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Constant Curvatures of Parallel Hypersurfaces in E_1^{n+1} Lorentz Space

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Abstract: In this paper generalized Gaussian and mean curvatures of a parallel hypersurface in E^{n+1} Euclidean space will be denoted respectively by $\overline{K}$ and $\overline{H}$, and Generalized Gaussian and mean curvatures of a parallel hypersurface in $E_1{}^{n+1}$ Lorentz space will be denoted respectively by $\overline{\overline{K}}$ and $\overline{\overline{H}}$.Generalized Gaussian curvature and mean curvatures, $\overline{K}$and $\overline{H}$ofaparallel hypersurface in E^{n+1}Euclidean space are givenin[2].Before nowwe studied relations between curvatures of a hypersurface in Lorentzian space and we introduced higher order Gaussian curvatures of hypersurfaces in Lorentzian space. In this paper, by considering our last studieson higher order Gaussian and mean curvatures, we calculate the generalized $\overline{\overline{K}}$and $\overline{\overline{H}}$ofaparallel hypersurface in $E_1{}^{n+1}$ Lorentz space and we prove theorems about generalized $\overline{\overline{K}}$and $\overline{\overline{H}}$ ofa parallel hypersurface in $E_1{}^{n+1}$ Lorentz space.

Keywords: Gaussian Curvatures, Mean Curvatures, Parallel Hypersurface, Higher Order Gaussian Curvatures

1. Introduction

Suppose that V is an n-dimensional vector space over the real numbers for $n = 1,2,\ldots$

A symmetric bilinear form $\beta: V \times V \to$R is

i) positive (resp.negative) definite if and only if $\vec{w} \neq 0$ implies $\beta(\vec{w},\vec{w}) > 0$ $(resp. \beta(w,w) < 0)$for all $\vec{w}$ in V,

ii) non-degenerate if and only if $\beta(\vec{w},\vec{z}) = 0$ for all $\vec{z}$ in V implies that $\vec{w} = \vec{0}$, and

iii) indefinite if and only if there exist $\vec{w}$and $\vec{z}$ in V with $\beta(\vec{w},\vec{w}) > 0$ and $\beta(\vec{z},\vec{z}) < 0$.

A non-degenerate, symmetric bilinear formβ is called a scalar product. For an indefinite scalar product βon V, a vector $\vec{w} \neq \vec{0}$ is said to be(see [5], p. 4)

a) spacelike if and only if $\beta(\vec{w},\vec{w}) > 0$,

b) timelike if and only if $\beta(\vec{w},\vec{w}) < 0$, and

c) null if and only if $\beta(\vec{w},\vec{w}) = 0$.

2. Basic Concepts

Definition 1.1.

Let U be a unit normal vector field on semi-Riemannianhypersurfaces$M \subset \overline{M}$.The$(1,1)$ tensor field S on M such that

$$\langle S(V), W\rangle = \langle II(V,W), U\rangle forall V, W \in \aleph(M)$$

is called the shape operator of $M \subset \overline{M}$ derived from U. (see [1], p. 107)

Definition 1.2.

Let M and $\overline{M}$ be two hypersurfaces in $E_1{}^{n+1}$ with unit normal vectors $N$of $M$and$\overline{N}$of $\overline{M}$.

$$N = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial_{x_i}}$$

where each α_i is a C^∞ function of M. If there exists a function$f$,from M to $\overline{M}$ such that

$$f: M \to \overline{M}$$

$$P \to f(P) = P + rN_P$$

Then$\overline{M}$ is called parallel hypersurfaceof M, where $r \in R$. (see[3])

Definition 1.3.

Let M be a hypersurface in ${E_1}^{n+1}$ and $T_M(P)$ be a tangent space on M, at $P \in M$. If S_P denotes the shape operator on M, then

$$S_P : T_M(P) \to T_M(P)$$

is a linear mapping. If we denote the characteristic vectors by $k_1, k_2, \ldots, k_n$ and the corresponding characteristic vectors by $x_1, x_2, \ldots, x_n$ of S_P then $k_1, k_2, \ldots, k_n$ are the principal curvatures and $x_1, x_2, \ldots, x_n$ are the principal directions of M, at $P \in M$. On the other hand, if we use the notions $\varepsilon_{i\neq 1}=+1$ and $\varepsilon_1=\pm 1$

$$K_1^{(n)}(k_1, k_2, \ldots, k_n) = \varepsilon_1 k_1 + \sum_{i=2}^{n} \varepsilon_i k_i$$

$$K_2^{(n)}(k_1, k_2, \ldots, k_n) = \sum_{i=1<j}^{n} \varepsilon_i k_i k_j + \sum_{i\neq 1<j}^{n} \varepsilon_i k_i k_j$$

$$K_3^{(n)}(k_1, k_2, \ldots, k_n) = \sum_{i=1<j<t}^{n} \varepsilon_i k_i k_j k_t + \sum_{i\neq 1<j<t}^{n} \varepsilon_i k_i k_j k_t$$

$$\vdots$$

$$K_n^{(n)}(k_1, k_2, \ldots, k_n) = \varepsilon_1 \prod_{i=1}^{n} k_i$$

then the characteristic polynomial of S(P) becomes

$$P_{S(P)}(k) = k^n + (-1){K_1}^{(n)} k^{n-1} + \ldots + (-1)^n {K_n}^{(n)}$$

and $K_1, K_2, \ldots, K_n$ are uniquely determined, where the functions K_i are called the higher ordered Gaussian curvatures of the hypersurface M (see[3]).

Theorem 1.1.

Let $\overline{M}$ be a parallel surface of the surface $M \subset E_1^3$. Let the Gaussian curvature and mean curvature of M be denoted by K and H at $P \in M$, respectively. Gaussian curvature and mean curvature of $\overline{M}$ are denoted by $\overline{\overline{K}}$ and $\overline{\overline{H}}$. Then we know that

i) N_P is timelike

$$\overline{\overline{K}} = \frac{K}{1 + 2rH - r^2 K}$$

$$\overline{\overline{H}} = \frac{H - rK}{1 + 2rH - r^2 K}$$

ii) N_P is spacelike

$$\overline{\overline{K}} = \frac{K}{1 + 2rH + r^2 K}$$

$$\overline{\overline{H}} = \frac{H + rK}{1 + 2rH + r^2 K}$$

[3].

Theorem 1.2.

Let M be a hypersurface in ${E_1}^{n+1}$, $K_1, K_2, \ldots, K_n$ the so-called higher order Gaussian curvatures and $k_1, k_2, \ldots, k_n$ the principal curvatures at the point $P \in M$. Let us define a function

$$\varphi : M \to R$$

$$P \to \varphi(P) = \varphi(r, k_1, k_2, \ldots, k_n)$$

$$= \prod_{i=1}^{n} 1 + \varepsilon_i r k_i$$

such that $\varepsilon_{i\neq 1}=+1$ and $\varepsilon_1=\pm 1$

$$\begin{aligned}
&\varphi(r, k_1, k_2, \ldots, k_n) \\
&= 1 + r\left(\varepsilon_1 k_1 + \sum_{i=2}^{n} \varepsilon_i k_i\right) \\
&+ r^2\left(\sum_{i=1<j}^{n} \varepsilon_i k_i k_j + \sum_{i\neq 1<j}^{n} \varepsilon_i k_i k_j\right) \\
&+ r^3\left(\sum_{i=1<j<t}^{n} \varepsilon_i k_i k_j k_t + \sum_{i\neq 1<j<t}^{n} \varepsilon_i k_i k_j k_t\right) \\
&+ \cdots + r^n\left(\varepsilon_1 \prod_{i=1}^{n} k_i\right)
\end{aligned}$$

or

$$\varphi(k_1, k_2, \ldots, k_n) = rK_1 + r^2 K_2 + \cdots + r^n K_n$$

[4].

Theorem 1.3.

Let M be a hypersurface in ${E_1}^{n+1}$, $K_1, K_2, \ldots, K_n$ are the so-called higher order Gaussian curvatures and $k_1, k_2, \ldots, k_n$ are the principal curvatures at the point $P \in M$. $\overline{\overline{K}}$ and $\overline{\overline{H}}$ are generalized Gaussian and mean curvatures of $\overline{M}$ at the point $f(P)$. Suppose that the function

$$\varphi : M \to R$$

$$P \to \varphi(P) = \varphi(r, k_1, k_2, \ldots, k_n)$$

$$= \prod_{i=1}^{n} 1 + \varepsilon_i r k_i$$

such that $\varepsilon_i=+1$ and $\varepsilon_1=\pm 1$ ($i\neq +1$).

Then we have

$$\overline{\overline{K}} = \varepsilon \frac{\frac{\partial^n \varphi(r, k_1, k_2, \ldots, k_n)}{\partial r^n}}{n!\, \varphi(r, k_1, k_2, \ldots, k_n)}$$

$$\overline{\overline{H}} = \frac{\frac{\partial \varphi(r,k_1,k_2,\dots,k_n)}{\partial r}}{\varphi(r,k_1,k_2,\dots,k_n)}.$$

Proof

If k is principal curvatures of M at the point P in direction X, then $\frac{k}{1+rk}$ is the principal curvatures of $\overline{M}$ at the point $f(P)$ in direction $f_*(X)$, that is, $\bar{S}(f_*(X)) = \frac{k}{1+rk} f_*(X)$ which means that f preserves principal directions, where f_* is the differential of f and we know that

$$\bar{S}(f_*(X_1)) = \frac{k_1}{1+rk_1} f_*(X_1)$$

$$\bar{S}(f_*(X_2)) = \frac{k_2}{1+rk_2} f_*(X_2)$$

$$\bar{S}(f_*(X_n)) = \frac{k_n}{1+rk_n} f_*$$

then we know that the shape operator of $\overline{M}$ is

$$\bar{S} = \begin{bmatrix} \frac{\varepsilon_1 k_1}{1+r\varepsilon_1 k_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\varepsilon_n k_n}{1+r\varepsilon_n k_n} \end{bmatrix}$$

and

$$\overline{\overline{K}} = \det S_r$$

$$= \varepsilon \left(\frac{\varepsilon_1 k_1}{1+r\varepsilon_1 k_1} \dots \frac{\varepsilon_n k_n}{1+r\varepsilon_n k_n} \right)$$

$$= \varepsilon \frac{\varepsilon_1 k_1 \varepsilon_2 k_2 \dots \varepsilon_n k_n}{\prod_{i=1}^{n}(1+\varepsilon_i r k_i)}$$

$$= \varepsilon \frac{\prod_{i=1}^{n} \varepsilon_i k_i}{\prod_{i=1}^{n}(1+\varepsilon_i r k_i)}$$

We multiply the right sides of the equation with $n!$

$$= \varepsilon \frac{n! \prod_{i=1}^{n} \varepsilon_i k_i}{n! \prod_{i=1}^{n}(1+\varepsilon_i r k_i)}$$

$$= \varepsilon \frac{n!\, K_n}{n!\, \varphi(\mathrm{r}, \mathrm{k}_1, \mathrm{k}_2, \dots, k_n)}$$

and we derivate to $\varphi(\mathrm{r}, \mathrm{k}_1, \mathrm{k}_2, \dots, k_n)$ order n according to r

$$\frac{\partial \varphi(\mathrm{r}, \mathrm{k}_1, \mathrm{k}_2, \dots, k_n)}{\partial r} = \frac{\partial(1 + \mathrm{r}K_1 + r^2 K_2 + \cdots + r^n K_n)}{\partial r}$$

$$= K_1 + 2rK_2 + \cdots + nr^{n-1}K_n$$

$$\frac{\partial^2 \varphi(\mathrm{r}, \mathrm{k}_1, \mathrm{k}_2, \dots, k_n)}{\partial r^2} = \frac{\partial(K_1 + 2rK_2 + \cdots + nr^{n-1}K_n)}{\partial r}$$

$$= 2K_2 + \cdots + n(n-1)r^{n-2}K_n$$

and we continue to derivation, we have

$$\frac{\partial^n \varphi(\mathrm{r}, \mathrm{k}_1, \mathrm{k}_2, \dots, k_n)}{\partial r^n} = n!\, K_n$$

and we obtain with implying equality

$$\overline{\overline{K}} = \varepsilon \frac{\frac{\partial^n \varphi(\mathrm{r}, \mathrm{k}_1, \mathrm{k}_2, \dots, k_n)}{\partial r^n}}{n!\, \varphi(\mathrm{r}, \mathrm{k}_1, \mathrm{k}_2, \dots, k_n)}.$$

We proof the other equality

$$\overline{\overline{H}} = \frac{1}{n} IzS_r$$

$$= \frac{1}{n}\left(\frac{\varepsilon_1 k_1}{1+r\varepsilon_1 k_1} + \cdots + \frac{\varepsilon_n k_n}{1+r\varepsilon_n k_n} \right)$$

$$= \frac{1}{n}\left(\frac{\varepsilon_1 k_1 \prod_{i=2}^{n}(1+\varepsilon_i r k_i) + \cdots + \varepsilon_n k_n \prod_{i=1}^{n-1}(\varepsilon_i r k_i)}{\prod_{i=1}^{n}(1+\varepsilon_i r k_i)} \right)$$

We derivate according to r

$$\frac{\partial \varphi(\mathrm{r}, \mathrm{k}_1, \mathrm{k}_2, \dots, k_n)}{\partial r} = \frac{\partial\left(\prod_{i=1}^{n}(1+\varepsilon_i r k_i)\right)}{\partial r}$$

$$= \frac{\partial\big((1+\varepsilon_1 r k_1)(1+\varepsilon_2 r k_2)\dots(1+\varepsilon_n r k_n)\big)}{\partial r}$$

$$= \varepsilon_1 k_1 (1+\varepsilon_2 r k_2)(1+\varepsilon_3 r k_3) \dots (1+\varepsilon_n r k_n)$$

$$+ (1+\varepsilon_1 r k_1)\varepsilon_2 k_2 (1+\varepsilon_3 r k_3) \dots (1+\varepsilon_n r k_n)$$

$$+ (1+\varepsilon_1 r k_1)(1+\varepsilon_2 r k_2)\varepsilon_3 k_3 \dots (1+\varepsilon_n r k_n)$$

$$\cdots$$

$$+ (1+\varepsilon_1 r k_1)(1+\varepsilon_2 r k_2)(1+\varepsilon_3 r k_3) \dots (1 + \varepsilon_{n-1} r k_{n-1})\varepsilon_n k_n$$

$$= \varepsilon_1 k_1 \prod_{i=2}^{n}(1+\varepsilon_i r k_i) + \varepsilon_2 k_2 (1+\varepsilon_1 r k_1) \prod_{i=3}^{n}(1+\varepsilon_i r k_i) + \cdots + \varepsilon_n k_n \prod_{i=1}^{n-1}(\varepsilon_i r k_i)$$

So we have last equation and we obtain that

$$\overline{\overline{H}} = \frac{1}{n} \frac{\frac{\partial \varphi(\mathrm{r}, \mathrm{k}_1, \mathrm{k}_2, \dots, k_n)}{\partial r}}{\varphi(\mathrm{r}, \mathrm{k}_1, \mathrm{k}_2, \dots, k_n)}.$$

3. Generalized Theorems

Theorem 2.1.

Let $\overline{M}$ be a parallel hypersurface in E_1^{n+1} and $K_1, K_2, \dots, K_n$ are the so-called higher order Gaussian curvatures of M at the point $P \in M$, and let

$$\sum_{i=1}^{n-1} r^i K_i = -1$$

thenthe generalized Gaussian curvature of $\overline{M}$ is

$$\overline{\overline{K}} = \varepsilon \frac{1}{r^n}.$$

Proof

We know that the generalized Gaussian curvature of a parallel hypersurface is given by

$$\overline{\overline{K}} = \varepsilon \frac{\frac{\partial^n \varphi(\mathrm{r}, k_1, k_2, \ldots, k_n)}{\partial r^n}}{n!\, \varphi(\mathrm{r}, k_1, k_2, \ldots, k_n)}$$

$$= \varepsilon \frac{\prod_{i=1}^{n} \varepsilon_i k_i}{1 + rK_1 + r^2 K_2 + \cdots + r^n K_n}$$

since we have,

$$\sum_{i=1}^{n-1} r^i K_\mathrm{i} = -1$$

then

$$\sum_{i=1}^{n-1} r^i K_i = 1 + rK_1 + r^2 K_2 + \cdots + r^{n-1} K_{n-1} = -1$$

andfinallyweget

$$\overline{\overline{K}} = \varepsilon \frac{\prod_{i=1}^{n} \varepsilon_i k_i}{r^n K_n}$$

$$\overline{\overline{K}} = \varepsilon \frac{\prod_{i=1}^{n} \varepsilon_i k_i}{r^n \prod_{i=1}^{n} \varepsilon_i k_i} = \varepsilon \frac{1}{r^n}.$$

Theorem 2.2.

Let $\overline{M}$ be a parallel hypersurface in E_1^{n+1} and $K_1, K_2, \ldots, K_n$ so-called higher order Gaussian curvatures of M, at the point $P \in M$ and let

$$\sum_{i=1}^{n} (i-1)\, r^i K_i = 1$$

then generalized Gaussian curvature of $\overline{M}$ is

$$\overline{\overline{H}} = \frac{1}{r}.$$

Proof:

We know that the generalized mean curvature of a parallel hypersurface is given by

$$\overline{\overline{H}} = \frac{\frac{\partial \varphi(\mathrm{r}, \mathrm{k}_1, \mathrm{k}_2, \ldots, k_n)}{\partial r}}{\varphi(\mathrm{r}, \mathrm{k}_1, \mathrm{k}_2, \ldots, k_n)}$$

$$= \frac{K_1 + 2rK_2 + \cdots + nr^{n-1} K_n}{1 + rK_1 + r^2 K_2 + \cdots + r^n K_n}$$

$$= \frac{1}{r} \frac{rK_1 + 2r^2 K_2 + \cdots + nr^n K_n}{1 + rK_1 + r^2 K_2 + \cdots + r^n K_n}$$

$$\sum_{i=1}^{n} r^i K_i = 1 + rK_1 + r^2 K_2 + \cdots + r^n K_n$$

since we have that

$$\overline{\overline{H}} = \frac{1}{r} \frac{\sum_{i=1}^{n} i r^i K_i}{1 + \sum_{i=1}^{n} r^i K_i}$$

we can write that

$$\sum_{i=1}^{n} r^i K_i = \sum_{i=1}^{n} (i-1)\, r^i K_i + \sum_{i=1}^{n} r^i K_i$$

so we obtain that

$$\overline{\overline{H}} = \frac{1}{r} \frac{\sum_{i=1}^{n} (i-1)\, r^i K_i + \sum_{i=1}^{n} r^i K_i}{1 + \sum_{i=1}^{n} r^i K_i}$$

If we add and subtract 1 inthe numerator, the above equality does not change

$$\overline{\overline{H}} = \frac{1}{r} \frac{1 + \sum_{i=1}^{n} (i-1)\, r^i K_i + \sum_{i=1}^{n} r^i K_i - 1}{1 + \sum_{i=1}^{n} r^i K_i}$$

$$= \frac{1}{r}\left(1 - \left(\frac{1 - \sum_{i=1}^{n} (i-1)\, r^i K_i}{1 + \sum_{i=1}^{n} r^i K_i}\right)\right)$$

$$= \frac{1}{r}\left(1 - \left(\frac{1 - 1}{1 + \sum_{i=1}^{n} r^i K_i}\right)\right)$$

and we have that

$$\overline{H} = \frac{1}{r}.$$

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