in \Gamma(D)$,

$$g([X, Y], \xi_i) = g(\overline{\overline{\nabla}}_X Y, \xi_i) - g(\overline{\overline{\nabla}}_Y X, \xi_i)$$

$$= -g(Y, \overline{\overline{\nabla}}_X \xi_i) + g(X, \overline{\overline{\nabla}}_Y \xi_i).$$

Using (9) and (12), we have

$$g([X, Y], \xi_i) = -g(Y, \overline{\nabla}_X \xi_i - X) + g(X, \overline{\nabla}_Y \xi_i - Y)$$

$$= -g(Y, 2X - \sum_{i=1}^{s} \eta^i(X)\xi_i - X)$$

$$+ g(X, 2Y - \sum_{i=1}^{s} \eta^i(Y)\xi_i - Y)$$

$$= 0.$$

So $\eta^i([X, Y]) = 0$ for *i=1,2,...s*. Then, we have $[X, Y] \in D$.

Theorem 4.2 Let M be a semi-invariant submanifold of a generalized Kenmotsu manifold $\overline{M}$ with semi-symmetric non-metric connection. The distribution $D \oplus Sp\{\xi_{1,...,}\xi_s\}$ is integrable if and only if

$$h(X, \varphi Y) = h(\varphi X, Y)$$

is satisfied.

Proof. Using (6) and (9), then

$$\varphi([X, Y]) = \varphi(\nabla_X^* Y - \nabla_Y^* X)$$

$$= \varphi(\overline{\overline{\nabla}}_X Y - h(X, Y) - \overline{\overline{\nabla}}_Y X + h(Y, X))$$

$$= \varphi(\overline{\nabla}_X Y - \sum_{i=1}^{s} \eta^i(Y)X - \overline{\nabla}_Y X + \sum_{i=1}^{s} \eta^i(X)Y)$$

$$= \overline{\nabla}_X \varphi Y - (\overline{\nabla}_X \varphi)Y - \sum_{i=1}^{s} \eta^i(Y)\varphi X$$

$$- \overline{\nabla}_Y \varphi X + (\overline{\nabla}_Y \varphi)X + \sum_{i=1}^{s} \eta^i(X)\varphi Y.$$

For (11) and (16), we have

$$\varphi([X,Y]) = \nabla_X \varphi Y - \nabla_Y \varphi X + \sum_{i=1}^{s}\{2g(X,\varphi Y)\xi_i + \eta^i(Y)\varphi X - \eta^i(X)\varphi Y\} + h(X,\varphi Y) - h(\varphi X, Y)$$

where $\varphi([X,Y])$ shows the component of $\nabla_X Y$ from the ortogonal complementary distribution of $D \oplus Sp\{\xi_{1,\ldots,}\xi_s\}$ in *M*. Then, we have $[X,Y] \in D \oplus Sp\{\xi_{1,\ldots,}\xi_s\}$ if and only if $h(X,\varphi Y) = h(Y,\varphi X)$.

Theorem 4.3 Let *M* be a semi-invariant submanifold of a generalized Kenmotsu manifold $\overline{M}$ with semi-symmetric non-metric connection. The distribution $D^{\perp} \oplus Sp\{\xi_{1,\ldots,}\xi_s\}$ is integrable if and only if

$$A_{\varphi X} Y = A_{\varphi Y} X$$

is satisfied.

Proof. We have for all $X, Y \in \Gamma(D^{\perp})$

$$g([X,Y],\xi_i) = g(\overline{\overline{\nabla}}_X Y, \xi_i) - g(\overline{\overline{\nabla}}_Y X, \xi_i) = -g(Y, \overline{\overline{\nabla}}_X \xi_i) + g(X, \overline{\overline{\nabla}}_Y \xi_i).$$

Using (9) and (12), we have

$$g([X,Y],\xi_i) = -g(Y, \overline{\nabla}_X \xi_i - X) + g(X, \overline{\nabla}_Y \xi_i - Y) = -g(Y, 2X - \sum_{i=1}^{s}\eta^i(X)\xi_i - X) + g(X, 2Y - \sum_{i=1}^{s}\eta^i(Y)\xi_i - Y) = 0.$$

Using (6) and (9) then

$$\varphi([X,Y]) = \varphi(\nabla^*_X Y - \nabla^*_Y X) = \overline{\nabla}_X \varphi Y - (\overline{\nabla}_X \varphi)Y - \sum_{i=1}^{s}\eta^i(Y)\varphi X - \overline{\nabla}_Y \varphi X + (\overline{\nabla}_Y \varphi)X + \sum_{i=1}^{s}\eta^i(X)\varphi Y.$$

For (11) and (17), we have

$$\varphi([X,Y]) = (-A_{\varphi Y} + a)X + \nabla^{\perp}_X \varphi Y - \sum_{i=1}^{s}\{g(\varphi X, Y)\xi_i + 2\eta^i(Y)\varphi X\} - (-A_{\varphi X} + a)Y - \nabla^{\perp}_Y \varphi X + \sum_{i=1}^{s}\{g(\varphi Y, X)\xi_i - 2\eta^i(X)\varphi Y + \eta^i(X)\varphi Y - \eta^i(Y)\varphi X\}$$

$$= \sum_{i=1}^{s}\{2g(X,\varphi Y)\xi_i + \eta^i(Y)\varphi X - \eta^i(X)\varphi Y\} + A_{\varphi X}Y - A_{\varphi Y}X + \nabla^{\perp}_X \varphi Y - \nabla^{\perp}_Y \varphi X.$$

Then we obtain,

$$[X,Y] \in D^{\perp} \oplus Sp\{\xi_1,\ldots,\xi_s\} \;\Rightarrow A_{\varphi X} Y = A_{\varphi Y} X.$$

Conversely

$$\varphi^2([X,Y]) = \sum_{i=1}^{s}\{2g(X,\varphi Y)\varphi\xi_i + \eta^i(Y)\varphi^2 X - \eta^i(X)\varphi^2 Y\} + A_{\varphi X}Y - A_{\varphi Y}X + \varphi(\nabla^{\perp}_X \varphi Y) - \varphi(\nabla^{\perp}_Y \varphi X)$$

$$[X,Y] = \sum_{i=1}^{s}\{-\eta^i(Y)X + \eta^i(X)Y\} + \sum_{i=1}^{s}\{\eta^k(X)\xi_k - \eta^k(X)\xi_k\} + \varphi(\nabla^{\perp}_X \varphi Y) - \varphi(\nabla^{\perp}_Y \varphi X)$$

then, we have

$$[X,Y] \in D^{\perp} \oplus Sp\{\xi_1,\ldots,\xi_s\}.$$

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