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On the Construction of Molaei's Generalized Hypergroups

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Abstract: The purpose of this paper is making a construction and generalization of Molaei's generalized groups by using construction of the Rees matrix semigroup over a polygroup *H* and a matrix with entries in *H*. We call it "Molaei's generalized hypergroups" and we give some examples.

Keywords: Hypergroup, Polygroup, Molaei's Generalized Hypergroup

1. Introduction

In [10] generalized groups or completely simple semigroups is introduced as a class of algebras of interest in physics and they are an interesting generalization of groups. In [1], it is proved the generalized groups are the completely simple semigroups. Aaújo and Konieczny used the Rees matrix semigroup, (see [8]) over a group and they showed that the Molaei's generalized groups are the completely simple semigroups. In this paper we change the group to the polygroup and we obtain a new construction, by using this construction we can define "Molaei's generalized hypergroup" and we give some examples.

Let H be a non-empty set. A hyperoperation on H is a function from $H\times H$ to $P^*(H)$, which $P^*(H)$ is the set of all non-empty subsets of H. A hypergroupoid is the couple $(H, *)$, where H is a non-empty set and "*" is a hyperoperation on H, i.e., $*: H\times H\to P^*(H)$. As usual, we write $a*b = *(a,b)$, for all a and b in H. If M and N belong to $P^*(H)$ and a be an element of H, we define:

$$M*N := \bigcup_{m\in M,\, n\in N} m*n, \quad M*a := M*\{a\},$$
$$a*N := \{a\}*N.$$

The relational notation $M \approx N$ is used to assert that M and N have an element in common, i.e., $M\cap N$ is non empty set.

We recalled the following definitions: [3, 4, 6, 9]

1) the hyperoperatoin ' * ' is associative, if for every elements a, b and c of H, $(a*b)*c=a*(b*c)$;

2) the hypergroupoid $(H, *)$ is semihypergroup, if the hyperoperation ' * ' is associative;

3) the hypergroupoid $(H, *)$ is quasihypergroup, if for all a of H, $a*H = H*a = H$;

4) the hypergroupoid $(H, *)$ is hypergroup if it is both quasihypergroup and semihypergroup,

5) the hypergroup $(H, *)$ is polygroup if there exist a unique element e in H, which for every a in H, $e * a = a * e = \{a\}$, and there exists a unitary operation ${}^{-1}: H \to H$, by a maps to a^{-1}, which for every elements a, b and c in H, if a be an element of $b * c$ then b be an element of $a * c^{-1}$ and c be an element of $b^{-1} * a$.

As usual, this polygroup is demonstrated by $< H, *, e, {}^{-1} >$. We refer to [2, 5, 6, 7], for more details about polygroups.

Let $< H, *, e, {}^{-1} >$ be a polygroup and K be a non-empty subset of H, we denoted $K^{-1} = \{k^{-1}: k$ be an element of $K\}$, it is easy to show that, the following axioms hold for every a and b in H,

$$(a^{-1})^{-1} = a,\ e^{-1} = e,\ e \in (a * a^{-1}) \cap (a^{-1} * a),\ (a * b)^{-1} = b^{-1} * a^{-1}.$$

2. Molaei's Generalized Hypergroups

In this section, we consider a polygroup and by using the Rees matrix semigroup's structure over polygroup, we construct a new structure and obtain three properties of this new structure. Theorem 2.1, 2.2 and 2.3 guide us inspire the definition of Molaei's generalized hypergroups.

Let $< H, *, e, {}^{-1} >$ be a polygroup and let I, Λ be non-empty sets and M be a map from $\Lambda \times I$ to H, by $M(\lambda, i) =$

$m_{\lambda i}$.

Assume that $\mathcal{MGH}(H; I, \Lambda, M) := I \times H \times \Lambda$, We define the following hyper-composition:

$$\circ: \mathcal{MGH}(H; I, \Lambda, M) \times \mathcal{MGH}(H; I, \Lambda, M) \to P^*(\mathcal{MGH}(H; I, \Lambda, M))$$

$$((i, x, \lambda), (j, y, \mu)) \mapsto (i, x, \lambda) \circ (j, y, \mu),$$

which for all i and j in I, for all x and y in H and for all λ and μ in Λ,

$$(i, x, \lambda) \circ (j, y, \mu) := \{i\} \times (x * m_{\lambda j} * y) \times \{\mu\}.$$

Theorem 2.1. $\mathcal{MGH}(H; I, \Lambda, M)$ is a semihypergroup.

Proof. Let i and j in I, λ and μ in Λ and a, b, c in H. Since $(a * m_{\lambda i} * b)$ is a non-empty subset of H, so

$$\{\} \neq (i, x, \lambda) \circ (j, y, \mu) \in P^*(\mathcal{MGH}(H; I, \Lambda, M)).$$

Therefore "$\circ$" is a hyperoperation. Now we check the associative property of hyperoperation "$\circ$".

We have the following equations:

$$\begin{aligned}
(i, a, \lambda) \circ ((j, b, \mu) \circ (k, c, \upsilon)) &= (i, a, \lambda) \circ (\{j\} \times (b * m_{\mu k} * c) \times \{\upsilon\})) \\
&= (i, a, \lambda) \circ \bigcup_{s \in b * m_{\mu k} * c} (j, s, \upsilon) \\
&= \bigcup_{s \in b * m_{\mu k} * c} (i, a, \lambda) \circ (j, s, \upsilon) \\
&= \bigcup_{s \in b * m_{\mu k} * c} \{i\} \times (a * m\lambda j * s) \times \{\upsilon\} \\
&= \{i\} \times (\bigcup_{s \in b * m_{\mu k} * c} a * m\lambda j * s) \times \{\upsilon\} \\
&= \{i\} \times ((a * m\lambda j) * (b * m\mu k * c)) \times \{\upsilon\} \\
&= \{i\} \times ((a * m\lambda j * b) * m\mu k * c)) \times \{\upsilon\} \\
&= \{i\} \times (\bigcup_{t \in a * m_{\lambda j} * b} t * m\mu k * c) \times \{\upsilon\} \\
&= \bigcup_{t \in a * m_{\lambda j} * b} \{i\} \times (t * m\mu k * c) \times \{\upsilon\} \\
&= \bigcup_{t \in a * m_{\lambda j} * b} (i, t, \mu) \circ (k, c, \upsilon) \\
&= (\bigcup_{t \in a * m_{\lambda j} * b} (i, t, \mu)) \circ (k, c, \upsilon) \\
&= (\{i\} \times (a * m\lambda j * b) \times \{\mu\}) \circ (k, c, \upsilon) \\
&= ((i, a, \lambda) \circ ((j, b, \mu)) \circ (k, c, \upsilon)
\end{aligned}$$

Therefore, $\mathcal{MGH}(H; I, \Lambda, M)$ is a semihypergroup.

Theorem 2.2. For every element $(i, a, \lambda) \in \mathcal{MGH}(H; I, \Lambda, M)$, there is a unique non-empty subset $E(i, a, \lambda) \subseteq \mathcal{MGH}(H; I, \Lambda, M)$, such that for every element (j, b, μ) of $E(i, a, \lambda)$, implies
$(i, a, \lambda) \in [(i, a, \lambda) \circ (j, b, \mu)] \cap [(j, b, \mu) \circ (i, a, \lambda)]$. Moreover,

$$E(i, a, \lambda) = \{i\} \times [(m_{\lambda i}{}^{-1} * a^{-1} * a) \cap (a * a^{-1} * m_{\lambda i}{}^{-1})] \times \{\lambda\}.$$

Proof. Since $m_{\lambda i}$ is an element of polygroup H, there exist $m_{\lambda i}{}^{-1}$, such that
$e \in [(m_{\lambda i}{}^{-1} * m_{\lambda i}) \cap (m_{\lambda i} * m_{\lambda i}{}^{-1})]$. Now, we have:

$$\begin{aligned}
(i, a, \lambda) \circ (i, m_{\lambda i}{}^{-1}, \lambda) &= \{i\} \times (a * m_{\lambda i} * m_{\lambda i}{}^{-1}) \times \{\lambda\} \\
&\supseteq \{i\} \times (a * e) \times \{\lambda\} = \{i\} \times \{a\} \times \{\lambda\} = \{(i, a, \lambda)\}. \\
\text{Also } \{(i, a, \lambda)\} &= \{i\} \times \{a\} \times \{\lambda\} = \{i\} \times (e * a) \times \{\lambda\} \\
&\subseteq \{i\} \times (m_{\lambda i}{}^{-1} * m_{\lambda i} * a) \times \{\lambda\} \\
&= (i, m_{\lambda i}{}^{-1}, \lambda) \circ (i, a, \lambda).
\end{aligned}$$

Therefore, $(i, m_{\lambda i}{}^{-1}, \lambda)$ is an element of $E(i, a, \lambda)$.

If (j, b, μ) be an arbitrary element of $E(i, a, \lambda)$, then we have:

$$\begin{aligned}
(i, a, \lambda) &\in [(i, a, \lambda) \circ (j, b, \mu)] \cap [(j, b, \mu) \circ (i, a, \lambda)] \\
&= [\{i\} \times (a * m_{\lambda j} * b) \times \{\mu\}] \cap [\{j\} \times (b * m_{\mu i} * a) \times \{\lambda\}].
\end{aligned}$$

Therefore, j = i and $\mu = \lambda$ and $a \in (a * m_{\lambda i} * b) \cap (b * m_{\lambda i} * a)$. Since,

$$a \in (a * m_{\lambda i} * b) \Leftrightarrow b \in (a * m_{\lambda i})^{-1} * a = m_{\lambda i}^{-1} * a^{-1} * a,$$

$$a \in (b * m_{\lambda i} * a) \Leftrightarrow b \in a * (m_{\lambda i} * a)^{-1} = a * a^{-1} * m_{\lambda i}^{-1},$$

therefore, $b \in [(m_{\lambda i}^{-1} * a^{-1} * a) \cap (a * a^{-1} * m_{\lambda i}^{-1})]$.

Conversely if $b \in [(m_{\lambda i}^{-1} * a^{-1} * a) \cap (a * a^{-1} * m_{\lambda i}^{-1})]$, then (i, b, λ) is an element of $E(i, a, \lambda)$ and the proof is complete.

Theorem 2.3. For every element $(i, a, \lambda) \in \mathcal{MGH}(H; I, \Lambda, M)$, there is a non-empty subset $\mathcal{I}(i, a, \lambda) \in \mathcal{MGH}(H; I, \Lambda, M)$, such that for all β in $\mathcal{I}(i, a, \lambda)$

$$[(i, a, \lambda) \circ \beta] \cap [\beta \circ (i, a, \lambda)] \approx E(i, a, \lambda).$$

Proof. Assume that $c = m_{\lambda i}$, by choosing $\mathcal{I}(i, a, \lambda) = \{i\} \times (c^{-1} * a^{-1} * c^{-1}) \times \{\lambda\}$. which, it is a non-empty subset of $\mathcal{MGH}(H; I, \Lambda, M)$, we show that it satisfies the condition of Theorem. Since H is a polygroup, hence, $\{a\} = a * e$ and $e \in c * c^{-1}$ and $e \in a * a^{-1}$ then we have:

$$e \in a * a^{-1} = (a * e) * a^{-1} = (a * e * a^{-1}) \subseteq (a * (c * c^{-1}) * a^{-1}) = (a * c * c^{-1} * a^{-1}).$$

Then $c^{-1} \in \{c^{-1}\} = e * c^{-1} \subseteq (a * c * c^{-1} * a^{-1}) * c^{-1} = (a * c) * (c^{-1} * a^{-1} * c^{-1})$.

Similarly,

$$e \in a^{-1} * a = (a^{-1} * e) * a = (a^{-1} * e * a) \subseteq (a^{-1} * (c^{-1} * c) * a) = (a^{-1} * c^{-1} * c * a).$$

Then $c^{-1} \in \{c^{-1}\} = c^{-1} * e \subseteq c^{-1} * (a^{-1} * c^{-1} * c * a) = (c^{-1} * a^{-1} * c^{-1}) * c * a$.

Also, $c^{-1} \in (c^{-1} * a^{-1} * a) \cap (a * a^{-1} * c^{-1})$, hence, c^{-1} is an element of the following set

$$[(a * c) * (c^{-1} * a^{-1} * c^{-1})] \cap [(c^{-1} * a^{-1} * c^{-1}) * c * a] \cap [(c^{-1} * a^{-1} * a)] \cap [(a * a^{-1} * c^{-1})],$$

then,

$$(i, m_{\lambda i}^{-1}, \lambda) \in [(i, a, \lambda) \circ \mathcal{I}(i, a, \lambda)] \cap [\mathcal{I}(i, a, \lambda) \circ (i, a, \lambda)] \cap E(i, a, \lambda).$$

Let $\beta = (i, x, \lambda) \in \mathcal{I}(i, a, \lambda)$, then $x \in c^{-1} * a^{-1} * c^{-1}$, hence $c^{-1} \in a * c * x \cap x * c * a$, therefore c^{-1} is an element of the set $[a * c * x] \cap [x * c * a] \cap [(c^{-1} * a^{-1} * a)] \cap [(a * a^{-1} * c^{-1})]$, then

$$(i, m_{\lambda i}^{-1}, \lambda) \in [(i, a, \lambda) \circ (i, x, \lambda)] \cap [(i, x, \lambda) \circ (i, a, \lambda)] \cap E(i, a, \lambda).$$

and the proof is complete.

Theorem 2.1, 2.2 and 2.3 guidance us to follows for definition of a generalization of Molaei's generalized group.

Definition 2.1. A semihypergroup $(\mathcal{H}, \circ)$ is called Molaei's generalized hypergroup, if it satisfies in the following conditions:

(MGH1) $\forall h \in \mathcal{H}$, $\exists!\ \mathcal{E}(h) \subseteq \mathcal{H}$, such that for every element $\alpha \in \mathcal{E}(h)$, $h \in [h \circ \alpha] \cap [\alpha \circ h]$,

(MGH1) $\forall h \in \mathcal{H}$, $\exists\ \mathcal{I}(h) \subseteq \mathcal{H}$, such that for every element $\beta \in \mathcal{I}(h)$, $[h \circ \beta] \cap [\beta \circ h] \approx \mathcal{E}(h)$,

(The symbole $\exists!$ means there is a unique.)

Example 2.1. If $< H, *, e, ^{-1} >$ be a polygroup and let I, Λ be non-empty sets and M be a map from $\Lambda \times I$ to H by $M(\lambda, i) = m_{\lambda i}$. Then, by use Theorems 2.1, 2.2 and 2.3, $\mathcal{MGH}(H; I, \Lambda, M) := I \times H \times \Lambda$, with hyperoperation "$\circ$" is a Molaei's generalized hypergroup.

Example 2.2. Every polygroup is a Molaei's generalized hypergroup. If $< H, *, e, ^{-1} >$ be a polygroup, it is semihypergroup and for every element $h \in H$,

$$\mathcal{E}(h) = (h * h^{-1}) \cap (h^{-1} * h) \text{ and } \mathcal{I}(h) = \{h^{-1}\}.$$

Example 2.3. Every Molaei's generalized group is a Molaei's generalized hypergroup. If G be a Molaei's generalized group, we consider the hyperoperation $x * y = \{xy\}$, then $(G, *)$ is a semihypergroup and for every $h \in G$,

$$\mathcal{E}(h) = \{e(h)\} \text{ is unique and } \mathcal{I}(h) = \{h^{-1}\}.$$

3. Conclusion

This paper deal with one of the newest construction of a generalization of hypergroups. We changed the group to the polygroup in the structure of Rees matrix semigroup and we obtained a new construction. By using this construction we defined "Molaei's generalized hypergroup" and we gave some examples.

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